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PIECEWISE SOLUTIONS OF EVOLUTIONARY VARIATIONAL INEQUALITIES. APPLICATION TO DOUBLE-LAYERED DYNAMICS MODELLING OF EQUILIBRIUM PROBLEMS

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ABSTRACT. This paper presents novel results about the structure of solutions for certain evolutionary variational inequality problems. We show that existence of piecewise solutions is dependant upon the form of the constraint set underlying the evolutionary variational inequality problem considered. We discuss our results in the context of double-layered dynamics theory and we apply them to the modelling of traffic network equilibrium problems, in particular to the study of the evolution of such problems in a neighbourhood of a steady state.

Key words and phrases: Evolutionary variational inequalities, Projected dynamical systems, Equilibrium.

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1. INTRODUCTION

This paper presents results concerning solution classes for certain evolutionary variational inequality (EVI) problems. The results are then used in the context of the double-layered dynamics (DLD) modelling of certain traffic network problems. The novelty of the results in the first part of the paper resides in showing how the structure of solutions of certain types of EVI is a direct consequence of the type of constraint sets involved in their formulation. In the second part, we use this information in the study of the evolution, in finite-time, from disequilibrium to equilibrium, of an applied equilibrium problem whose steady states are modelled by an EVI. Such a study, started in [7], is made possible by the recently introduced theory of double-layered dynamics. The question of finite-time dynamics is extended here by introducing the concept of r-strongly pseudo-monotone mappings. The paper contains novel illustrative examples and an application to traffic network problems which is more detailed than the one in [7].

Evolutionary variational inequalities were first introduced in the 1960's ([4, 26, 32]), and have been used in the study of partial differential equations and boundary value problems. They are

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part of general variational inequalities theory, a large area of research with important applications in control theory, optimization, operations research, economics theory and transportation science (see for example [2, 11, 12, 13, 14, 16, 17, 19, 20, 22, 25, 28, 30] and the references therein). The form of EVI problems we consider in the present paper represents a unified formulation coming from applied problems in traffic, spatial price and financial equilibrium problems [11, 12, 13, 14] and were introduced first in [6]. The existence and uniqueness theory for EVI problems has been studied in many contexts; here we use the result in [13]. In [8] the authors give a refinement of this existence result showing under what conditions continuous solutions exist. In [3, 7] the authors present computational procedures for obtaining approximate solutions of an EVI problem of the type considered here.

Building upon the existence results of [13, 8] we show under what conditions solutions to EVI problems are expected to be piecewise functions. This depends directly upon the form of the constraint set we work with; in particular, we consider various forms of demand constraints (piecewise continuous functions, step functions) and draw new conclusions about the type of solutions in each case.

In [6, 7] the authors introduce double-layered dynamics theory as the natural combination of the theories of EVI and projected dynamical systems (PDS). PDS theory has started to develop in the context of differential inclusions [18, 10, 1], but was first formalized in [16] on the Euclidean space and in [20, 5] on arbitrary Hilbert spaces. In essence, DLD consists of associating to an EVI on the Hilbert space $L^2([0,T],\mathbb{R}^q)$, an infinite-dimensional PDS, whose critical points coincide exactly with the solutions of the EVI problem and vice versa. In this paper we use DLD theory to study the structure of EVI solutions for particular (step) demand functions. DLD is further used to show how some equilibrium states can be reached in finite time under suitable conditions.

We recall that variational inequalities theory has been used to formulate, qualitatively analyze, and solve a number of network equilibrium problems [14, 27, 28, 29, 30]. However DLD theory is also attractive for the modelling and analysis of equilibrium problems because it allows the study of applications involving two types of time dependency: one represented by the time-dependent equilibria (that can be predicted for a given problem via EVI theory), and the other represented by the time-dependent behavior of the application around the predicted equilibrium curve (obtained via PDS theory). The interpretation of the two timescales in DLD theory was discussed in [7] and it is further deepened in this paper with the help of what we call *the prediction timescale* and *the adjustment timescale*.

The structure of the paper is as follows: in Section 2 we present our results about piecewise solutions of EVI in the generic context of L^p-spaces and a theoretical example. In Section 3 we give brief introductions to PDS and DLD, and show that step function solutions for EVI are possible. In Section 4 we discuss the relation between the prediction and adjustment timescales in DLD theory. We introduce here the concept of r-strong pseudo-monotonicity with degree α (which is similar but more general than that of strong pseudo-monotonicity with degree α [24, 28, 20]) and we show how this concept is useful in determining when an EVI solution can be reached in finite-time. Section 5 presents a dynamic traffic equilibrium example following a computational procedure as in [6] (using an original MAPLE 8 code), illustrating the theoretical results of the previous sections and their possible consequences for traffic control. We close with conclusions and acknowledgements in Section 6.

2. PIECEWISE SOLUTIONS OF EVOLUTIONARY VARIATIONAL INEQUALITIES

Evolutionary variational inequalities were originally introduced by Lions and Stampacchia [26] and Brezis [4]. In this paper we use an EVI in the form initially proposed in [13] but using the unified framework proposed first in [6, 7]. These EVI come from traffic network problems and economic equilibrium problems (see [6, 11, 12, 13]) and are presented next. We consider a nonempty, convex, closed, bounded subset of the reflexive Banach space $L^p([0,T],\mathbb{R}^q)$ given by:

$$\mathbb{K} = \Big\{ u \in L^p([0,T],\mathbb{R}^q) \mid \lambda(t) \le u(t) \le \mu(t) \text{ a.e. in } [0,T]; \quad \sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \quad \xi_{ji} \in \{-1,0,1\}, i \in \{1,\dots,q\}, j \in \{1,\dots,l\} \Big\}. \tag{2.1}$$

Recall that

$$\langle\langle \phi, u \rangle\rangle := \int_0^T \langle \phi(u)(t), u(t) \rangle dt$$

is the duality mapping on $L^p([0,T],\mathbb{R}^q)$, where $\phi \in (L^p([0,T],\mathbb{R}2^q))^*$ and $u \in L^p([0,T],\mathbb{R}^q)$. Let $F : \mathbb{K} \to (L^p([0,T],\mathbb{R}^q))^*$; the standard form of the EVI we work with is therefore:

$$\text{find } u \in \mathbb{K} \text{ such that } \langle\langle F(u), v-u \rangle\rangle \ge 0,\ \forall v \in \mathbb{K}. \tag{2.2}$$

Theorem 2.1. *If F in (2.2) satisfies either of the following conditions:*

1. *F is hemicontinuous with respect to the strong topology on $\mathbb{K}$, and there exist $A \subseteq \mathbb{K}$ nonempty, compact, and $B \subseteq \mathbb{K}$ compact such that, for every $v \in \mathbb{K} \setminus A$, there exists $u \in B$ with $\langle\langle F(u), v-u \rangle\rangle < 0$;*
2. *F is hemicontinuous with respect to the weak topology on $\mathbb{K}$;*
3. *F is pseudo-monotone and hemicontinuous along line segments,*

then the EVI problem (2.2) admits a solution over the constraint set $\mathbb{K}$.

For a proof, see [13]. If F is strictly monotone, then the solution of (2.2) is unique. Another result about uniqueness of solutions to (2.2) can be found in [7] and we recall it in the next section.

Remark 2.2. Theorem 2.1 simply states that a measurable solution can be found for an EVI problem of type (2.2). We show next that this problem admits a piecewise solution, provided the constraint functions λ, μ, ρ, satisfy λ, $\mu \in L^p([0,T],\mathbb{R}^q)$ and $\rho_j(t)$, $j \in \{1,\dots,l\}$, are piecewise functions as presented below.

We consider sets

$$\mathbb{K} = \Big\{ u \in L^p([0,T],\mathbb{R}^q) \mid \lambda(t) \le u(t) \le \mu(t) \text{ a.e. in } [0,T]; \quad \sum_{i=1}^{q} \xi_{ji} u_i(t) = \rho_j(t) \text{ a.e. in } [0,T], \quad \xi_{ji} \in \{-1,0,1\}, i \in \{1,\dots,q\}, j \in \{1,\dots,l\} \Big\}, \tag{2.3}$$

where ρ_j are given by

$$\rho_j(t) = \begin{cases} c_1(t), & \text{if } 0 \le t \le t_1 \\ c_2(t), & \text{if } t_1 < t \le t_2 \\ \dots & \dots \\ c_{k_j}(t), & \text{if } t_{k_j-1} < t \le t_{k_j} = T \end{cases},$$

$$c_n^j \in L^p([t_{n-1},t_n],\mathbb{R}^q), \ \text{ for any } n \in \{1,\dots,k_j\}$$

Remark 2.3. Without loss of generality, we can consider that all $\rho_j(t)$ partition the interval $[0,T]$ in the same number of subintervals. Otherwise, we consider the set

$$\Delta := \bigcup_{j=1}^{l} \{0, t_1, t_2, , t_{k_j-1}, T\}$$

and we partition $[0,T]$ according to the division set Δ, possibly rewriting the functions $\rho_j(t)$. Therefore, we consider sets $\mathbb{K}$ as in (2.3) with

$$\rho_j(t) = \begin{cases} c_1(t), & \text{if } 0 \le t \le t_1 \\ c_2(t), & \text{if } t_1 < t \le t_2 \\ \dots & \dots \\ c_k(t), & \text{if } t_{k-1} < t \le t_k = T \end{cases},$$

$$c_n^j \in L^p([t_{n-1}, t_n], \mathbb{R}^q), \quad \text{for any } n \in \{1, \dots, k\}.$$

Theorem 2.4. *Assume $\mathbb{K}$ is of the form (2.3) and assume that $F : \mathbb{K} \to L^p([0,T], \mathbb{R}^q)^*$ is strictly monotone and continuous. Then EVI (2.2) admits a unique piecewise solution.*

Proof. We first prove the result for the case of sets $\mathbb{K}$ as in (2.3) where $j := 1$. These are therefore of the form

$$\mathbb{K} = \Bigg\{ u \in L^p([0,T], \mathbb{R}^q) \mid \lambda(t) \le u(t) \le \mu(t) \quad \text{and} \quad \sum_{i=1}^{q} \xi_{1i} u_i(t) = \rho_1(t) \text{ a.e. on } [0,T] \Bigg\}, \tag{2.4}$$

where $\xi_{1i} \in \{0,1\}$ and ρ_1 is given by

$$\rho_1(t) = \begin{cases} c_1(t), & \text{if } 0 \le t \le t_1 \\ c_2(t), & \text{if } t_1 < t \le t_2 \\ \dots & \dots \\ c_k(t), & \text{if } t_{k-1} < t \le t_k = T \end{cases},$$

$$c_n \in L^p([t_{n-1}, t_n], \mathbb{R}^q), \quad \text{for any } n \in \{1, \dots, k\}.$$

For each $n \in \{1, \dots, k\}$ we consider the following set

$$\mathbb{K}_n := \{u|_{[t_{n-1}, t_n]} \mid u \in \mathbb{K}\} \text{ which has the property that}$$

$$\mathbb{K}_n \subseteq \Bigg\{ z \in L^p([t_{n-1}, t_n], \mathbb{R}^q) \mid \lambda(t) \le z(t) \le \mu(t) \quad \text{and} \quad \sum_{i=1}^{q} \xi_{1i} z_i(t) = c_n(t) \text{ a.a. } t \in [t_{n-1}, t_n] \Bigg\}.$$

We also consider the evolutionary variational inequality EVI_n on the set $\mathbb{K}_n$, namely

$$\text{find } u \in \mathbb{K}_n \text{ s.t. } \int_{t_{n-1}}^{t_n} \langle F(u)(t), v(t) - u(t) \rangle dt \ge 0, \quad \forall v \in \mathbb{K}_n.$$

Each of the sets $\mathbb{K}_n$ is closed, convex and bounded, and the mapping F satisfies Theorem 2.1(3) on $\mathbb{K}_n$. According to this theorem each EVI_n has a unique measurable solution. Let us denote it by u_n^*. We then consider the mapping $u^* : [0,T] \to \mathbb{R}^q$ given by:

$$u^*(t) = \begin{cases} u_1^*(t), & \text{if } 0 \le t \le t_1 \\ u_2^*(t), & \text{if } t_1 < t \le t_2 \\ \dots & \dots \\ u_k^*(t), & \text{if } t_{k-1} < t \le t_k = T \end{cases} . \tag{2.5}$$

We show that $u^* \in \mathbb{K}$. By the definition of u^* we see that $\lambda(t) \le u^*(t) \le \mu(t)$, and

$$\sum_{i=1}^{q} \xi_{1i} u_i^*(t) = \rho_1(t) \quad \text{a.e. on} \quad [0,T].$$

It remains to show that $u^* \in L^p([0,T],\mathbb{R}^q)$. This follows from the fact that

$$\mu \in L^p([0,T],\mathbb{R}^q) \quad \text{and} \quad ||u^*(t)||_p \le ||\mu(t)||_p < \infty,$$

thus $u^* \in \mathbb{K}$.

Suppose now that u^* is not a solution of the EVI problem (2.2). Then there exists $v \in \mathbb{K}$ so that

$$\langle\langle F(u^*), v - u^* \rangle\rangle < 0 \Longleftrightarrow \int_0^T \langle F(u^*)(t), v(t) - u^*(t) \rangle dt < 0.$$

This is further equivalent to

$$\sum_{n=1}^{k} \int_{t_{n-1}}^{t_n} \langle F(u_n^*)(t), v(t) - u_n^*(t) \rangle dt < 0.$$

Let $w_n := v \mid_{[t_{n-1},t_n]}$; we subsequently get

$$\sum_{n=1}^{k} \int_{t_{n-1}}^{t_n} \langle F(u_n^*)(t), w_n(t) - u_n^*(t) \rangle dt < 0. \tag{2.6}$$

But on each set $\mathbb{K}_n$ we have that EVI_n is solvable and so

$$\begin{aligned} &\langle\langle F(u_n^*), z - u_n^* \rangle\rangle \ge 0, \, \forall z \in \mathbb{K}_n \\ &\qquad \Longleftrightarrow \int_{t_{n-1}}^{t_n} \langle F(u_n^*)(t), z(t) - u_n^*(t)) \rangle dt \ge 0, \, \forall z \in \mathbb{K}_n. \end{aligned} \tag{2.7}$$

We note that w_n defined above is an element of $\mathbb{K}_n$, so let $z := w_n$ in (2.7). Since we can do this for each $n \in \{1,\dots,k\}$, we get that

$$\sum_{n=1}^{k} \int_{t_{n-1}}^{t_n} \langle F(u_n^*)(t), w_n(t) - u_n^*(t)) \rangle dt \ge 0, \qquad \forall n \in \{1,\dots,k\}. \tag{2.8}$$

We see now that (2.6) and (2.8) lead to a contradiction. Hence $u^* \in \mathbb{K}$ is a piecewise solution of EVI (2.2).

Keeping in mind Remark 2.3, the case $j > 1$ can be shown in a similar manner, by defining, for each $n \in \{1,\dots,k\}$, the set

$$\mathbb{K}_n := \{u \mid_{[t_{n-1},t_n]} \mid u \in \mathbb{K}\} \text{ where}$$

$$\mathbb{K}_n \subseteq \Bigg\{ z \in L^p([t_{n-1}, t_n], \mathbb{R}^q) \mid \lambda \leq z \leq \mu \quad \text{and} \quad \sum_{i=1}^{q} \xi_{ji} z_i(t) = c_n^j(t) \text{ a.a. } t \in [t_{n-1}, t_n], j \in \{1, \dots, l\} \Bigg\}.$$

□

Next we prove more about the structure of the solutions of an EVI problem (2.2) for the case of $L^2([0,T], \mathbb{R}^q)$.

Corollary 2.5. *Assume the hypotheses of Theorem 2.4, where $p := 2$, λ, μ are continuous functions, ρ_j are piecewise continuous and F is given by $F(u)(t) = A(t)u(t) + B(t)$, where $A(t)$ is a positive definite matrix for each $t \in [0,T]$ and A, B are continuous. Then EVI (2.2) admits a piecewise continuous solution.*

Proof. Each EVI_n has, under the present hypotheses, a continuous solution $u_n^*(t)$. This follows from [3]. Then by Theorem 2.4 the solution of the EVI (2.2) is piecewise continuous. □

Corollary 2.5 is also important from a computational point of view. We obtain the solution $u^*(t)$ by computing the piecewise components $u_n^*(t)$, as shown in [8], or using the computational procedure in [6].

Example 2.1. Let $p := 2$, $q := 4$, $T := 90$, $j := 2$ and $\xi_{ji} := 1$ for $i, j \in \{1, 2\}$. We set $\lambda(t) = (0,0,0,0)$ and $\mu(t) = (100, 100, 100, 100)$ for t in $[0, 90]$, hence

$$\mathbb{K} = \Bigg\{ u \in L^2([0,90], \mathbb{R}^4) | 0 \leq u_i^j(t) \leq 100 \quad \text{a.e. in } [0,90],\ i \in \{1,2\},\ j \in \{1,2\} \quad \text{and} \quad \sum_{i=1}^{2} u_i^j(t) = \rho_j(t) \text{ a.e. in } [0,90],\ j \in \{1,2\} \Bigg\},$$

where

$$\rho_1(t) = \begin{cases} 2t, & \text{if } 0 \leq t \leq 30 \\ -2t + 220, & \text{if } 30 < t \leq 90 \end{cases}, \quad \rho_2(t) = \begin{cases} t^2, & \text{if } 0 \leq t \leq 30 \\ t, & \text{if } 30 < t \leq 90 \end{cases}.$$

We consider:

$$F\left(u_1^1, u_2^1, u_1^2, u_2^2\right)(t) = (u_1^1(t) - 120, u_2^1(t) - 120, 2u_1^2(t) + u_2^2(t) - 330, u_1^2(t) + 2u_2^2(t) - 330),$$

$F : \mathbb{K} \to L^2([0,90], \mathbb{R}^4)$ and the following EVI:

$$\langle\langle F(u), v - u \rangle\rangle \geq 0, \quad \forall v \in \mathbb{K}.$$

We remark that $F : \mathbb{K} \to L^2([0,90], \mathbb{R}^4)$ satisfies the hypotheses of Corollary 2.5. Using a computational procedure as in [6], we obtain that the unique equilibrium curve of this problem is given by the piecewise continuous function

$$u^*(t) = \begin{cases} \left(t, t, \frac{t^2}{2}, \frac{t^2}{2}\right), & \text{if } 0 \leq t \leq 30 \\ \left(-t + 110, -t + 110, \frac{t}{2}, \frac{t}{2}\right), & \text{if } 30 < t \leq 90. \end{cases}$$

In the next section we further refine our results by studying the structure of solutions to EVI (2.2) in the context of double-layered dynamics theory.

3. DOUBLE-LAYERED DYNAMICS

In essence, EVI problems of the type considered in this paper can be viewed as a 1-parameter family of a static variational inequality, with parameter t. From here on, we consider that our EVI (2.2) represents the model of an equilibrium problem (as, for example, in [14]). In this context, the parameter t will be taken to mean physical time. As t varies over $[0, T]$, the constraints of the equilibrium problem change, and so the static states describe a curve of equilibria. Such an equilibrium curve can be of the form (2.5), as in Theorem 2.4. DLD was introduced in [6, 7] as a unifying tool for deepening the study of an EVI problem with constraint sets $\mathbb{K} \subseteq L^2([0,T], \mathbb{R}^q)$.

3.1. **PDS.** A thorough introduction to both theories and applications of EVI and PDS can be found in [6]. DLD theory is presented in detail in [7]. In this section we outline only the necessary theoretical facts in order to insure a self-contained presentation of this work. Let X be a Hilbert space of arbitrary (finite or infinite) dimension and let $K \subset X$ be a non-empty, closed, convex subset. We assume that the reader is familiar with the concepts of *tangent and normal cones to* K *at* $x \in K$ ($T_K(x)$, respectively $N_K(x)$), and *the projection operator of* X *onto* K, $P_K : X \to K$ given by $||P_K(z) - z|| = \inf_{x \in K} ||x - z||$.

The properties of projection operators on Hilbert spaces are well-known (see for instance [33]). The directional Gateaux derivative of the operator P_K is defined, for any $x \in K$ and any element $v \in X$, as the limit (for a proof see [33]):

$$\Pi_K(x, v) := \lim_{\delta \to 0^+} \frac{P_K(x + \delta v) - x}{\delta}; \quad \text{moreover,} \quad \Pi_K(x, v) = P_{T_K(x)}(v).$$

Let $\Pi_K : K \times X \to X$ be the operator given by $(x, v) \mapsto \Pi_K(x, v)$. Note that Π_K is discontinuous on the boundary of the set K. In [15, 21], several characterizations of Π_K are given.

Theorem 3.1. *Let* X *be a Hilbert space and* K *be a non-empty, closed, convex subset. Let* $F : K \to X$ *be a Lipschitz continuous vector field and* $x_0 \in K$. *Then the initial value problem*

$$\frac{dx(\tau)}{d\tau} = \Pi_K(x(\tau), -F(x(\tau))), \qquad x(0) = x_0 \in K \tag{3.1}$$

has a unique absolutely continuous solution on the interval $[0, \infty)$.

For a proof, see [9, 5]. This result is a generalization of the one in [16], where $X := \mathbb{R}^n$, K was a convex polyhedron and F had linear growth.

Definition 3.1. A *projected dynamical system* is given by a mapping $\phi : \mathbb{R}_+ \times K \to K$ which solves the initial value problem:

$$\dot{\phi}(\tau, x) = \Pi_K(\phi(\tau, x), -F(\phi(\tau, x))), \quad \phi(0, x) = x_0 \in K.$$

3.2. **DLD.** Double-layer dynamics consists of intertwining an EVI problem and a PDS as follows: we let $p := 2$, $X := L^2([0,T], \mathbb{R}^q)$ and we consider sets $\mathbb{K} \subseteq L^2([0,T], \mathbb{R}^q)$, as given by (2.1). Further, we consider the infinite-dimensional PDS defined on $\mathbb{K}$ by

$$\frac{du(\cdot, \tau)}{d\tau} = \Pi_{\mathbb{K}}(u(\cdot, \tau), -F(u)(\cdot, \tau)), \qquad u(\cdot, 0) = u(\cdot) \in \mathbb{K}, \tag{3.2}$$

where we assume the following hypothesis: $F : \mathbb{K} \to L^2([0,T], \mathbb{R}^q)$ is strictly pseudo-monotone and Lipschitz continuous. Note that this hypothesis is in the scope of both Theorems 2.1 and 2.4. The following results hold (see [8] for a proof of the first and see [6] for a proof of the second):

Theorem 3.2.

(1) *Assuming that F is strictly pseudo-monotone and Lipschitz continuous, the solutions of the EVI problem (2.2) are the same as the critical points of PDS (3.2). The converse is also true.*
(2) *EVI (2.2) has a unique solution.*

DLD theory helps establish the long time behaviour of the applied problem with respect to its curve of equilibria. This has been done in [7], where infinite-dimensional PDS theory was used to draw conclusions about the stability of such a curve. Next we use a DLD setting to prove a new result about the solution structure of an EVI problem.

Theorem 3.3. *Assume a set $\mathbb{K}$ is as in (2.4), where $p = 2$, $\lambda(t) := \lambda$, $\mu(t) := \mu$ are constant functions, and $\rho_j(t)$ are step functions. Let $F : \mathbb{K} \to L^2([0,T],\mathbb{R})$, $F(u)(t) = Au(t) + B$ be strictly pseudo-monotone and Lipschitz continuous on $\mathbb{K}$. Then the unique solution of EVI (2.2) is a step function.*

Proof. From Corollary 2.5 and Theorem 3.2(2), we have that the unique solution of the EVI problem (2.2) is of the form:

$$u^*(t) = \begin{cases} u_1^*(t), & \text{if } 0 \le t \le t_1 \\ u_2^*(t), & \text{if } t_1 < t \le t_2 \\ \cdots & \cdots \\ u_k^*(t), & \text{if } t_{k-1} < t \le t_k = T \end{cases} \quad \text{, where each } u_n^* \text{ is continuous, } n \in \{1,\dots,k\}.$$

From Theorem 3.2(1), we have that this solution curve constitutes the unique equilibrium of PDS (3.2). Let us now arbitrarily fix $n \in \{1,2,\dots,k\}$ and $t \in (t_{n-1}, t_n]$. We denote by PDS_t the finite-dimensional projected dynamical system given by the flow of the equation:

$$\frac{dw(\tau)}{d\tau} = \Pi_{\mathbb{K}(t)}(w(\tau), -F_t(w(\tau))), \tag{3.3}$$

where

$$\mathbb{K}(t) := \left\{ w := u(t) \in \mathbb{R}^q \mid \lambda \le w \le \mu, \text{ and} \sum_{i=1}^{k} \xi_{ji} w_i = c_n^j,\ j \in \{1,\dots,l\} \right\} \text{ and}$$

$$F_t : \mathbb{K}(t) \to \mathbb{R}^q, \text{ given by } F_t(w) := Aw + B.$$

DLD theory implies that the unique equilibrium point of this system is $u_n^*(t)$. Similarly, choosing $t' \in (t_{n-1}, t_n]$ and $t \neq t'$, the unique equilibrium point of $PDS_{t'}$ is $u_n^*(t')$. However, the constraint sets $\mathbb{K}(t)$ and $\mathbb{K}(t')$ coincide, and the mappings F_t and $F_{t'}$ are the same, hence PDS_t and $PDS_{t'}$ are given by the same differential equation (3.3). Therefore $u_n^*(t) = u_n^*(t')$. Since t, t' were arbitrarily chosen on $(t_{n-1}, t_n]$, then $u_n^*(t)$ =constant=: u_n^* on the interval $(t_{n-1}, t_n]$. Since n was also arbitrarily chosen in $\{1,\dots,k\}$, the solution u^* is a step function. □

4. ADJUSTMENT TO EQUILIBRIA IN DOUBLE-LAYERED DYNAMICS

Recall that we consider an EVI (2.2) as the model of an equilibrium problem. The solution of this EVI is interpreted as a curve of equilibrium states of the underlying problem over the time interval $[0,T]$. These are all the *potential* equilibrium states the problem can reach. Therefore we call $[0,T]$ *the prediction timescale*.

We further associate to EVI (2.2) a PDS (3.2). By Theorem 3.2, the equilibrium curve is stationary in the projected dynamics (3.2), hence $\tau \in [0,\infty)$ represents the evolution time of the problem from disequilibrium to equilibrium. Therefore we call $[0, \infty)$ *the adjustment scale*. Our DLD models include the following assumptions:

(1) $t,\ \tau$ represent physical time;
(2) time unit is the same;
(3) time flows forward.

The modelling questions we want to answer here are of the following type: does an equilibrium problem modelled via DLD reach one of its predicted equilibrium states *in finite time*, starting from an observed initial state $u(t_0)$, at some $t_0 \in [0, T]$?

A first answer to this question was given in [7] (Theorem 4.2), where it is shown that for a fixed t_0, under strong pseudo-monotonicity with degree $\alpha < 2$ of F_{t_0}, the PDS_{t_0} (as defined in (3.3) above) admits a finite-time attractor, namely $u^*(t_0)$. An estimate for the time necessary for a trajectory of the PDS_{t_0} to reach $u^*(t_0)$ is given and is denoted by l_{t_0}. In [7], l_{t_0} is interpreted as an instantaneous adjustment of the dynamics at time t_0 to its corresponding equilibrium at t_0. This kind of interpretation may be applicable to problems where the adjustment dynamics take place very rapidly, for example internet traffic problems. Here, in the first subsection below, we give a new more general time estimate, more readily applicable to the modelling of equilibrium problems. In the second subsection we use this estimate in the context of the two timescales (prediction and adjustment).

In this part of the paper we prove a generalization of our result in [7] (see Lemma 4.2 below). In order to do so, we need to introduce first a new concept, that of r-strong pseudo-monotonicity as follows:

Definition 4.1. Let $K \subseteq X$ be closed, convex, where X is a generic Hilbert space. Let $\langle\langle \cdot, \cdot \rangle\rangle$ be the inner product on X and $f : K \to X$ a mapping. Then:

(1) f is called *locally r-strongly pseudo-monotone with degree α at $x^* \in K$* if, for a given $r > 0$, there exists a neighbourhood $N(x^*) \subset K$ of the point x^* with the property that for any point $x \in N(x^*) \backslash B[x^*, r]$, there exists a positive scalar $\eta(r) > 0$ so that
$$\langle\langle f(x^*), x - x^* \rangle\rangle \geq 0 \implies \langle\langle f(x), x - x^* \rangle\rangle \geq \eta(r) ||x - x^*||^{\alpha}.$$
(2) f is called *r-strongly pseudo-monotone with degree α at $x^* \in K$* if the above holds for all $x \in K \backslash B[x^*, r]$.

Remark 4.1.

(1) Definition 4.1 is a generalization of strong pseudo-monotonicity with degree α at x^* (first introduced in [20]); strong pseudo-monotonicity with degree α is itself a generalization of the notions of local and global strong monotonicity with degree α introduced in [24, 28].
(2) Definition 4.1 is not vacuous; note that any (locally) strongly pseudo-monotone mapping f with degree α at x^* satisfies Definition 4.1.
(3) There exist mappings satisfying r-strong pseudo-monotonicity with α at x^*, but which do not satisfy strong pseudo-monotonicity with α at that point (see our example in Section 5 and the justification in the Appendix).

We are ready to prove the following:

Lemma 4.2. *Assume that $f : K \to X$ satisfies condition (1) (respectively (2)) of Definition 4.1 with degree $0 < \alpha < 2$, is Lipschitz continuous, and that x^* is a critical point of the projected dynamical system given by $-f$ on K:*
$$\frac{dx(\tau)}{d\tau} = \Pi_K(x(\tau), -f(x(\tau))).$$

Given an initial state $x(0) \in N(x^*)\backslash B[x^*, r]$ (respectively $x(0) \in K\backslash B[x^*, r]$), the unique trajectory of the projected system starting at $x(0)$ reaches $\partial B[x^*, r]$ after

$$\tau := \frac{||x(0) - x^*||_X^{2-\alpha} - r^{2-\alpha}}{\eta(r)(2-\alpha)} \text{ units of time.}$$

Proof. Assume f to be locally r-strongly pseudo-monotone with degree $\alpha < 2$ at $x^* \in K$; there exists a neighbourhood $N(x^*)$ and $\eta(r) \geq 0$ so that

$$\langle\langle f(x^*), x - x^* \rangle\rangle \geq 0 \implies \langle\langle f(x), x - x^* \rangle\rangle \, \eta(r)||x - x^*||^\alpha.$$

Let $x(0) \in N(x^*)\backslash B[x^*, r]$ and $x(\tau)$ the unique trajectory of PDS starting at $x(0)$. Assume that

$$\|x(\tau) - x^*\| - r > 0, \, \forall \tau \geq 0 \implies \|x(\tau) - x^*\| > r > 0. \tag{4.1}$$

This implies that

$$D(\tau) := \frac{1}{2}||x(\tau) - x^*||^2 > 0, \quad \forall \tau > 0.$$

We have

$$\begin{aligned} \frac{d}{d\tau}D(\tau) &= \left\langle\left\langle \frac{d}{d\tau}(x(\tau) - x^*), x(\tau) - x^* \right\rangle\right\rangle \\ &= \langle\langle \Pi_K(x(\tau), -f(x(\tau))), x(\tau) - x^* \rangle\rangle \\ &\leq -\langle\langle f(x(\tau)), x(\tau) - x^* \rangle\rangle \, . \end{aligned}$$

Since x^* is an equilibrium point, then $\Pi_K(x^*, -f(x^*)) = 0 \Leftrightarrow -f(x^*) \in N_K(x^*)$, hence

$$-\langle\langle f(x^*), x(\tau) - x^* \rangle\rangle \leq 0. \tag{4.2}$$

Based on (4.2), from the hypothesis we have that

$$-\langle\langle f(x(\tau)), x(\tau) - x^* \rangle\rangle \leq -\eta(r)||x(\tau) - x^*||^\alpha \tag{4.3}$$

and so from (4.2) and (4.3) we have that

$$\frac{d}{d\tau}D(\tau) \leq -\eta(r)||x(\tau) - x^*||^\alpha \leq 0 \implies \tau \mapsto \|x(\tau) - x^*\| \text{ is decreasing.}$$

Following a similar computation as in [7] (proof of Theorem 4.2), integrating from 0 to τ, we obtain

$$\|x(\tau) - x^*\|^{2-\alpha} \leq \|x(0) - x^*\|^{2-\alpha} - \tau\eta(r)[2-\alpha].$$

The last inequality is equivalent to

$$\|x(\tau) - x^*\| - r \leq \left[\|x(0) - x^*\|^{2-\alpha} - \tau\eta(r)[2-\alpha]\right]^{\frac{1}{2-\alpha}} - r,$$

and we see that our assumption (4.1) is contradicted because we can find a moment $\tau > 0$, (which we will denote from now on by l_0^r to keep a notation consistency with [7]) so that

$$\|x(\tau) - x^*\| - r \leq 0,$$

namely

$$l_0^r := \frac{\|x(0) - x^*\|_X^{2-\alpha} - r^{2-\alpha}}{\eta(r)[2-\alpha]}. \tag{4.4}$$

□

Remark 4.3. The result of Lemma 4.2 is a generalization of the one in Theorem 4.1 in [7]. In that case, we simply have $r = 0$ and a strong pseudo-monotone mapping with $\alpha < 2$, thus x^* is a finite-time attractor and the adjustment time of the dynamics from the initial state $x(0)$ to the equilibrium x^* is given by

$$l_0 := \frac{||x(0) - x^*||_{\mathbb{R}^q}^{2-\alpha}}{(2-\alpha)\eta}. \tag{4.5}$$

We return now to the study of an equilibrium problem modelled with DLD. We assume that we start observing the problem at some $t_0 \in [0, T]$, with initial data $u(t_0, 0) \in \mathbb{K}(t_0)$, where

$$\mathbb{K}(t_0) = \left\{ w := u(t_0) \in \mathbb{R}^q \mid \lambda(t_0) \le w \le \mu(t_0), \sum_{i=1}^{q} \xi_{ji} w_i = \rho_j(t_0),\ j \in \{1, \dots, l\} \right\}.$$

We consider PDS_{t_0} and $F_{t_0} : \mathbb{K}(t_0) \to \mathbb{R}^q$ as in (3.3); according to DLD theory, its unique equilibrium is $u^*(t_0)$. Let $w(\tau) := u(t_0, \tau)$ be the solution of the PDS_{t_0} starting at $u(t_0, 0)$. Then Lemma 4.2 implies that: whenever F_{t_0} is r-strongly pseudo-monotone with degree $\alpha < 2$ at $u^*(t_0)$, then by (4.4) we have that

$$\|u(t_0, l_{t_0}^r) - u^*(t_0)\| = r. \tag{4.6}$$

However, time passes uniformly on both prediction and adjustment scales. Formula (4.6) indicates that $l_{t_0}^r$ units have passed on the adjustment scale, but none have passed on the prediction scale. Thus (4.6) makes sense only if there exists $\Delta t > 0$ so that

$$u^*(t_0) = u^*(t_0 + \Delta t) \text{ and } \Delta t = l_{t_0}^r.$$

The last formula gives us the following interpretation:

the r-neighbourood of the equilibrium $u^*(t_0 + \Delta t)$ is reached in finite time starting from the disequilibrium state $u^*(t_0, 0)$, if $l_{t_0}^r = \Delta t$.

Remark 4.4. In the more particular case of a mapping F which is strongly pseudo-monotone with degree $\alpha < 2$ at x^*, keeping in mind (4.5), we have that the equilibrium $u^*(t_0 + \Delta t)$ is reached in finite time starting from the disequilibrium state $u^*(t_0, 0)$, if $l_{t_0} = \Delta t$.

In the next section we present a novel traffic network example to illustrate our results.

5. APPLICATION TO TRAFFIC NETWORK EQUILIBRIUM PROBLEMS

In the example below we consider that the demand ρ on the network is a piecewise continuous function of t and we illustrate our interpretation of adjustment to the neighbourhood of a predicted equilibrium state of the network. Such an example represents a novelty for the DLD applications present in the literature so far. Moreover, in this example we present a new use of formula (4.6) as follows: if an equilibrium state takes place twice in the time interval $[0, T]$, namely in our previous notation $u^*(t_0) = u^*(t_0 + \Delta t)$, then we can determine for which initial states $u(t_0)$, formula (4.6) takes place. In other words, we can determine from which initial disequilibrium states the traffic will adjust to (a neighbourhood of) the equilibrium $u^*(t_0 + \Delta t)$.

We consider a traffic network with one origin destination pair having two links (as depicted in Figure 1) and the following constraint set corresponding to this network configuration

$$\mathbb{K} := \{u \in L^2([0, 110], \mathbb{R}^2) \mid 0 \le u(t) \le 120,\ u_1(t) + u_2(t) = \rho_1(t) \text{ a.a. } t \in [0, 110]\},$$

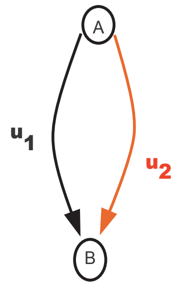

Figure 1

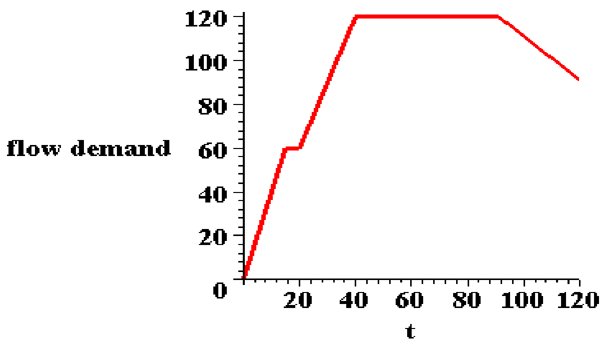

Figure 2

where

$$\rho_1(t) = \begin{cases} 4t, & t \in [0, 15], \\ 60, & t \in (15, 20], \\ 3t, & t \in (20, 40], \\ 120, & t \in (40, 91], \\ -t + 211, & t \in (91, 110]. \end{cases}$$

We consider the time unit to be a minute and the time interval $[0, 110]$ to correspond to 6:30 am - 8:20 am during a weekday. Let the flows on each link be denoted by u_1, u_2 and the demand by ρ_1 (Figure 2 depicts the demand). We see that during the hight of rush hour, 7:10-8:00 am (i.e., $t \in (40, 91]$) the demand is highest.

Let us also consider the cost on each link to be given by the mapping

$$F : \mathbb{K} \to L^2([0, 110], \mathbb{R}^2), \;\; F((u_1, u_2)) = (u_1 + 151, u_2 + 60).$$

The dynamic equilibria for such a problem are given by the EVI (see also [14, 7])

$$\int_0^{110} \langle F(u)(t), v(t) - u(t) \rangle \, dt \geq 0, \qquad \forall v \in \mathbb{K}.$$

The mapping F is Lipschitz continuous with constant 1 and $F(u) := Au + B$ with A positive definite; by Corollary 2.5, the unique solution of the above EVI is piecewise continuous; moreover, by Theorem 3.3, the solution has a constant value over the intervals $[15, 20]$ and $[40, 91]$. By the method proposed in [6], implemented with a MAPLE 8 code, we compute an

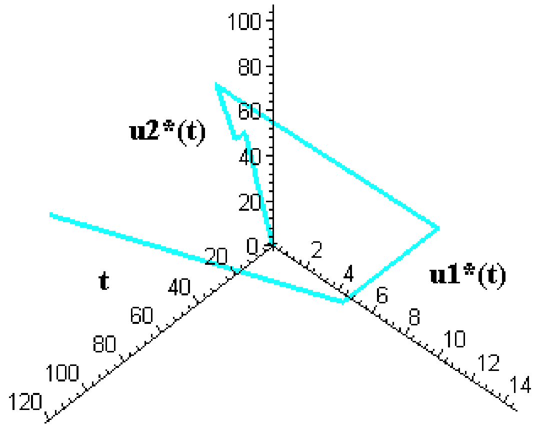

Figure 3

approximate solution to be

$$u^*(t) = \begin{cases} (0, 4t), & t \in [0, 15], \\ (0, 60), & t \in (15, 20], \\ (0, 3t), & t \in \left(20, \frac{91}{3}\right], \\ \left(\frac{3t-91}{2}, \frac{3t+91}{2}\right), & t \in \left(\frac{91}{3}, 40\right], \\ (14.5, 105.5), & t \in (40, 91], \\ \left(\frac{-t+120}{2}, \frac{-t+302}{2}\right), & t \in (91, 110]. \end{cases}$$

The graph of this solution is presented in Figure 3. We note that the Wardrop equilibrium conditions are satisfied for this solution, namely all paths with positive flow in equilibrium have equal minimal costs, as can be seen below:

$$F(u^*)(t) = \begin{cases} (151, 4t + 60), & t \in [0, 15], \\ (151, 120), & t \in (15, 20], \\ (151, 3t + 60), & t \in \left(20, \frac{91}{3}\right], \\ \left(\frac{3t+211}{2}, \frac{3t+211}{2}\right), & t \in \left(\frac{91}{3}, 40\right], \\ (165.5, 165.5), & t \in (40, 91], \\ \left(\frac{-t+422}{2}, \frac{-t+422}{2}\right), & t \in (91, 110]. \end{cases}$$

We see here that users prefer the second road to the first, however, during the rush hour peak, they will use both routes, as they become equally expensive.

So far, the EVI model of this problem has provided the approximate equilibrium curve for the traffic, given a certain structure of the demand function. In general however, the traffic may be in disequilibrium, in which case we want to know if/how it will evolve towards a steady state. This type of question is answered via the DLD model of this network, namely considering the PDS:

$$\frac{u(t,\tau)}{d\tau} = \Pi_{\mathbb{K}}(u(t,\tau), -F(u)(t,\tau)).$$

Let $t_0 \in [0, 110]$ be fixed and consider the projected dynamics at t_0, PDS_{t_0}, given by

$$\frac{dw(\tau)}{d\tau} = \Pi_{\mathbb{K}(t_0)}(w(\tau), -F_{t_0}(w(\tau))),$$

where

$$\mathbb{K}(t_0) = \{u(t) := w \in \mathbb{R}^2 \mid (0,0) \le (w_1, w_2) \le (120, 120), w_1 + w_2 = \rho_1(t)\},$$

and $F_{t_0} : \mathbb{K}(t) \to \mathbb{R}^2$, $F_{t_0}(w) = (w_1 + 151, w_2 + 60)$.

We can study whether the traffic approaches a small given neighbourhood of a steady state. Moreover, as we show below, this has consequences for traffic control, as one could find a flow distribution at the initial time t_0 so that at a later time the traffic will adjust "close enough" to an equilibrium.

Let $t_0 := 35$ (i.e. 7:05 am) and we have that $u^*(35) = (7, 98)$ cars/min; but we also note that there exists $\Delta t := 71$ min with the property that

$$u^*(35) = u^*(35 + \Delta t) = u^*(106) = (7, 98).$$

The mapping $F_{t_0=35}$ is 1-strongly pseudo-monotone with degree $\alpha := 1$ and $\eta := \sqrt{2}$ at $u^*(35)$ (see Appendix for a proof). Using formula (4.4), we can find a flow distribution at $t_0 = 35$ so that

$$l_{35}^1 = \Delta t \Leftrightarrow \frac{\|u(35,0) - u^*(35)\| - 1}{\sqrt{2}} = 71 \text{ min} \implies \|u(35,0) - u^*(35)\| \approx 71.7.$$

This means that if at $t_0 = 35$ the flow distribution is, for example, $u(35, 0) = (79, 26)$ cars/min, the traffic could adjust close to $u^*(35 + \Delta t) = u^*(106) = (7, 98)$ cars/min after approximately 71 minutes.

5.1. **Appendix.** We remark that for the set $\mathbb{K}$ in our application the following holds: for any $u \neq v \in \mathbb{K}$ with $u := (u_1, u_2)$ and $v := (v_1, v_2)$, it is always the case that $u_1(t) + u_2(t) = v_1(t) + v_2(t) = \rho_1(t)$. This implies that

$$u_1(t) - v_1(t) = -(u_2(t) - v_2(t)), \text{ for a.a. } t \in [0, 110]. \tag{5.1}$$

This further implies that a pair $u \neq v \in \mathbb{K}$ satisfies $u_1 \neq v_1$ and $u_2 \neq v_2$ a.a. on $[0, 110]$.

1. We show first that $F_t(w) = (w_1 + 151, w_2 + 60)$ is strongly pseudo-monotone with $\alpha := 1$ and $\eta := \frac{1}{\sqrt{2}}$ whenever $w_2 \le 90$, for a.a. $t \in [0, \frac{91}{3}]$. Let $w, v \in \mathbb{K}(t)$ and we evaluate

$$\langle F_t(v), w - v\rangle = (v_1 + 151)(w_1 - v_1) + (v_2 + 60)(w_2 - v_2) \overset{\text{by (5.1)}}{=} (v_1 - v_2 + 91)(w_1 - v_1).$$

Then $(v_1 - v_2 + 91)(w_1 - v_1) \ge 0$ if and only if $v_1 - v_2 + 91 \ge 0$ and $w_1 - v_1 \ge 0$. Now, we evaluate

$$\langle F_t(w), w - v\rangle \overset{\text{by (5.1)}}{=} (w_1 - w_2 + 91)(w_1 - v_1).$$

We take $\alpha = 1$ and want to find $\eta > 0$ so that

$$(w_1 - w_2 + 91)(w_1 - v_1) \ge \eta\sqrt{(w_1 - v_1)^2 + (w_2 - v_2)^2} \overset{\text{by (5.1)}}{=} \sqrt{2}\eta|w_1 - v_1|.$$

Since $w_1 \ge v_1$ and $w \neq v$, we can divide the last inequality by $w_1 - v_1$ and so

$$(w_1 - w_2 + 91) \ge \sqrt{2}\eta.$$

But $w_1 \ge 0$ from the hypothesis, so if $w_2 \le 90 \implies -w_2 \ge -90$, then we find $\eta := \frac{1}{\sqrt{2}}$.

2. Here we show that choosing $r = 1$, $F_{t=35}(w) := (w_1 + 151, w_2 + 60)$ is 1-strongly pseudo-monotone with degree $\alpha = 1$ and $\eta(1) = \sqrt{2}$ at $u^*(35) = (7, 98)$. We have that

$$\langle F_t((7,98)), (w_1 - 7, w_2 - 98)\rangle$$
$$= \langle (158, 158), (w_1 - 7, w_2 - 98)\rangle \overset{\text{by (5.1)}}{=} 0 : F_t(7,98), (w_1 - 7, w_2 - 98),$$

therefore

$$\langle F_t(w), (w_1 - 7, w_2 - 98)\rangle \overset{\text{by (5.1)}}{=} (w_1 - w_2 + 91)(w_1 - 7) \overset{w_1+w_2=105}{=} 2(w_1 - 7)^2.$$

We find $\eta(1) > 0$ so that

$$2|w_1 - 7|^2 \geq \sqrt{2}\eta(1)|w_1 - 7| \implies \eta(1) := \min\left\{\sqrt{2}|w_1 - 7|\right\} = \sqrt{2}r = \sqrt{2}.$$

Note that F_{35} is not strongly pseudo-monotone with $\alpha < 2$ at $u^*(35)$.

6. CONCLUSIONS AND ACKNOWLEDGEMENTS

In this paper we presented new results about the solution form of an EVI (2.2) subject to various types of constraint sets. These results have consequences for the study and modelling of equilibrium problems, in particular here, traffic network equilibrium problems. We have further demonstrated how the recently developed theory of double-layered dynamics, which combines evolutionary variational inequalities and projected dynamical systems over a unified constraint set, can be used for the modelling, analysis, and computation of solutions to time-dependent equilibrium problems; concretely, we presented here a novel interpretation of the timescales present in a DLD model of an equilibrium problem, more general than the one in [7]. We also answered questions regarding the finite-time adjustment to equilibrium states for traffic network problems by the introduction of a new type of monotonicity. This type, called r-strong pseudo-monotonicity, implies a stability property of a small neighbourhood around an equilibrium of a projected dynamical system.

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