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PARTITIONED CYCLIC FUNCTIONAL EQUATIONS

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Abstract

We prove the generalized Hyers-Ulam-Rassias stability of a partitioned functional equation. It is applied to show the stability of algebra homomorphisms between Banach algebras associated with partitioned functional equations in Banach algebras.

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Contents

1. Partitioned Cyclic Functional Equations

Let E_1 and E_2 be Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th. M. Rassias [4] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}||x||^p$$

for all $x \in E_1$.

Recently, T. Trif [5] proved that, for vector spaces V and W, a mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation

$$n\ {}_{n-2}C_{k-2} f\left(\frac{x_1 + \cdots + x_n}{n}\right) + {}_{n-2}C_{k-1} \sum_{i=1}^{n} f(x_i) = k \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right)$$

for all $x_1, \ldots, x_n \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

Throughout this paper, let V and W be real normed vector spaces with norms $\|\cdot\|$ and $\|\cdot\|$, respectively, and let p, k and n be positive integers with $k \leq p^n$.

Lemma 1.1. *A mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation*

$$p^n f\left(\frac{x_1 + \cdots + x_{p^n}}{p^n}\right) + p(k-1)\sum_{i=1}^{p^{n-1}} f\left(\frac{x_{pi-p+1} + \cdots + x_{pi}}{p}\right) = k\sum_{i=1}^{p^n} f\left(\frac{x_i + \cdots + x_{i+k-1}}{k}\right) \tag{1.1}$$

for all $x_1 = x_{p^n+1}, \ldots, x_{k-1} = x_{p^n+k-1}, x_k, \ldots, x_{p^n} \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in V$.

Proof. Assume that a mapping $f : V \to W$ satisfies (1.1). Put $x_1 = x,\ x_2 = y$ and $x_3 = \cdots = x_{p^n} = 0$ in (1.1), then

$$p^n f\left(\frac{x+y}{p^n}\right) + p(k-1) f\left(\frac{x+y}{p}\right) = k\left[(k-1) f\left(\frac{x+y}{k}\right) + f\left(\frac{x}{k}\right) + f\left(\frac{y}{k}\right)\right]. \tag{1.2}$$

Putting $y = 0$ in (1.2),

$$p^n f\left(\frac{x}{p^n}\right) + p(k-1) f\left(\frac{x}{p}\right) = k^2 f\left(\frac{x}{k}\right). \tag{1.3}$$

Replacing x by kx and y by ky in (1.2),

$$p^n f\left(\frac{kx+ky}{p^n}\right) + p(k-1) f\left(\frac{kx+ky}{p}\right) \tag{1.4}$$

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$$= k[(k-1)f(x+y) + f(x) + f(y)].$$

Replacing x by $kx + ky$ in (1.3),

$$p^n f\left(\frac{kx+ky}{p^n}\right) + p(k-1)f\left(\frac{kx+ky}{p}\right) = k^2 f(x+y). \tag{1.5}$$

From (1.4) and (1.5),

$$0 = -kf(x+y) + k[f(x) + f(y)].$$

Hence f is additive.

The converse is obvious. □

The main purpose of this paper is to prove the generalized Hyers-Ulam-Rassias stability of the functional equation (1.1).

2. Stability of Partitioned Cyclic Functional Equations

From now on, let W be a Banach space.

We are going to prove the generalized Hyers-Ulam-Rassias stability of the functional equation (1.1). From now on, $n \geq 2$. For a given mapping $f : V \to W$, we set

$$Df(x_1, \dots, x_{p^n}) := p^n f\left(\frac{x_1 + \cdots + x_{p^n}}{p^n}\right) + p(p^2-1)\sum_{i=1}^{p^{n-1}} f\left(\frac{x_{pi-p+1} + \cdots + x_{pi}}{p}\right) - p^2 \sum_{i=1}^{p^n} f\left(\frac{x_i + \cdots + x_{i+p^2-1}}{p^2}\right) \tag{2.1}$$

for all $x_1 = x_{p^n+1}, \dots, x_{p^2-1} = x_{p^n+p^2-1}, x_{p^2}, \dots, x_{p^n} \in V$.

Theorem 2.1. *Let $f : V \to W$ be a mapping with $f(0) = 0$ for which there exists a function $\varphi : V^{p^n} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x) := \sum_{j=0}^{\infty} p^j \varphi\Big(\underbrace{\frac{x}{p^j}, \cdots, \frac{x}{p^j}}_{p \text{ times}}, \underbrace{0, \dots, 0}_{p^2-p \text{ times}}, \ \dots\ , \underbrace{\frac{x}{p^j}, \dots, \frac{x}{p^j}}_{p \text{ times}}, \underbrace{0, \dots, 0}_{p^2-p \text{ times}}\Big) < \infty \tag{2.2}$$

and

$$\|Df(x_1,\dots,x_{p^n})\| \le \varphi(x_1,\dots,x_{p^n}) \tag{2.3}$$

for all x, $x_1 = x_{p^n+1},\dots,x_{p^2-1} = x_{p^n+p^2-1}, x_{p^2},\dots,x_{p^n} \in V$. *Then there exists a unique additive mapping* $T : V \to W$ *such that*

$$\|f(x) - T(x)\| \le \frac{1}{(p^2-1)p^{n-1}}\widetilde{\varphi}(x) \tag{2.4}$$

for all $x \in V$. *Furthermore, if* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in V$, *then* T *is linear.*

Proof. Let

$$\begin{gathered} x_1 = \cdots = x_p = x, \quad x_{p+1} = \cdots = x_{p^2} = 0, \\ x_{p^2+1} = \cdots = x_{p^2+p} = x, \quad x_{p^2+p+1} = \cdots = x_{2p^2} = 0, \\ \cdots\cdots, \\ x_{p^n-p^2+1} = \cdots = x_{p^n-p^2+p} = x, \quad x_{p^n-p^2+p+1} = \cdots = x_{p^n} = 0 \end{gathered}$$

in (2.3). Then we get

$$\begin{aligned} &\left\|p^n f\left(\frac{x}{p}\right) + p^{n-1}(p^2-1)f(x) - p^2\cdot p^n f\left(\frac{x}{p}\right)\right\| \\ &\qquad\qquad \le \varphi(x,\dots,x,\ 0,\dots,0,\ \dots\ ,x,\dots,x,\ 0,\dots,0) \end{aligned} \tag{2.5}$$

for all $x \in V$. So one can obtain

$$\left\|f(x) - pf\left(\frac{x}{p}\right)\right\| \le \frac{1}{(p^2-1)p^{n-1}}\varphi\left(x,\dots,x,\ 0,\dots,0,\ ,x,\dots,x,\ 0,\dots,0\right)$$

for all $x \in V$. We prove by induction on j that

$$\left\| p^j f\left(\frac{1}{p^j}x\right) - p^{j+1} f\left(\frac{1}{p^{j+1}}x\right)\right\| \tag{2.6}$$
$$\leq \frac{p^j}{(p^2-1)p^{n-1}}\varphi\left(\frac{x}{p^j},\dots,\frac{x}{p^j}, 0,\dots,0, \ \dots\ ,\frac{x}{p^j},\dots,\frac{x}{p^j}, 0,\dots,0\right)$$

for all $x \in V$. So we get

$$\left\| f(x) - p^j f\left(\frac{1}{p^j}x\right)\right\| \tag{2.7}$$
$$\leq \frac{1}{(p^2-1)p^{n-1}}\sum_{m=0}^{j-1} p^m \varphi\left(\frac{x}{p^m},\dots,\frac{x}{p^m},0,\dots,0,\dots,\frac{x}{p^m},\dots,\frac{x}{p^m},0,\dots,0\right)$$

for all $x \in V$.

Let x be an element in V. For positive integers l and m with $l > m$,

$$\left\| p^l f\left(\frac{1}{p^l}x\right) - p^m f\left(\frac{1}{p^m}x\right)\right\| \tag{2.8}$$
$$\leq \frac{1}{(p^2-1)p^{n-1}}\sum_{j=m}^{l-1} p^j \varphi\left(\frac{x}{p^j},\dots,\frac{x}{p^j},0,\dots,0,\dots,\frac{x}{p^j},\dots,\frac{x}{p^j},0,\dots,0\right),$$

which tends to zero as $m \to \infty$ by (2.2). So $\left\{p^j f\left(\frac{1}{p^j}x\right)\right\}$ is a Cauchy sequence for all $x \in V$. Since W is complete, the sequence $\left\{p^j f\left(\frac{1}{p^j}x\right)\right\}$ converges for all $x \in V$. We can define a mapping $T : V \to W$ by

$$T(x) = \lim_{j\to\infty} p^j f\left(\frac{1}{p^j}x\right) \qquad \text{for all} \qquad x \in V. \tag{2.9}$$

By (2.3) and (2.9), we get

$$\begin{aligned} \|DT(x,\dots,x_{p^n})\| &= \lim_{j\to\infty} p^j \left\| Df\left(\frac{1}{p^j}x_1,\dots,\frac{1}{p^j}x_{p^n}\right)\right\| \\ &\le \lim_{j\to\infty} p^j \varphi\left(\frac{1}{p^j}x_1,\dots,\frac{1}{p^j}x_{p^n}\right) \\ &= 0 \end{aligned}$$

for all $x_1,\dots,x_{p^n} \in V$. Hence $T(x_1,\dots,x_{p^n}) = 0$ for all $x_1,\dots,x_{p^n} \in V$. By Lemma A, T is additive. Moreover, by passing to the limit in (2.7) as $j \to \infty$, we get the inequality (2.4).

Now let $L : V \to W$ be another additive mapping satisfying

$$\|f(x) - L(x)\| \le \frac{1}{(p^2-1)p^{n-1}}\widetilde{\varphi}(x)$$

for all $x \in V$.

$$\begin{aligned} \|T(x) - L(x)\| &= p^j \left\| T\left(\frac{1}{p^j}x\right) - L\left(\frac{1}{p^j}x\right)\right\| \\ &\le p^j \left\| T\left(\frac{1}{p^j}x\right) - f\left(\frac{1}{p^j}x\right)\right\| + p^j \left\| f\left(\frac{1}{p^j}x\right) - L\left(\frac{1}{p^j}x\right)\right\| \\ &\le \frac{2}{(p^2-1)p^{n-1}} p^j \widetilde{\varphi}\left(\frac{1}{p^j}x\right), \end{aligned}$$

which tends to zero as $j \to \infty$ by (2.2). Thus $T(x) = L(x)$ for all $x \in V$. This proves the uniqueness of T. Assume that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in V$. The additive mapping T given above is the same as the additive mapping T given in [4]. By the same reasoning as [4], the additive mapping $T : V \to W$ is linear. □

Corollary 2.2. *If a mapping $f : V \to W$ satisfies*

$$\|Df(x_1, \dots, x_{2^n})\| \leq \varepsilon(\|x_1\|^p + \cdots + \|x_{2^n}\|^p) \tag{2.10}$$

for some $p > 1$ and for all $x_1, \dots, x_{2^n} \in V$, then there exists a unique additive mapping $T : V \to W$ such that

$$\|T(x) - f(x)\| \leq \frac{2^{p-1}\varepsilon}{3(2^{p-1} - 1)}\|x\|^p \tag{2.11}$$

for all $x \in V$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in V$, then the function T is linear.

Proof. Since $\varphi(x_1, \dots, x_{2^n}) = \varepsilon(\|x_1\|^p + \cdots + \|x_{2^n}\|^p)$ satisfies the condition (2.2), Theorem 2.1 says that there exists a unique additive mapping $T : V \to W$ such that

$$\begin{aligned}
\|T(x) - f(x)\| &\leq \frac{1}{3 \cdot 2^{n-1}}\widetilde{\varphi}(x) \\
&= \frac{1}{3 \cdot 2^{n-1}} \sum_{j=0}^{\infty} 2^j \varepsilon \left(\left\| \frac{x}{2^j} \right\|^p + \cdots + \left\| \frac{x}{2^j} \right\|^p \right) \\
&= \frac{2^{p-1}\varepsilon}{3(2^{p-1} - 1)}\|x\|^p
\end{aligned}$$

for all $x \in V$. $\square$

Theorem 2.3. *Let $f : V \to W$ be a continuous mapping with $f(0) = 0$ such that* (2.2) *and* (2.3) *for all $x_1, \dots, x_{2^n} \in V$. If the sequence $\left\{2^j f\left(\frac{1}{2^j}x\right)\right\}$ converges uniformly on V, then there exists a unique continuous linear mapping $T : V \to W$ satisfying* (2.4).

Proof. By Theorem 2.1, there exists a unique linear mapping $T : V \to W$ satisfying (2.2). By the continuity of f, the uniform convergence and the definition of T, the linear mapping $T : V \to W$ is continuous, as desired. $\square$

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3. Approximate Algebra Homomorphisms in Banach Algebras

In this section, let $\mathbb{A}$ and $\mathbb{B}$ be Banach algebras with norms $\|\cdot\|$ and $\|\cdot\|$, respectively.

D.G. Bourgin [3] proved the stability of ring homomorphisms between Banach algebras. In [1], R. Badora generalized the Bourgin's result.

We prove the generalized Hyers-Ulam-Rassias stability of algebra homomorphisms between Banach algebras associated with the functional equation (1.1).

Theorem 3.1. *Let $\mathbb{A}$ and $\mathbb{B}$ be real Banach algebras, and $f : \mathbb{A} \to \mathbb{B}$ a mapping with $f(0) = 0$ for which there exist functions $\varphi : \mathbb{A}^{2^n} \to [0, \infty)$ and $\psi : \mathbb{A} \times \mathbb{A} \to [0, \infty)$ such that (2.2),*

$$\|Df(x_1, \dots, x_{2^n})\| \le \varphi(x_1, \dots, x_{2^n}), \tag{3.1}$$

$$\widetilde{\psi}(x, y) := \sum_{j=0}^{\infty} 2^j \psi\left(\frac{1}{2^j} x, y\right) < \infty \tag{3.2}$$

and

$$\|f(xy) - f(x)f(y)\| \le \psi(x, y) \tag{3.3}$$

for all $x, y, x_1, \dots, x_{2^n} \in \mathbb{A}$, where D is in (2.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{A}$, then there exists a unique algebra homomorphism

$T : \mathbb{A} \to \mathbb{B}$ *satisfying* (2.4)*. Further, if* $\mathbb{A}$ *and* $\mathbb{B}$ *are unital, then* f *itself is an algebra homomorphism.*

Proof. By the same method as the proof of Theorem 2.1, one can show that there exists a unique linear mapping $T : \mathbb{A} \to \mathbb{B}$ satisfying (2.4). The linear mapping $T : \mathbb{A} \to \mathbb{B}$ was given by

$$T(x) = \lim_{j\to\infty} 2^j f\left(\frac{1}{2^j}x\right) \tag{3.4}$$

for all $x \in \mathbb{A}$. Let

$$R(x, y) = f(x \cdot y) - f(x)f(y) \tag{3.5}$$

for all $x, y \in \mathbb{A}$. By (3.2), we get

$$\lim_{j\to\infty} 2^j R\left(\frac{1}{2^j}x, y\right) = 0 \tag{3.6}$$

for all $x, y \in \mathbb{A}$. So

$$\begin{aligned} T(xy) &= \lim_{j\to\infty} 2^j f\left(\frac{1}{2^j}(xy)\right) \\ &= \lim_{j\to\infty} 2^j f\left[\left(\frac{1}{2^j}x\right) y\right] \\ &= \lim_{j\to\infty} 2^j \left[f\left(\frac{1}{2^j}x\right) f(y) + R\left(\frac{1}{2^j}x, y\right)\right] \\ &= T(x)f(y) \end{aligned} \tag{3.7}$$

for all $x, y \in \mathbb{A}$. Thus

$$T(x)f\left(\frac{1}{2^j}y\right) = T\left[x\left(\frac{1}{2^j}y\right)\right] = T\left[\left(\frac{1}{2^j}x\right)y\right] = T\left(\frac{1}{2^j}x\right)f(y) = \frac{1}{2^j}T(x)f(y) \tag{3.8}$$

for all $x, y \in \mathbb{A}$. Hence

$$T(x)2^j f\left(\frac{1}{2^j}y\right) = T(x)f(y) \tag{3.9}$$

for all $x, y \in \mathbb{A}$. Taking the limit in (3.9) as $j \to \infty$, we obtain

$$T(x)T(y) = T(x)f(y) \tag{3.10}$$

for all $x, y \in \mathbb{A}$. Therefore,

$$T(xy) = T(x)T(y) \tag{3.11}$$

for all $x, y \in \mathbb{A}$. So $T : \mathbb{A} \to \mathbb{B}$ is an algebra homomorphism.

Now assume that $\mathbb{A}$ and $\mathbb{B}$ are unital. By (3.7),

$$T(y) = T(1y) = T(1)f(y) = f(y) \tag{3.12}$$

for all $y \in \mathbb{A}$. So $f : \mathbb{A} \to \mathbb{B}$ is an algebra homomorphism, as desired. □

Corollary 3.2. *Let $f : \mathbb{A} \to \mathbb{B}$ be a mapping such that (3.2), (3.3) and*

$$\|Df(x_1, \dots, x_{2^n})\| \le \varepsilon(\|x_1\|^p + \dots + \|x_{2^n}\|^p) \tag{3.13}$$

for some $p > 1$ and for all $x, y, x_1, \dots, x_{2^n} \in \mathbb{A}$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathbb{A}$, then there exists a unique algebra homomorphism $T : \mathbb{A} \to \mathbb{B}$ such that

$$\|T(x) - f(x)\| \le \frac{2^{p-1}\varepsilon}{3(2^{p-1}-1)}\|x\|^p \tag{3.14}$$

for all $x \in \mathbb{A}$.

Proof. By Corollary 2.2, there exists a unique linear mapping $T : \mathbb{A} \to \mathbb{B}$ such that (3.14). By Theorem 3.1, the linear mapping T is an algebra homomorphism. □

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