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A GEOMETRICAL PROOF OF A NEW INEQUALITY FOR THE GAMMA FUNCTION

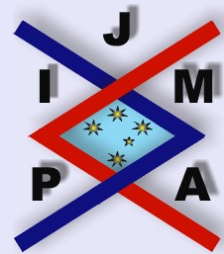
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Abstract

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Abstract

Using the inclusions between the unit balls for the p -norms, we obtain a new inequality for the gamma function.

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Since the gamma function

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt, \quad x > 0$$

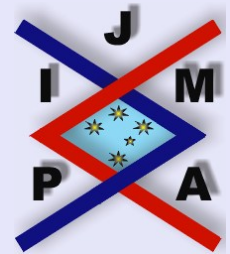
is one of the most important functions in Mathematics, there exists an extensive literature on its inequalities (see [1], [2]).

Our aim here is to present and prove the inequalities

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1 \quad x \in [0, 1], \quad n \in \mathbb{N}.$$

As we will show the above inequalities follow immediately from a key geometrical argument. From now on for any $r > 0$, $p, n \geq 1$ we will consider the notation:

$$D_{\|\cdot\|_p}^{n,r} = \{(x_1, \dots, x_n) \in \mathbb{R}^n / \|(x_1, \dots, x_n)\|_p < r\}$$



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for the n -ball of radius r for the p -norm $\|(x_1, \dots, x_n)\|_p = (|x_1|^p + \dots + |x_n|^p)^{1/p}$.

To this end, we need to prove the following:

Lemma 1. For all n in \mathbb{N} , $p \geq 1$ and $r > 0$ we have:

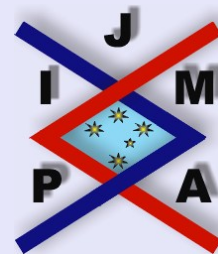
$$(1) \quad \text{Volume} \left(D_{\|\cdot\|_p}^{n,r} \right) = 2^n \frac{\Gamma \left(1 + \frac{1}{p} \right)^n}{\Gamma \left(1 + \frac{n}{p} \right)} r^n.$$

Proof. For $n = 1$, $D_{\|\cdot\|_p}^{1,r}$ is the interval $(-r, r)$, whose measure is $2r$, i.e.,

$$2r = 2 \frac{\Gamma \left(1 + \frac{1}{p} \right)}{\Gamma \left(1 + \frac{1}{p} \right)} r$$

and (1) holds. By induction, let us assume that (1) holds for $n-1$. Then we note that $|x_1|^p + \dots + |x_n|^p < r^p$ is equivalent to $|x_1|^p + \dots + |x_{n-1}|^p < r^p - |x_n|^p$ and by virtue of the induction hypothesis we have

$$\begin{aligned} \text{Volume} \left(D_{\|\cdot\|_p}^{n,r} \right) &= \int_{D_{\|\cdot\|_p}^{n,r}} dx_1 \dots dx_n \\ &= 2 \int_0^r \left(\int_{D_{\|\cdot\|_p}^{n-1, (r^p - |x_n|^p)^{1/p}} dx_1 \dots dx_{n-1}} \right) dx_n \end{aligned}$$



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$$\begin{aligned}
&= 2 \int_0^r 2^{n-1} \frac{\Gamma\left(1 + \frac{1}{p}\right)^{n-1}}{\Gamma\left(1 + \frac{n-1}{p}\right)} (r^p - x_n^p)^{\frac{n-1}{p}} dx_n \\
&= 2^n \frac{\Gamma\left(1 + \frac{1}{p}\right)^{n-1}}{\Gamma\left(1 + \frac{n-1}{p}\right)} r^n \int_0^1 (1 - z^p)^{\frac{n-1}{p}} dz,
\end{aligned}$$

where $z = x_n/r$.

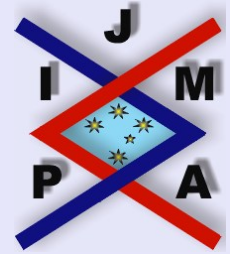
If we consider $F(a, b, c, z)$ the first hypergeometric function (see [3]), then

$$\int (1 - z^p)^{\frac{n-1}{p}} dz = z F\left(\frac{1}{p}, -\frac{n-1}{p}, 1 + \frac{1}{p}, z^n\right)$$

and by well-known properties of the hypergeometric function we deduce:

$$\begin{aligned}
\text{Volume } \left(D_{\|\cdot\|_p}^{n,r}\right) &= 2^n \frac{\Gamma\left(1 + \frac{1}{p}\right)^{n-1}}{\Gamma\left(1 + \frac{n-1}{p}\right)} r^n F\left(\frac{1}{p}, -\frac{n-1}{p}, 1 + \frac{1}{p}, 1\right) \\
&= 2^n \frac{\Gamma\left(1 + \frac{1}{p}\right)^{n-1}}{\Gamma\left(1 + \frac{n-1}{p}\right)} r^n \frac{\Gamma\left(1 + \frac{1}{p}\right) \Gamma\left(1 + \frac{n-1}{p}\right)}{\Gamma\left(1 + \frac{n}{p}\right)} \\
&= 2^n \frac{\Gamma\left(1 + \frac{1}{p}\right)^n}{\Gamma\left(1 + \frac{n}{p}\right)} r^n.
\end{aligned}$$

□



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Therefore we have

Theorem 2. For all $n \in \mathbb{N}$ and x in $(0, 1)$ we have

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1.$$

Proof. For all n in \mathbb{N} and $p \geq 1$, from the inclusions

$$D_{\|\cdot\|_1}^{n,1} \subseteq D_{\|\cdot\|_p}^{n,1} \subseteq D_{\|\cdot\|_\infty}^{n,1},$$

we deduce

$$\text{Volume} \left(D_{\|\cdot\|_1}^{n,1} \right) \leq \text{Volume} \left(D_{\|\cdot\|_p}^{n,1} \right) \leq \text{Volume} \left(D_{\|\cdot\|_\infty}^{n,1} \right),$$

so by Lemma 1:

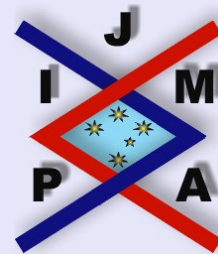
$$2^n \frac{\Gamma(2)^n}{\Gamma(n+1)} \leq 2^n \frac{\Gamma\left(1 + \frac{1}{p}\right)^n}{\Gamma\left(1 + \frac{n}{p}\right)} \leq 2^n$$

and with $1/p = x$, bearing in mind that $\Gamma(2) = 1$, $\Gamma(n+1) = n!$,

$$\frac{1}{n!} \leq \frac{\Gamma(1+x)^n}{\Gamma(1+nx)} \leq 1.$$

□

From this it follows immediately that the function $\Gamma(1+x)^n/\Gamma(1+nx)$ is strictly decreasing on $(0, 1]$.



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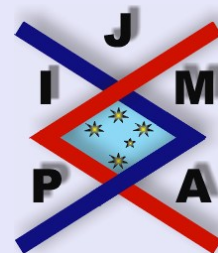
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