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PARTITIONED CYCLIC FUNCTIONAL EQUATIONS

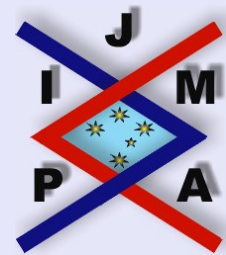
JAE-HYEONG BAE AND WON-GIL PARK

Department of Mathematics
Chungnam National University
Daejeon 305-764, Korea.

EMail: jhbae@math.cnu.ac.kr

EMail: wgpark@math.cnu.ac.kr

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Abstract

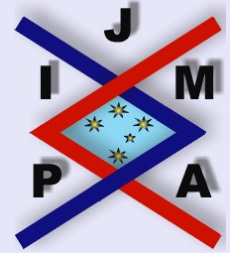
We prove the generalized Hyers-Ulam-Rassias stability of a partitioned functional equation. It is applied to show the stability of algebra homomorphisms between Banach algebras associated with partitioned functional equations in Banach algebras.

2000 Mathematics Subject Classification: 39B05, 39B82

Key words: Stability, Partitioned functional equation, Algebra homomorphism.

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1. Partitioned Cyclic Functional Equations

Let E_1 and E_2 be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f : E_1 \rightarrow E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in E_1$. Th. M. Rassias [4] showed that there exists a unique \mathbb{R} -linear mapping $T : E_1 \rightarrow E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p} \|x\|^p$$

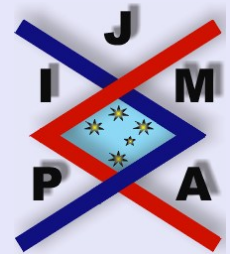
for all $x \in E_1$.

Recently, T. Trif [5] proved that, for vector spaces V and W , a mapping $f : V \rightarrow W$ with $f(0) = 0$ satisfies the functional equation

$$\begin{aligned} n_{n-2} C_{k-2} f\left(\frac{x_1 + \cdots + x_n}{n}\right) + n_{n-2} C_{k-1} \sum_{i=1}^n f(x_i) \\ = k \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right) \end{aligned}$$

for all $x_1, \dots, x_n \in V$ if and only if the mapping $f : V \rightarrow W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

Throughout this paper, let V and W be real normed vector spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively, and let p, k and n be positive integers with $k \leq p^n$.



Lemma 1.1. A mapping $f : V \rightarrow W$ with $f(0) = 0$ satisfies the functional equation

$$(1.1) \quad p^n f \left(\frac{x_1 + \cdots + x_{p^n}}{p^n} \right) + p(k-1) \sum_{i=1}^{p^n-1} f \left(\frac{x_{pi-p+1} + \cdots + x_{pi}}{p} \right) \\ = k \sum_{i=1}^{p^n} f \left(\frac{x_i + \cdots + x_{i+k-1}}{k} \right)$$

for all $x_1 = x_{p^n+1}, \dots, x_{k-1} = x_{p^n+k-1}, x_k, \dots, x_{p^n} \in V$ if and only if the mapping $f : V \rightarrow W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

Proof. Assume that a mapping $f : V \rightarrow W$ satisfies (1.1). Put $x_1 = x, x_2 = y$ and $x_3 = \cdots = x_{p^n} = 0$ in (1.1), then

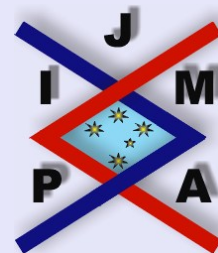
$$(1.2) \quad p^n f \left(\frac{x+y}{p^n} \right) + p(k-1) f \left(\frac{x+y}{p} \right) \\ = k \left[(k-1) f \left(\frac{x+y}{k} \right) + f \left(\frac{x}{k} \right) + f \left(\frac{y}{k} \right) \right].$$

Putting $y = 0$ in (1.2),

$$(1.3) \quad p^n f \left(\frac{x}{p^n} \right) + p(k-1) f \left(\frac{x}{p} \right) = k^2 f \left(\frac{x}{k} \right).$$

Replacing x by kx and y by ky in (1.2),

$$(1.4) \quad p^n f \left(\frac{kx+ky}{p^n} \right) + p(k-1) f \left(\frac{kx+ky}{p} \right)$$



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$$= k[(k-1)f(x+y) + f(x) + f(y)].$$

Replacing x by $kx + ky$ in (1.3),

$$(1.5) \quad p^n f\left(\frac{kx + ky}{p^n}\right) + p(k-1)f\left(\frac{kx + ky}{p}\right) = k^2 f(x+y).$$

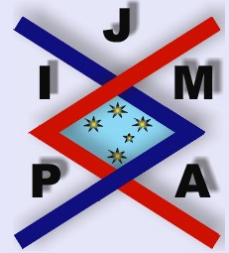
From (1.4) and (1.5),

$$0 = -kf(x+y) + k[f(x) + f(y)].$$

Hence f is additive.

The converse is obvious. □

The main purpose of this paper is to prove the generalized Hyers-Ulam-Rassias stability of the functional equation (1.1).



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2. Stability of Partitioned Cyclic Functional Equations

From now on, let W be a Banach space.

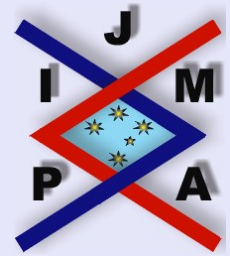
We are going to prove the generalized Hyers-Ulam-Rassias stability of the functional equation (1.1). From now on, $n \geq 2$. For a given mapping $f : V \rightarrow W$, we set

$$(2.1) \quad Df(x_1, \dots, x_{p^n}) := p^n f\left(\frac{x_1 + \dots + x_{p^n}}{p^n}\right) \\ + p(p^2 - 1) \sum_{i=1}^{p^{n-1}} f\left(\frac{x_{pi-p+1} + \dots + x_{pi}}{p}\right) \\ - p^2 \sum_{i=1}^{p^n} f\left(\frac{x_i + \dots + x_{i+p^2-1}}{p^2}\right)$$

for all $x_1 = x_{p^{n+1}}, \dots, x_{p^{2-1}} = x_{p^n+p^2-1}, x_{p^2}, \dots, x_{p^n} \in V$.

Theorem 2.1. *Let $f : V \rightarrow W$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : V^{p^n} \rightarrow [0, \infty)$ such that*

$$(2.2) \quad \tilde{\varphi}(x) \\ := \sum_{j=0}^{\infty} p^j \varphi \left(\underbrace{\frac{x}{p^j}, \dots, \frac{x}{p^j}}_{p \text{ times}}, \underbrace{0, \dots, 0}_{p^2-p \text{ times}}, \dots, \underbrace{\frac{x}{p^j}, \dots, \frac{x}{p^j}}_{p \text{ times}}, \underbrace{0, \dots, 0}_{p^2-p \text{ times}} \right) < \infty$$



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and

$$(2.3) \quad \|Df(x_1, \dots, x_{p^n})\| \leq \varphi(x_1, \dots, x_{p^n})$$

for all x , $x_1 = x_{p^{n+1}}, \dots, x_{p^{2-1}} = x_{p^n+p^2-1}, x_{p^2}, \dots, x_{p^n} \in V$. Then there exists a unique additive mapping $T : V \rightarrow W$ such that

$$(2.4) \quad \|f(x) - T(x)\| \leq \frac{1}{(p^2 - 1)p^{n-1}} \tilde{\varphi}(x)$$

for all $x \in V$. Furthermore, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in V$, then T is linear.

Proof. Let

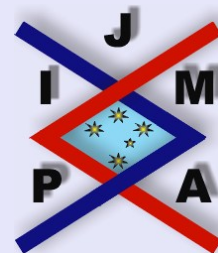
$$\begin{aligned} x_1 = \dots = x_p = x, \quad x_{p+1} = \dots = x_{p^2} = 0, \\ x_{p^2+1} = \dots = x_{p^2+p} = x, \quad x_{p^2+p+1} = \dots = x_{2p^2} = 0, \\ \dots, \\ x_{p^n-p^2+1} = \dots = x_{p^n-p^2+p} = x, \quad x_{p^n-p^2+p+1} = \dots = x_{p^n} = 0 \end{aligned}$$

in (2.3). Then we get

$$(2.5) \quad \left\| p^n f\left(\frac{x}{p}\right) + p^{n-1}(p^2 - 1)f(x) - p^2 \cdot p^n f\left(\frac{x}{p}\right) \right\| \leq \varphi(x, \dots, x, 0, \dots, 0, \dots, x, \dots, x, 0, \dots, 0)$$

for all $x \in V$. So one can obtain

$$\left\| f(x) - pf\left(\frac{x}{p}\right) \right\| \leq \frac{1}{(p^2 - 1)p^{n-1}} \varphi(x, \dots, x, 0, \dots, 0, \dots, x, \dots, x, 0, \dots, 0)$$



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for all $x \in V$. We prove by induction on j that

$$(2.6) \quad \left\| p^j f \left(\frac{1}{p^j} x \right) - p^{j+1} f \left(\frac{1}{p^{j+1}} x \right) \right\| \\ \leq \frac{p^j}{(p^2 - 1)p^{n-1}} \varphi \left(\frac{x}{p^j}, \dots, \frac{x}{p^j}, 0, \dots, 0, \dots, \frac{x}{p^j}, \dots, \frac{x}{p^j}, 0, \dots, 0 \right)$$

for all $x \in V$. So we get

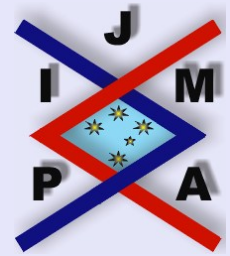
$$(2.7) \quad \left\| f(x) - p^j f \left(\frac{1}{p^j} x \right) \right\| \\ \leq \frac{1}{(p^2 - 1)p^{n-1}} \sum_{m=0}^{j-1} p^m \varphi \left(\frac{x}{p^m}, \dots, \frac{x}{p^m}, 0, \dots, 0, \dots, \frac{x}{p^m}, \dots, \frac{x}{p^m}, 0, \dots, 0 \right)$$

for all $x \in V$.

Let x be an element in V . For positive integers l and m with $l > m$,

$$(2.8) \quad \left\| p^l f \left(\frac{1}{p^l} x \right) - p^m f \left(\frac{1}{p^m} x \right) \right\| \\ \leq \frac{1}{(p^2 - 1)p^{n-1}} \sum_{j=m}^{l-1} p^j \varphi \left(\frac{x}{p^j}, \dots, \frac{x}{p^j}, 0, \dots, 0, \dots, \frac{x}{p^j}, \dots, \frac{x}{p^j}, 0, \dots, 0 \right),$$

which tends to zero as $m \rightarrow \infty$ by (2.2). So $\left\{ p^j f \left(\frac{1}{p^j} x \right) \right\}$ is a Cauchy sequence for all $x \in V$. Since W is complete, the sequence $\left\{ p^j f \left(\frac{1}{p^j} x \right) \right\}$ converges for



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all $x \in V$. We can define a mapping $T : V \rightarrow W$ by

$$(2.9) \quad T(x) = \lim_{j \rightarrow \infty} p^j f \left(\frac{1}{p^j} x \right) \quad \text{for all } x \in V.$$

By (2.3) and (2.9), we get

$$\begin{aligned} \|DT(x, \dots, x_{p^n})\| &= \lim_{j \rightarrow \infty} p^j \left\| Df \left(\frac{1}{p^j} x_1, \dots, \frac{1}{p^j} x_{p^n} \right) \right\| \\ &\leq \lim_{j \rightarrow \infty} p^j \varphi \left(\frac{1}{p^j} x_1, \dots, \frac{1}{p^j} x_{p^n} \right) \\ &= 0 \end{aligned}$$

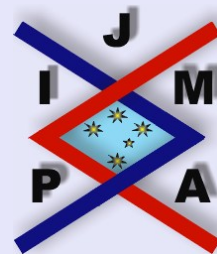
for all $x_1, \dots, x_{p^n} \in V$. Hence $T(x_1, \dots, x_{p^n}) = 0$ for all $x_1, \dots, x_{p^n} \in V$. By Lemma A, T is additive. Moreover, by passing to the limit in (2.7) as $j \rightarrow \infty$, we get the inequality (2.4).

Now let $L : V \rightarrow W$ be another additive mapping satisfying

$$\|f(x) - L(x)\| \leq \frac{1}{(p^2 - 1)p^{n-1}} \tilde{\varphi}(x)$$

for all $x \in V$.

$$\begin{aligned} \|T(x) - L(x)\| &= p^j \left\| T \left(\frac{1}{p^j} x \right) - L \left(\frac{1}{p^j} x \right) \right\| \\ &\leq p^j \left\| T \left(\frac{1}{p^j} x \right) - f \left(\frac{1}{p^j} x \right) \right\| + p^j \left\| f \left(\frac{1}{p^j} x \right) - L \left(\frac{1}{p^j} x \right) \right\| \\ &\leq \frac{2}{(p^2 - 1)p^{n-1}} p^j \tilde{\varphi} \left(\frac{1}{p^j} x \right), \end{aligned}$$



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which tends to zero as $j \rightarrow \infty$ by (2.2). Thus $T(x) = L(x)$ for all $x \in V$. This proves the uniqueness of T . Assume that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in V$. The additive mapping T given above is the same as the additive mapping T given in [4]. By the same reasoning as [4], the additive mapping $T : V \rightarrow W$ is linear. \square

Corollary 2.2. *If a mapping $f : V \rightarrow W$ satisfies*

$$(2.10) \quad \|Df(x_1, \dots, x_{2^n})\| \leq \varepsilon(\|x_1\|^p + \dots + \|x_{2^n}\|^p)$$

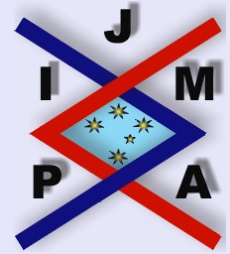
for some $p > 1$ and for all $x_1, \dots, x_{2^n} \in V$, then there exists a unique additive mapping $T : V \rightarrow W$ such that

$$(2.11) \quad \|T(x) - f(x)\| \leq \frac{2^{p-1}\varepsilon}{3(2^{p-1} - 1)} \|x\|^p$$

for all $x \in V$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in V$, then the function T is linear.

Proof. Since $\varphi(x_1, \dots, x_{2^n}) = \varepsilon(\|x_1\|^p + \dots + \|x_{2^n}\|^p)$ satisfies the condition (2.2), Theorem 2.1 says that there exists a unique additive mapping $T : V \rightarrow W$ such that

$$\begin{aligned} \|T(x) - f(x)\| &\leq \frac{1}{3 \cdot 2^{n-1}} \tilde{\varphi}(x) \\ &= \frac{1}{3 \cdot 2^{n-1}} \sum_{j=0}^{\infty} 2^j \varepsilon \left(\left\| \frac{x}{2^j} \right\|^p + \dots + \left\| \frac{x}{2^j} \right\|^p \right) \\ &= \frac{2^{p-1}\varepsilon}{3(2^{p-1} - 1)} \|x\|^p \end{aligned}$$



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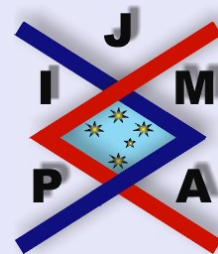
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for all $x \in V$. □

Theorem 2.3. *Let $f : V \rightarrow W$ be a continuous mapping with $f(0) = 0$ such that (2.2) and (2.3) for all $x_1, \dots, x_{2^n} \in V$. If the sequence $\{2^j f(\frac{1}{2^j}x)\}$ converges uniformly on V , then there exists a unique continuous linear mapping $T : V \rightarrow W$ satisfying (2.4).*

Proof. By Theorem 2.1, there exists a unique linear mapping $T : V \rightarrow W$ satisfying (2.2). By the continuity of f , the uniform convergence and the definition of T , the linear mapping $T : V \rightarrow W$ is continuous, as desired. □



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3. Approximate Algebra Homomorphisms in Banach Algebras

In this section, let \mathbb{A} and \mathbb{B} be Banach algebras with norms $\|\cdot\|$ and $\|\cdot\|$, respectively.

D.G. Bourgin [3] proved the stability of ring homomorphisms between Banach algebras. In [1], R. Badora generalized the Bourgin's result.

We prove the generalized Hyers-Ulam-Rassias stability of algebra homomorphisms between Banach algebras associated with the functional equation (1.1).

Theorem 3.1. *Let \mathbb{A} and \mathbb{B} be real Banach algebras, and $f : \mathbb{A} \rightarrow \mathbb{B}$ a mapping with $f(0) = 0$ for which there exist functions $\varphi : \mathbb{A}^{2^n} \rightarrow [0, \infty)$ and $\psi : \mathbb{A} \times \mathbb{A} \rightarrow [0, \infty)$ such that (2.2),*

$$(3.1) \quad \|Df(x_1, \dots, x_{2^n})\| \leq \varphi(x_1, \dots, x_{2^n}),$$

$$(3.2) \quad \tilde{\psi}(x, y) := \sum_{j=0}^{\infty} 2^j \psi\left(\frac{1}{2^j}x, y\right) < \infty$$

and

$$(3.3) \quad \|f(xy) - f(x)f(y)\| \leq \psi(x, y)$$

for all $x, y, x_1, \dots, x_{2^n} \in \mathbb{A}$, where D is in (2.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{A}$, then there exists a unique algebra homomorphism



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$T : \mathbb{A} \rightarrow \mathbb{B}$ satisfying (2.4). Further, if \mathbb{A} and \mathbb{B} are unital, then f itself is an algebra homomorphism.

Proof. By the same method as the proof of Theorem 2.1, one can show that there exists a unique linear mapping $T : \mathbb{A} \rightarrow \mathbb{B}$ satisfying (2.4). The linear mapping $T : \mathbb{A} \rightarrow \mathbb{B}$ was given by

$$(3.4) \quad T(x) = \lim_{j \rightarrow \infty} 2^j f \left(\frac{1}{2^j} x \right)$$

for all $x \in \mathbb{A}$. Let

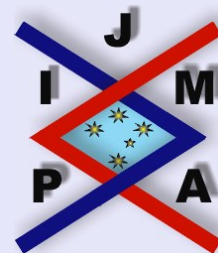
$$(3.5) \quad R(x, y) = f(x \cdot y) - f(x)f(y)$$

for all $x, y \in \mathbb{A}$. By (3.2), we get

$$(3.6) \quad \lim_{j \rightarrow \infty} 2^j R \left(\frac{1}{2^j} x, y \right) = 0$$

for all $x, y \in \mathbb{A}$. So

$$(3.7) \quad \begin{aligned} T(xy) &= \lim_{j \rightarrow \infty} 2^j f \left(\frac{1}{2^j} (xy) \right) \\ &= \lim_{j \rightarrow \infty} 2^j f \left[\left(\frac{1}{2^j} x \right) y \right] \\ &= \lim_{j \rightarrow \infty} 2^j \left[f \left(\frac{1}{2^j} x \right) f(y) + R \left(\frac{1}{2^j} x, y \right) \right] \\ &= T(x)f(y) \end{aligned}$$



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for all $x, y \in \mathbb{A}$. Thus

$$(3.8) \quad T(x)f\left(\frac{1}{2^j}y\right) = T\left[x\left(\frac{1}{2^j}y\right)\right] \\ = T\left[\left(\frac{1}{2^j}x\right)y\right] = T\left(\frac{1}{2^j}x\right)f(y) = \frac{1}{2^j}T(x)f(y)$$

for all $x, y \in \mathbb{A}$. Hence

$$(3.9) \quad T(x)2^j f\left(\frac{1}{2^j}y\right) = T(x)f(y)$$

for all $x, y \in \mathbb{A}$. Taking the limit in (3.9) as $j \rightarrow \infty$, we obtain

$$(3.10) \quad T(x)T(y) = T(x)f(y)$$

for all $x, y \in \mathbb{A}$. Therefore,

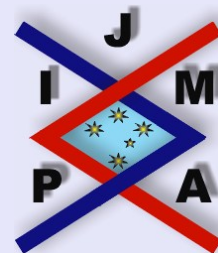
$$(3.11) \quad T(xy) = T(x)T(y)$$

for all $x, y \in \mathbb{A}$. So $T : \mathbb{A} \rightarrow \mathbb{B}$ is an algebra homomorphism.

Now assume that \mathbb{A} and \mathbb{B} are unital. By (3.7),

$$(3.12) \quad T(y) = T(1y) = T(1)f(y) = f(y)$$

for all $y \in \mathbb{A}$. So $f : \mathbb{A} \rightarrow \mathbb{B}$ is an algebra homomorphism, as desired. \square



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Corollary 3.2. Let $f : \mathbb{A} \rightarrow \mathbb{B}$ be a mapping such that (3.2), (3.3) and

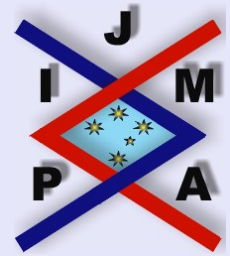
$$(3.13) \quad \|Df(x_1, \dots, x_{2^n})\| \leq \varepsilon(\|x_1\|^p + \dots + \|x_{2^n}\|^p)$$

for some $p > 1$ and for all $x, y, x_1, \dots, x_{2^n} \in \mathbb{A}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{A}$, then there exists a unique algebra homomorphism $T : \mathbb{A} \rightarrow \mathbb{B}$ such that

$$(3.14) \quad \|T(x) - f(x)\| \leq \frac{2^{p-1}\varepsilon}{3(2^{p-1} - 1)} \|x\|^p$$

for all $x \in \mathbb{A}$.

Proof. By Corollary 2.2, there exists a unique linear mapping $T : \mathbb{A} \rightarrow \mathbb{B}$ such that (3.14). By Theorem 3.1, the linear mapping T is an algebra homomorphism. \square



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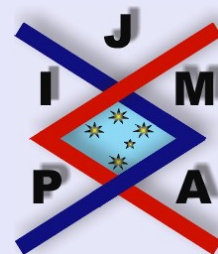
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