journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 8 (2007), Issue 4, Article 102, 6 pp.

TWO NEW ALGEBRAIC INEQUALITIES WITH $2n$ VARIABLES

XIAO-GUANG CHU, CHENG-EN ZHANG, AND FENG QI

SUZHOU HENGTIAN TRADING CO. LTD.
EASTERN BAODAI ROAD, SUZHOU CITY,
JIANGSU PROVINCE, 215128, CHINA
srr345@163.com

DEPARTMENT OF INFORMATION ENGINEERING,
XUCHANG VOCATIONAL TECHNICAL COLLEGE,
XUCHANG CITY, HENAN PROVINCE,
461000, CHINA
zhangchengen63@163.com

RESEARCH INSTITUTE OF MATHEMATICAL INEQUALITY THEORY,
HENAN POLYTECHNIC UNIVERSITY,
JIAOZUO CITY, HENAN PROVINCE,
454010, CHINA
qifeng618@gmail.com
URL: http://rgmia.vu.edu.au/qi.html

Received 12 March, 2007; accepted 05 December, 2007
Communicated by D. Hinton

ABSTRACT. In this paper, by proving a combinatorial identity and an algebraic identity and by using Cauchy's inequality, two new algebraic inequalities involving $2n$ positive variables are established.

Key words and phrases: Algebraic inequality, Cauchy's inequality, Combinatorial identity, Algebraic identity.

2000 *Mathematics Subject Classification.* 26D05; 26D07; 26D15.

1. MAIN RESULTS

When solving Question CIQ-103 in [2] and Question CIQ-142 in [5], the following two algebraic inequalities involving $2n$ variables were posed.

Theorem 1.1. *Let $n \ge 2$ and x_i for $1 \le i \le 2n$ be positive real numbers. Then*

$$\sum_{i=1}^{2n} \frac{x_i^{2n-1}}{\sum_{k\neq i}^{2n}(x_i+x_k)^{2n-1}} \ge \frac{n}{2^{2n-2}(2n-1)}. \tag{1.1}$$

Equality in (1.1) *holds if and only if $x_i = x_j$ for all $1 \le i, j \le 2n$.*

The authors would like to express heartily their thanks to the anonymous referees for their valuable corrections and comments on the original version of this paper.
078-07

Theorem 1.2. *Let $n \geq 2$ and y_i for $1 \leq i \leq 2n$ be positive real numbers. Then*

$$\sum_{i=1}^{2n} \frac{y_i^2}{y_{i-1|2n} \sum_{k=i}^{i+n-2} y_{k|2n}} \geq \frac{2n}{n-1}, \tag{1.2}$$

where $m|2n$ means $m \mod 2n$ for all nonnegative integers m. Equality in (1.2) *holds if and only if $y_i = y_j$ for all $1 \leq i, j \leq 2n$.*

The notation $\sum_{k=i}^{i+n-2} y_{k|2n}$ in Theorem 1.2 could be illustrated with an example to clarify the meaning: If $n = 5$ then $\sum_{k=9}^{12} y_{k|10} = y_9 + y_{10} + y_1 + y_2$.

In this article, by proving a combinatorial identity and an algebraic identity and by using Cauchy's inequality, these two algebraic inequalities (1.1) and (1.2) involving $2n$ positive variables are proved.

Moreover, as a by-product of Theorem 1.1, the following inequality is deduced.

Theorem 1.3. *For $n \geq 2$ and $1 \leq k \leq n-1$,*

$$\sum_{p=1}^{k} p(p+1)\binom{2n}{k-p} < \frac{2^{2(n-1)}k(k+1)}{n}. \tag{1.3}$$

2. TWO LEMMAS

In order to prove inequalities (1.1) and (1.2), the following two lemmas are necessary.

Lemma 2.1. *Let n and k be natural numbers such that $n > k$. Then*

$$\sum_{k=0}^{n-1} (n-k)^2 \binom{2n}{k} = 4^{n-1} n. \tag{2.1}$$

Proof. It is well known that

$$\binom{n}{k} = \binom{n}{n-k}, \quad k\binom{n}{k} = n\binom{n-1}{k-1},$$
$$k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}, \quad \sum_{i=0}^{2n}\binom{2n}{i} = 4^n.$$

Then

$$\begin{aligned}
&\sum_{k=0}^{n-1} (n-k)^2 \binom{2n}{k} \\
&= n^2 \sum_{k=0}^{n-1} \binom{2n}{k} - (2n-1) \sum_{k=0}^{n-1} k\binom{2n}{k} + \sum_{k=0}^{n-1} k(k-1)\binom{2n}{k} \\
&= n^2 \sum_{k=0}^{n-1} \binom{2n}{k} - 2n(2n-1) \sum_{k=1}^{n-1} \binom{2n-1}{k-1} + 2n(2n-1) \sum_{k=2}^{n-1} \binom{2n-2}{k-2} \\
&= n^2 \sum_{k=0}^{n-1} \binom{2n}{k} - 2n(2n-1) \sum_{k=0}^{n-2} \binom{2n-1}{k} + 2n(2n-1) \sum_{k=0}^{n-3} \binom{2n-2}{k} \\
&= n^2 \frac{4^n - \binom{2n}{n}}{2} - 2n(2n-1)\frac{2^{2n-1} - 2\binom{2n-1}{n-1}}{2} + 2n(2n-1)\frac{4^{n-1} - \binom{2n-2}{n-1} - 2\binom{2n-2}{n-2}}{2}
\end{aligned}$$

$$= 4^{n-1}n + \frac{4n(2n-1)\binom{2n-1}{n} - n^2\binom{2n}{n} - 2n(2n-1)\left[\binom{2n-2}{n-1} + 2\binom{2n-2}{n-2}\right]}{2}$$
$$= 4^{n-1}n + \left[2(2n-1)^2 - n(2n-1) - n(2n-1) - 2(2n-1)(n-1)\right]\binom{2n-2}{n-1}$$
$$= 4^{n-1}n.$$

The proof of Lemma 2.1 is complete. □

Lemma 2.2. *Let $n \geq 2$ and y_i for $1 \leq i \leq 2n$ be positive numbers. Denote $x_i = y_i + y_{n+i}$ for $1 \leq i \leq n$ and*

$$A_n = \sum_{i=1}^{2n} y_i \sum_{k=i+1}^{n-1+i} y_{k|2n}, \tag{2.2}$$

where $m|2n$ means $m \mod 2n$ for all nonnegative integers m. Then

$$A_n = \sum_{1\leq i<j\leq n} x_i x_j. \tag{2.3}$$

Proof. Formula (2.2) can be written as

$$A_n = y_1(y_2 + \cdots + y_n) + y_2(y_3 + \cdots + y_{n+1}) + \cdots + y_{2n}(y_1 + \cdots + y_{n-1}). \tag{2.4}$$

From this, it is obtained readily that

$$A_n = \sum_{1\leq i<j\leq 2n} y_i y_j - \sum_{i=1}^{n} y_i y_{n+i}$$

by induction on n. Since

$$\sum_{1\leq i<j\leq n} x_i x_j = \sum_{1\leq i<j\leq n} (y_i + y_{i+n})(y_j + y_{j+n}),$$

then

$$A_n = \sum_{1\leq i<j\leq 2n} y_i y_j - \sum_{i=1}^{n} y_i y_{i+n} = \sum_{1\leq i<j\leq n} (y_i + y_{i+n})(y_j + y_{j+n}) = \sum_{1\leq i<j\leq n} x_i x_j,$$

which means that identity (2.3) holds. The proof of Lemma 2.2 is complete. □

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1.1. By Cauchy's inequality [1, 4], it follows that

$$\sum_{i=1}^{2n} \frac{x_i^{2n-1}}{\sum_{k\neq i}^{2n}(x_i + x_k)^{2n-1}} \sum_{i=1}^{2n}\sum_{j\neq i}^{2n} x_i(x_i + x_j)^{2n-1} \geq \left(\sum_{i=1}^{2n} x_i^n\right)^2. \tag{3.1}$$

Consequently, it suffices to show

$$(2n-1)4^{n-1}\left(\sum_{i=1}^{2n} x_i^n\right)^2 \geq n\sum_{i=1}^{2n}\sum_{j\neq i}^{2n} x_i(x_i + x_j)^{2n-1}$$
$$\iff (2n-1)4^{n-1}\sum_{i=1}^{2n} x_i^{2n} + (2n-1)2^{2n-1}\sum_{1\leq i<j\leq 2n} x_i^n x_j^n$$
$$\geq n\sum_{k=0}^{n}\left[\binom{2n-1}{k} + \binom{2n-1}{2n-k}\right]\sum_{i=1}^{2n}\sum_{j\neq i}^{2n} x_i^{2n-k}x_j^k$$

$$\begin{aligned}\iff (2n-1)\left(2^{2n-2}-n\right)\sum_{i=1}^{2n}x_i^{2n} \\ +\left[(2n-1)2^{2n-1}-2n\binom{2n-1}{n}\right]\sum_{1\le i<j\le 2n}x_i^nx_j^n \\ \ge n\sum_{k=1}^{n-1}\left[\binom{2n-1}{k}+\binom{2n-1}{2n-k}\right]\sum_{i=1}^{2n}\sum_{j\ne i}^{2n}x_i^{2n-k}x_j^k.\end{aligned}$$

Since $\binom{n}{k}=\binom{n}{n-k}$ and $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$, the above inequality becomes

$$\begin{aligned}\left[(2n-1)2^{2n-1}-n\binom{2n}{n}\right]\sum_{1\le i<j\le 2n}x_i^nx_j^n \\ +(2n-1)\left(2^{2n-2}-n\right)\sum_{i=1}^{2n}x_i^{2n}-n\sum_{k=1}^{n-1}\binom{2n}{k}\sum_{i=1}^{2n}\sum_{j\ne i}^{2n}x_i^{2n-k}x_j^k\ge 0.\end{aligned}\tag{3.2}$$

Utilization of $\sum_{k=0}^{2n}\binom{2n}{k}=2^{2n}$ and $\binom{2n}{0}=\binom{2n}{2n}=1$ yields

$$\begin{aligned}2\left(2^{2n-2}-n\right)+(2n-1)2^{2n-1}-n\binom{2n}{n}-n\sum_{k=1,k\ne n}^{2n-1}\binom{2n}{k} \\ =2^{2n}n-2n-n\left[\sum_{k=0}^{2n}\binom{2n}{k}-2\right]=0.\end{aligned}$$

Substituting this into (3.2) gives

$$\begin{aligned}\sum_{i=1,j=1,i\ne j}^{2n}\left\{\sum_{q=0}^{n-1}\left[2^{2n-2}-n-n\sum_{k=1}^{q}\binom{2n}{k}\right]x_i^qx_j^q\sum_{k=0}^{2n-2q-2}x_i^{2n-2q-k-2}x_j^k\right\} \\ \times(x_i-x_j)^2\ge 0,\end{aligned}\tag{3.3}$$

where $\sum_{k=1}^{q}\binom{2n}{k}=0$ for $q=0$. Employing (2.1) in the above inequality leads to

$$\begin{aligned}\sum_{p=0}^{n-1}(2n-2p-1)\left[2^{2n-2}-n\sum_{k=0}^{p}\binom{2n}{k}\right]=2^{2n-2}n^2-n\left\{\sum_{p=0}^{n-1}(2n-2p-1)\sum_{k=0}^{p}\binom{2n}{k}\right\} \\ =2^{2n-2}n^2-n\sum_{k=0}^{n-1}(n-k)^2\binom{2n}{k}=0.\end{aligned}$$

This implies that inequality (3.3) is equivalent to

$$\begin{aligned}\sum_{i=1,j=1,i\ne j}^{2n}(x_i-x_j)^2\left\{\sum_{k=1}^{n}\left[k2^{2n-2}-n\sum_{q=0}^{k-1}(k-q)\binom{2n}{q}\right]x_i^{2n-k-1}x_j^{k-1}\right. \\ \left.+\sum_{k=n+1}^{2n}\left[(2n-k+1)2^{2n-2}-n\sum_{q=k}^{2n}(2n-q+1)\binom{2n}{q-k}\right]x_i^{2n-k}x_j^{k-2}\right\}\ge 0,\end{aligned}\tag{3.4}$$

$$\sum_{i=1,j=1,i\neq j}^{2n}\left\{\sum_{k=1}^{n-1}\left[\frac{k(k+1)}{2}2^{2n-2}-n\sum_{p=1}^{k}\frac{p(p+1)}{2}\binom{2n}{k-p}\right]\right.$$
$$\left.\times x_i^{k-1}x_j^{k-1}\sum_{p=0}^{2n-2k-2}x_i^{2n-p-4}x_j^{p}\right\}(x_i-x_j)^4\geq 0.$$

In order to prove (3.4), it is sufficient to show

$$\frac{(n-1)[(n-1)+1]}{2}2^{2n-2}-n\sum_{p=1}^{n-1}\frac{p(p+1)}{2}\binom{2n}{n-p-1}>0. \tag{3.5}$$

Considering (2.1), it is sufficient to show

$$\sum_{k=0}^{n-1}(n-k)\binom{2n}{k}>2^{2n-2}. \tag{3.6}$$

By virtue of $\binom{n}{k}=\binom{n}{n-k}$ and $\sum_{k=0}^{2n}\binom{2n}{k}=2^{2n}$, inequality (3.6) can be rearranged as

$$\sum_{k=0}^{n-1}(n-k)\binom{2n}{k}+\sum_{k=n+1}^{2n}(k-n)\binom{2n}{k}>2^{2n-1},$$
$$\sum_{k=0}^{n-1}(2n-2k-1)\binom{2n}{k}+\sum_{k=n+1}^{2n}(2k-2n-1)\binom{2n}{k}>\binom{2n}{n}. \tag{3.7}$$

Since $n\geq 2$ and $\binom{2n}{n-1}+\binom{2n}{n+1}>\binom{2n}{n}$ is equivalent to $2>\frac{n+1}{n}$, then inequalities (3.7), (3.6) and (3.5) are valid. The proof of Theorem 1.1 is complete. □

Proof of Theorem 1.2. By Lemma 2.2, it is easy to see that $\sum_{i=1}^{2n}y_i=\sum_{i=1}^{n}x_i$. From Cauchy's inequality [1, 4], it follows that

$$A_n\sum_{i=1}^{2n}\frac{y_i^2}{y_{i-1|2n}\sum_{k=i}^{i+n-2}y_{k|2n}}\geq\left(\sum_{i=1}^{2n}y_i\right)^2,$$

where A_n is defined by (2.2) or (2.3) in Lemma 2.2. Therefore, it is sufficient to prove

$$(n-1)\left(\sum_{i=1}^{2n}y_i\right)^2\geq 2nA_n\iff (n-1)\left(\sum_{i=1}^{n}x_i\right)^2\geq 2n\sum_{1\leq i<j\leq n}x_ix_j$$
$$\iff (n-1)\sum_{i=1}^{n}x_i^2\geq 2\sum_{1\leq i<j\leq 2n}x_ix_j$$
$$\iff \sum_{1\leq i<j\leq 2n}(x_i-x_j)^2\geq 0.$$

The proof of Theorem 1.2 is complete. □

Proof of Theorem 1.3. Let

$$B_n=2^{2(n-1)}k(k+1)-n\sum_{p=1}^{k}p(p+1)\binom{2n}{k-p}. \tag{3.8}$$

Then

$$
\begin{aligned}
B_{n+1} &= k(k+1)2^{2n-2}2^2 - (n+1)\sum_{p=1}^{k} p(p+1)\binom{2n+2}{k-p} \\
&= 4B_n + \sum_{p=1}^{k} p(p+1)\left[4n\binom{2n}{k-p} - (n+1)\binom{2n+2}{k-p}\right] \qquad (3.9)\\
&\triangleq 4B_n + \sum_{p=1}^{k} p(p+1)C_{k-p}
\end{aligned}
$$

and

$$
\begin{aligned}
C_q - C_{q+1} &= 4n\left[\binom{2n}{q} - \binom{2n}{q+1}\right] - (n+1)\left[\binom{2n+2}{q} - \binom{2n+2}{q+1}\right]\\
&= 4n\binom{2n}{q}\left(1 - \frac{2n-q}{q+1}\right) - (n+1)\binom{2n+2}{q}\left(1 - \frac{2n+2-q}{q+1}\right)\\
&= 4n\binom{2n}{q}\frac{2q-2n+1}{q+1} - (n+1)\binom{2n+2}{q}\frac{2q-2n-1}{q+1}\\
&> \frac{2q-2n+1}{q+1}C_q
\end{aligned}
$$

for $0 \le q \le k-1$. Hence,

$$\frac{2n-q}{q+1}C_q > C_{q+1}. \qquad (3.10)$$

From the above inequality and the facts that

$$C_n = \frac{2(2n-1)(n+1)}{n+2}\binom{2n}{n} > 0 \qquad (3.11)$$

and $\frac{2n-q}{q+1} > 0$, it follows easily that $C_q > 0$. Consequently, we have $B_{n+1} > 4B_n$, and then $B_{k+2} > 4B_{k+1}$. As a result, utilization of (3.5) gives

$$B_{k+1} > 0, \quad B_{k+2} > 0, \quad B_{k+3} > 0, \quad B_{k+4} > 0, \quad \cdots, \quad B_{k+(n-k)} = B_n > 0.$$

The proof of inequality (1.3) is complete. □

REFERENCES

[1] J.-CH. KUANG, *Chángyòng Bùděngshì* (*Applied Inequalities*), 3rd ed., Shāndōng Kēxué Jìshù Chūbǎn Shè (Shandong Science and Technology Press), Jinan City, Shandong Province, China, 2004. (Chinese)

[2] J.-P. LI, Question CIQ-103, *Communications in Studies of Inequalities*, **11**(2) (2004), 277. (Chinese)

[3] ZH.-P. LIU, X.-G. CHU AND B.-N. GUO, On two algebraic inequalities with even variables, *J. Henan Polytech. Univ.*, **24**(5) (2005), 410–414. (Chinese)

[4] D. S. MITRINOVIĆ, J. E. PEČARIĆ, AND A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, 1993.

[5] W.-J. ZHANG, Question CIQ-142, *Communications in Studies of Inequalities*, **12**(1) (2005), 93. (Chinese)