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GENERALIZED RELATIVE INFORMATION AND INFORMATION INEQUALITIES

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ABSTRACT. In this paper, we have obtained bounds on Csiszár's *f-divergence* in terms of *relative information of type s* using Dragomir's [9] approach. The results obtained in particular lead us to bounds in terms of χ^2-*Divergence*, Kullback-Leibler's *relative information* and Hellinger's *discrimination.*

Key words and phrases: Relative information; Csiszár's $f-$divergence; χ^2-divergence; Hellinger's discrimination; Relative information of type s; Information inequalities.

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1. INTRODUCTION

Let

$$\Delta_n = \left\{ P = (p_1, p_2, \ldots, p_n) \middle| p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\}, \quad n \geq 2,$$

be the set of complete finite discrete probability distributions.

The Kullback Leibler's [13] *relative information* is given by

$$K(P||Q) = \sum_{i=1}^{n} p_i \ln\left(\frac{p_i}{q_i}\right), \tag{1.1}$$

for all $P, Q \in \Delta_n$.

In Δ_n, we have taken all $p_i > 0$. If we take $p_i \geq 0, \forall i = 1, 2, \ldots, n$, then in this case we have to suppose that $0 \ln 0 = 0 \ln\left(\frac{0}{0}\right) = 0$. From the *information theoretic* point of view we generally take all the logarithms with base 2, but here we have taken only natural logarithms.

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We observe that the measure (1.1) is not symmetric in P and Q. Its symmetric version, famous as *J-divergence* (Jeffreys [12]; Kullback and Leiber [13]), is given by

$$J(P||Q) = K(P||Q) + K(Q||P) = \sum_{i=1}^{n} (p_i - q_i) \ln\left(\frac{p_i}{q_i}\right). \tag{1.2}$$

Let us consider the one parametric generalization of the measure (1.1), called *relative information of type* s given by

$$K_s(P||Q) = [s(s-1)]^{-1}\left[\sum_{i=1}^{n} p_i^s q_i^{1-s} - 1\right], \quad s \neq 0, 1. \tag{1.3}$$

In this case we have the following limiting cases

$$\lim_{s\to 1} K_s(P||Q) = K(P||Q),$$

and

$$\lim_{s\to 0} K_s(P||Q) = K(Q||P).$$

The expression (1.3) has been studied by Vajda [22]. Previous to it many authors studied its characterizations and applications (ref. Taneja [20] and on line book Taneja [21]).

We have some interesting particular cases of the measure (1.3).

(i) When $s = \frac{1}{2}$, we have

$$K_{1/2}(P||Q) = 4\left[1 - B(P||Q)\right] = 4h(P||Q)$$

where

$$B(P||Q) = \sum_{i=1}^{n} \sqrt{p_i q_i}, \tag{1.4}$$

is the famous as Bhattacharya's [1] *distance*, and

$$h(P||Q) = \frac{1}{2}\sum_{i=1}^{n} (\sqrt{p_i} - \sqrt{q_i})^2, \tag{1.5}$$

is famous as Hellinger's [11] *discrimination*.

(ii) When $s = 2$, we have

$$K_2(P||Q) = \frac{1}{2}\chi^2(P||Q),$$

where

$$\chi^2(P||Q) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{q_i} = \sum_{i=1}^{n} \frac{p_i^2}{q_i} - 1, \tag{1.6}$$

is the χ^2-*divergence* (Pearson [16]).

(iii) When $s = -1$, we have

$$K_{-1}(P||Q) = \frac{1}{2}\chi^2(Q||P),$$

where

$$\chi^2(Q||P) = \sum_{i=1}^{n} \frac{(p_i - q_i)^2}{p_i} = \sum_{i=1}^{n} \frac{q_i^2}{p_i} - 1. \tag{1.7}$$

For simplicity, let us write the measures (1.3) in the unified way:

$$\Phi_s(P||Q) = \begin{cases} K_s(P||Q), & s \neq 0,1, \\ K(Q||P), & s = 0, \\ K(P||Q), & s = 1. \end{cases} \tag{1.8}$$

Summarizing, we have the following particular cases of the measures (1.8):

(i) $\Phi_{-1}(P||Q) = \frac{1}{2}\chi^2(Q||P)$.
(ii) $\Phi_0(P||Q) = K(Q||P)$.
(iii) $\Phi_{1/2}(P||Q) = 4\left[1 - B(P||Q)\right] = 4h(P||Q)$.
(iv) $\Phi_1(P||Q) = K(P||Q)$.
(v) $\Phi_2(P||Q) = \frac{1}{2}\chi^2(P||Q)$.

2. CSISZÁR'S $f-$DIVERGENCE AND INFORMATION BOUNDS

Given a convex function $f : [0,\infty) \to \mathbb{R}$, the $f-$divergence measure introduced by Csiszár [4] is given by

$$C_f(p,q) = \sum_{i=1}^{n} q_i f\left(\frac{p_i}{q_i}\right), \tag{2.1}$$

where $p,q \in \mathbb{R}^n_+$.

The following two theorems can be seen in Csiszár and Körner [5].

Theorem 2.1. *(Joint convexity). If $f : [0,\infty) \to \mathbb{R}$ be convex, then $C_f(p,q)$ is jointly convex in p and q, where $p,q \in \mathbb{R}^n_+$.*

Theorem 2.2. *(Jensen's inequality). Let $f : [0,\infty) \to \mathbb{R}$ be a convex function. Then for any $p,q \in \mathbb{R}^n_+$, with $P_n = \sum_{i=1}^n p_i > 0, Q_n = \sum_{i=1}^n p_i > 0$, we have the inequality*

$$C_f(p,q) \geq Q_n f\left(\frac{P_n}{Q_n}\right).$$

The equality sign holds for strictly convex functions iff

$$\frac{p_1}{q_i} = \frac{p_2}{q_2} = \cdots = \frac{p_n}{q_n}.$$

In particular, for all $P,Q \in \Delta_n$, we have

$$C_f(P||Q) \geq f(1),$$

with equality iff $P = Q$.

In view of Theorems 2.1 and 2.2, we have the following result.

Result 1. For all $P,Q \in \Delta_n$, we have

(i) $\Phi_s(P||Q) \geq 0$ for any $s \in \mathbb{R}$, with equality iff $P = Q$.
(ii) $\Phi_s(P||Q)$ is convex function of the pair of distributions $(P,Q) \in \Delta_n \times \Delta_n$ and for any $s \in \mathbb{R}$.

Proof. Take

$$\phi_s(u) = \begin{cases} [s(s-1)]^{-1}\left[u^s - 1 - s(u-1)\right], & s \neq 0,1; \\ u - 1 - \ln u, & s = 0; \\ 1 - u + u\ln u, & s = 1 \end{cases} \tag{2.2}$$

for all $u > 0$ in (2.1), we have

$$C_f(P||Q) = \Phi_s(P||Q) = \begin{cases} K_s(P||Q), & s \neq 0, 1; \\ K(Q||P), & s = 0; \\ K(P||Q), & s = 1. \end{cases}$$

Moreover,

$$\phi_s'(u) = \begin{cases} (s-1)^{-1}\left(u^{s-1} - 1\right), & s \neq 0, 1; \\ 1 - u^{-1}, & s = 0; \\ \ln u, & s = 1 \end{cases} \tag{2.3}$$

and

$$\phi_s''(u) = \begin{cases} u^{s-2}, & s \neq 0, 1; \\ u^{-2}, & s = 0; \\ u^{-1}, & s = 1. \end{cases} \tag{2.4}$$

Thus we have $\phi_s''(u) > 0$ for all $u > 0$, and hence, $\phi_s(u)$ is strictly convex for all $u > 0$. Also, we have $\phi_s(1) = 0$. In view of Theorems 2.1 and 2.2 we have the proof of parts (i) and (ii) respectively. □

For some studies on the measure (2.2) refer to Liese and Vajda [15], Österreicher [17] and Cerone et al. [3].

The following theorem summarizes some of the results studies by Dragomir [7], [8]. For simplicity we have taken $f(1) = 0$ and $P, Q \in \Delta_n$.

Theorem 2.3. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be differentiable convex and normalized i.e., $f(1) = 0$. If $P, Q \in \Delta_n$ are such that*

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

for some r and R with $0 < r \leq 1 \leq R < \infty$, then we have the following inequalities:

$$0 \leq C_f(P||Q) \leq \frac{1}{4}(R-r)\left(f'(R) - f'(r)\right), \tag{2.5}$$

$$0 \leq C_f(P||Q) \leq \beta_f(r, R), \tag{2.6}$$

and

$$\begin{aligned} 0 &\leq \beta_f(r,R) - C_f(P||Q) \\ &\leq \frac{f'(R) - f'(r)}{R - r}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\leq \frac{1}{4}(R-r)\left(f'(R) - f'(r)\right), \end{aligned} \tag{2.7}$$

where

$$\beta_f(r, R) = \frac{(R-1)f(r) + (1-r)f(R)}{R - r}, \tag{2.8}$$

and $\chi^2(P||Q)$ and $C_f(P||Q)$ are as given by (1.6) and (2.1) respectively.

In view of above theorem, we have the following result.

Result 2. Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. If there exists r, R such that

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \dots, n\},$$

with $0 < r \leq 1 \leq R < \infty$, then we have

$$0 \leq \Phi_s(P||Q) \leq \mu_s(r, R), \tag{2.9}$$

$$0 \leq \Phi_s(P||Q) \leq \phi_s(r, R), \tag{2.10}$$

and

$$\begin{aligned} 0 &\leq \phi_s(r, R) - \Phi_s(P||Q) \\ &\leq k_s(r, R)\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\leq \mu_s(r, R), \end{aligned} \tag{2.11}$$

where

$$\mu_s(r, R) = \begin{cases} \frac{1}{4}\frac{(R-r)\left(R^{s-1}-r^{s-1}\right)}{(s-1)}, & s \neq 1; \\ \frac{1}{4}(R-r)\ln\left(\frac{R}{r}\right), & s = 1 \end{cases} \tag{2.12}$$

$$\begin{aligned} \phi_s(r, R) &= \frac{(R-1)\phi_s(r) + (1-r)\phi_s(R)}{R-r} \\ &= \begin{cases} \frac{(R-1)(r^s-1)+(1-r)(R^s-1)}{(R-r)s(s-1)}, & s \neq 0, 1; \\ \frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{(R-r)}, & s = 0; \\ \frac{(R-1)r\ln r+(1-r)R\ln R}{(R-r)}, & s = 1, \end{cases} \end{aligned} \tag{2.13}$$

and

$$k_s(r, R) = \frac{\phi_s'(R) - \phi_s'(r)}{R-r} = \begin{cases} \dfrac{R^{s-1} - r^{s-1}}{(R-r)(s-1)}, & s \neq 1; \\ \dfrac{\ln R - \ln r}{R-r}, & s = 1. \end{cases} \tag{2.14}$$

Proof. The above result follows immediately from Theorem 2.3, by taking $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then in this case we have $C_f(P||Q) = \Phi_s(P||Q)$. □

Moreover,

$$\mu_s(r, R) = \frac{1}{4}(R-r)^2\, k_s(r, R),$$

where

$$k_s(r, R) = \begin{cases} \left[L_{s-2}(r, R)\right]^{s-2}, & s \neq 1; \\ \left[L_{-1}(r, R)\right]^{-1} & s = 1, \end{cases}$$

and $L_p(a,b)$ is the famous (ref. Bullen, Mitrinović and Vasić [2]) *p-logarithmic mean* given by

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq -1, 0; \\ \frac{b-a}{\ln b - \ln a}, & p = -1; \\ \frac{1}{e}\left[\frac{b^b}{a^a}\right]^{\frac{1}{b-a}}, & p = 0, \end{cases}$$

for all $p \in \mathbb{R}$, $a, b \in \mathbb{R}_+$, $a \neq b$.

We have the following corollaries as particular cases of Result 2.

Corollary 2.4. *Under the conditions of Result 2, we have*

$$0 \le \chi^2(Q||P) \le \frac{1}{4}(R+r)\left(\frac{R-r}{rR}\right)^2, \tag{2.15}$$

$$0 \le K(Q||P) \le \frac{(R-r)^2}{4Rr}, \tag{2.16}$$

$$0 \le K(P||Q) \le \frac{1}{4}(R-r)\ln\left(\frac{R}{r}\right), \tag{2.17}$$

$$0 \le h(P||Q) \le \frac{(R-r)\left(\sqrt{R}-\sqrt{r}\right)}{8\sqrt{Rr}} \tag{2.18}$$

and

$$0 \le \chi^2(P||Q) \le \frac{1}{2}(R-r)^2. \tag{2.19}$$

Proof. (2.15) follows by taking $s = -1$, (2.16) follows by taking $s = 0$, (2.17) follows by taking $s = 1$, (2.18) follows by taking $s = \frac{1}{2}$ and (2.19) follows by taking $s = 2$ in (2.9). □

Corollary 2.5. *Under the conditions of Result 2, we have*

$$0 \le \chi^2(Q||P) \le \frac{(R-1)(1-r)}{rR}, \tag{2.20}$$

$$0 \le K(Q||P) \le \frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r}, \tag{2.21}$$

$$0 \le K(P||Q) \le \frac{(R-1)r\ln r + (1-r)R\ln R}{R-r}, \tag{2.22}$$

$$0 \le h(P||Q) \le \frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} \tag{2.23}$$

and

$$0 \le \chi^2(P||Q) \le (R-1)(1-r). \tag{2.24}$$

Proof. (2.20) follows by taking $s = -1$, (2.21) follows by taking $s = 0$, (2.22) follows by taking $s = 1$, (2.23) follows by taking $s = \frac{1}{2}$ and (2.24) follows by taking $s = 2$ in (2.10). □

In view of (2.16), (2.17), (2.21) and (2.22), we have the following bounds on *J-divergence*:

$$0 \le J(P||Q) \le \min\{t_1(r,R), t_2(r,R)\}, \tag{2.25}$$

where

$$t_1(r,R) = \frac{1}{4}(R-r)^2\left[(rR)^{-1} + (L_{-1}(r,R))^{-1}\right]$$

and

$$t_2(r,R) = (R-1)(1-r)\left(L_{-1}(r,R)\right)^{-1}.$$

The expression $t_1(r,R)$ is due to (2.16) and (2.17) and the expression $t_2(r,R)$ is due to (2.21) and (2.22).

Corollary 2.6. *Under the conditions of Result 2, we have*

$$\begin{aligned} 0 &\le \frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \\ &\le \frac{R+r}{(rR)^2}\left[(R-1)(1-r) - \chi^2(P||Q)\right], \end{aligned} \tag{2.26}$$

$$\begin{aligned} 0 &\le \frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P) \\ &\le \frac{1}{rR}\left[(R-1)(1-r) - \chi^2(P||Q)\right], \end{aligned} \tag{2.27}$$

$$\begin{aligned} 0 &\le \frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q) \\ &\le \frac{\ln R - \ln r}{R-r}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} 0 &\le \frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\left(\sqrt{R}+\sqrt{r}\right)} - h(P||Q) \\ &\le \frac{1}{2\sqrt{rR}\left(\sqrt{R}+\sqrt{r}\right)}\left[(R-1)(1-r) - \chi^2(P||Q)\right]. \end{aligned} \tag{2.29}$$

Proof. (2.26) follows by taking $s = -1$, (2.27) follows by taking $s = 0$, (2.28) follows by taking $s = 1$, (2.29) follows by taking $s = \frac{1}{2}$ in (2.11). ☐

3. MAIN RESULTS

In this section, we shall present a theorem generalizing the one obtained by Dragomir [9]. The results due to Dragomir [9] are limited only to χ^2- *divergence*, while the theorem established here is given in terms of *relative information of type* s, that in particular lead us to bounds in terms of χ^2-*divergence*, Kullback-Leibler's *relative information* and Hellinger's *discrimination*.

Theorem 3.1. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ the generating mapping be normalized, i.e., $f(1) = 0$ and satisfy the assumptions:*

(i) *f is twice differentiable on (r,R), where $0 \le r \le 1 \le R \le \infty$;*
(ii) *there exists the real constants m, M with $m < M$ such that*

$$m \le x^{2-s} f''(x) \le M, \quad \forall x \in (r,R), \quad s \in \mathbb{R}. \tag{3.1}$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \le \frac{p_i}{q_i} \le R < \infty,$$

then we have the inequalities:

$$m\left[\phi_s(r,R) - \Phi_s(P||Q)\right] \le \beta_f(r,R) - C_f(P||Q) \le M\left[\phi_s(r,R) - \Phi_s(P||Q)\right], \tag{3.2}$$

where $C_f(P||Q), \Phi_s(P||Q), \beta_f(r,R)$ *and* $\phi_s(r,R)$ *are as given by (2.1), (1.8), (2.8) and (2.13) respectively.*

Proof. Let us consider the functions $F_{m,s}(\cdot)$ and $F_{M,s}(\cdot)$ given by

$$F_{m,s}(u) = f(u) - m\phi_s(u), \tag{3.3}$$

and

$$F_{M,s}(u) = M\phi_s(u) - f(u), \tag{3.4}$$

respectively, where m and M are as given by (3.1) and function $\phi_s(\cdot)$ is as given by (2.3).

Since $f(u)$ and $\phi_s(u)$ are normalized, then $F_{m,s}(\cdot)$ and $F_{M,s}(\cdot)$ are also normalized, i.e., $F_{m,s}(1) = 0$ and $F_{M,s}(1) = 0$. Moreover, the functions $f(u)$ and $\phi_s(u)$ are twice differentiable. Then in view of (2.4) and (3.1), we have

$$F''_{m,s}(u) = f''(u) - mu^{s-2} = u^{s-2}\left(u^{2-s}f''(u) - m\right) \geq 0$$

and

$$F''_{M,s}(u) = Mu^{s-2} - f''(u) = u^{s-2}\left(M - u^{2-s}f''(u)\right) \geq 0,$$

for all $u \in (r,R)$ and $s \in \mathbb{R}$. Thus the functions $F_{m,s}(\cdot)$ and $F_{M,s}(\cdot)$ are convex on (r,R).

We have seen above that the real mappings $F_{m,s}(\cdot)$ and $F_{M,s}(\cdot)$ defined over $\mathbb{R}_+$ given by (3.3) and (3.4) respectively are normalized, twice differentiable and convex on (r,R). Applying the r.h.s. of the inequality (2.6), we have

$$C_{F_{m,s}}(P||Q) \leq \beta_{F_{m,s}}(r,R), \tag{3.5}$$

and

$$C_{F_{m,s}}(P||Q) \leq \beta_{F_{M,s}}(r,R), \tag{3.6}$$

respectively.

Moreover,

$$C_{F_{m,s}}(P||Q) = C_f(P||Q) - m\Phi_s(P||Q), \tag{3.7}$$

and

$$C_{F_{M,s}}(P||Q) = M\Phi_s(P||Q) - C_f(P||Q). \tag{3.8}$$

In view of (3.5) and (3.7), we have

$$C_f(P||Q) - m\Phi_s(P||Q) \leq \beta_{F_{m,s}}(r,R),$$

i.e.,

$$C_f(P||Q) - m\Phi_s(P||Q) \leq \beta_f(r,R) - m\phi_s(r,R)$$

i.e.,

$$m\left[\phi_s(r,R) - \Phi_s(P||Q)\right] \leq \beta_f(r,R) - C_f(P||Q).$$

Thus, we have the l.h.s. of the inequality (3.2).

Again in view of (3.6) and (3.8), we have

$$M\Phi_s(P||Q) - C_f(P||Q) \leq \beta_{F_{M,s}}(r,R),$$

i.e.,

$$M\Phi_s(P||Q) - C_f(P||Q) \leq M\phi_s(r,R) - \beta_f(r,R),$$

i.e.,

$$\beta_f(r,R) - C_f(P||Q) \leq M\left[\phi_s(r,R) - \Phi_s(P||Q)\right].$$

Thus we have the r.h.s. of the inequality (3.2). □

Remark 3.2. For similar kinds of results in comparing the $f-$divergence with Kullback-Leibler *relative information* see the work by Dragomir [10]. The case of Hellinger *discrimination* is discussed in Dragomir [6].

We shall now present some particular case of the Theorem 3.1.

3.1. **Information Bounds in Terms of χ^2-Divergence.** In particular for $s = 2$, in Theorem 3.1, we have the following proposition:

Proposition 3.3. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ the generating mapping be normalized, i.e., $f(1) = 0$ and satisfy the assumptions:*

(i) *f is twice differentiable on (r, R), where $0 < r \leq 1 \leq R < \infty$;*
(ii) *there exists the real constants m, M with $m < M$ such that*

$$m \leq f''(x) \leq M, \quad \forall x \in (r, R). \tag{3.9}$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty,$$

then we have the inequalities:

$$\begin{aligned} &\frac{m}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\leq \beta_f(r, R) - C_f(P||Q) \\ &\leq \frac{M}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right], \end{aligned} \tag{3.10}$$

where $C_f(P||Q), \beta_f(r, R)$ and $\chi^2(P||Q)$ are as given by (2.1), (2.8) and (1.6) respectively.

The above proposition was obtained by Dragomir in [9]. As a consequence of the above Proposition 3.3, we have the following result.

Result 3. Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exist r, R $(0 < r \leq 1 \leq R < \infty)$ such that

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

then in view of Proposition 3.3, we have

$$\begin{aligned} &\frac{R^{s-2}}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\leq \phi_s(r, R) - \Phi_s(P||Q) \\ &\leq \frac{r^{s-2}}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right], \quad s \leq 2 \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &\frac{r^{s-2}}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\leq \phi_s(r, R) - \Phi_s(P||Q) \\ &\leq \frac{R^{s-2}}{2}\left[(R-1)(1-r) - \chi^2(P||Q)\right], \quad s \geq 2. \end{aligned} \tag{3.12}$$

Proof. Let us consider $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then according to expression (2.4), we have

$$\phi_s''(u) = u^{s-2}.$$

Now if $u \in [r, R] \subset (0, \infty)$, then we have

$$R^{s-2} \le \phi_s''(u) \le r^{s-2}, \quad s \le 2,$$

or accordingly, we have

$$\phi_s''(u) \begin{cases} \le r^{s-2}, & s \le 2; \\ \ge r^{s-2}, & s \ge 2 \end{cases} \tag{3.13}$$

and

$$\phi_s''(u) \begin{cases} \le R^{s-2}, & s \ge 2; \\ \ge R^{s-2}, & s \le 2, \end{cases} \tag{3.14}$$

where r and R are as defined above. Thus in view of (3.9), (3.13) and (3.14), we have the proof. □

In view of Result 3, we have the following corollary.

Corollary 3.4. *Under the conditions of Result 3, we have*

$$\begin{aligned} &\frac{1}{R^3}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\quad \le \frac{(R-1)(1-r)}{rR} - \chi^2(Q||P) \\ &\quad \le \frac{1}{r^3}\left[(R-1)(1-r) - \chi^2(P||Q)\right], \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\frac{1}{2R^2}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\quad \le \frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P) \\ &\quad \le \frac{1}{2r^2}\left[(R-1)(1-r) - \chi^2(P||Q)\right], \end{aligned} \tag{3.16}$$

$$\begin{aligned} &\frac{1}{2R}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\quad \le \frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q) \\ &\quad \le \frac{1}{2r}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} &\frac{1}{8\sqrt{R^3}}\left[(R-1)(1-r) - \chi^2(P||Q)\right] \\ &\quad \le \frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \\ &\quad \le \frac{1}{8\sqrt{r^3}}\left[(R-1)(1-r) - \chi^2(P||Q)\right]. \end{aligned} \tag{3.18}$$

Proof. (3.15) follows by taking $s = -1$, (3.16) follows by taking $s = 0$, (3.17) follows by taking $s = 1$, (3.18) follows by taking $s = \frac{1}{2}$ in Result 3. While for $s = 2$, we have equality sign. □

Proposition 3.5. *Let $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ the generating mapping be normalized, i.e., $f(1) = 0$ and satisfy the assumptions:*

(i) *f is twice differentiable on (r, R), where $0 < r \leq 1 \leq R < \infty$;*
(ii) *there exists the real constants m, Msuch that $m < M$ and*

$$m \leq x^3 f''(x) \leq M, \quad \forall x \in (r, R). \tag{3.19}$$

If $P, Q \in \Delta_n$ are discrete probability distributions satisfying the assumption

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty,$$

then we have the inequalities:

$$\begin{aligned} &\frac{m}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right] \\ &\leq \beta_f(r, R) - C_f(P||Q) \\ &\leq \frac{m}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right], \end{aligned} \tag{3.20}$$

where $C_f(P||Q)$, $\beta_f(r, R)$ and $\chi^2(Q||P)$ are as given by (2.1), (2.8) and (1.7) respectively.

As a consequence of above proposition, we have the following result.

Result 4. Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exist r, R $(0 < r \leq 1 \leq R < \infty)$ such that

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

then in view of Proposition 3.5, we have

$$\begin{aligned} &\frac{R^{s+1}}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right] \\ &\leq \phi_s(r, R) - \Phi_s(P||Q) \\ &\leq \frac{r^{s+1}}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right], \quad s \leq -1 \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} &\frac{r^{s+1}}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right] \\ &\leq \phi_s(r, R) - \Phi_s(P||Q) \\ &\leq \frac{R^{s+1}}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right], \quad s \geq -1. \end{aligned} \tag{3.22}$$

Proof. Let us consider $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then according to expression (2.4), we have

$$\phi_s''(u) = u^{s-2}.$$

Let us define the function $g : [r, R] \to \mathbb{R}$ such that $g(u) = u^3\phi_s''(u) = u^{s+1}$, then we have

$$\sup_{u\in[r,R]} g(u) = \begin{cases} R^{s+1}, & s \geq -1; \\ r^{s+1}, & s \leq -1 \end{cases} \tag{3.23}$$

and

$$\inf_{u\in[r,R]} g(u) = \begin{cases} r^{s+1}, & s \geq -1; \\ R^{s+1}, & s \leq -1. \end{cases} \tag{3.24}$$

In view of (3.23) , (3.24) and Proposition 3.5, we have the proof of the result. □

In view of Result 4, we have the following corollary.

Corollary 3.6. *Under the conditions of Result 4, we have*

$$\begin{aligned} &\frac{r}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right] \\ &\leq \frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P) \\ &\leq \frac{R}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right], \end{aligned} \tag{3.25}$$

$$\begin{aligned} &\frac{r^2}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right] \\ &\leq \frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q) \\ &\leq \frac{R^2}{2}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right], \end{aligned} \tag{3.26}$$

$$\begin{aligned} &\frac{\sqrt{r^3}}{8}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right] \\ &\leq \frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}} - h(P||Q) \\ &\leq \frac{\sqrt{R^3}}{8}\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right] \end{aligned} \tag{3.27}$$

and

$$\begin{aligned} &r^3\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right] \\ &\leq (R-1)(1-r) - \chi^2(P||Q) \\ &\leq R^3\left[\frac{(R-1)(1-r)}{rR} - \chi^2(Q||P)\right]. \end{aligned} \tag{3.28}$$

Proof. (3.25) follows by taking $s = 0$, (3.26) follows by taking $s = 1$, (3.27) follows by taking $s = \frac{1}{2}$ and (3.28) follows by taking $s = 2$ in Result 4. While for $s = -1$, we have equality sign. □

3.2. **Information Bounds in Terms of Kullback-Leibler Relative Information.** In particular for $s = 1$, in the Theorem 3.1, we have the following proposition (see also Dragomir [10]).

Proposition 3.7. *Let* $f : I \subset \mathbb{R}_+ \to \mathbb{R}$ *the generating mapping be normalized, i.e.,* $f(1) = 0$ *and satisfy the assumptions:*

(i) f *is twice differentiable on* (r, R) *, where* $0 < r \leq 1 \leq R < \infty$*;*
(ii) *there exists the real constants* m, M *with* $m < M$ *such that*

$$m \leq x f''(x) \leq M, \quad \forall x \in (r, R). \tag{3.29}$$

If $P, Q \in \Delta_n$ *are discrete probability distributions satisfying the assumption*

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty,$$

then we have the inequalities:

$$\begin{aligned} & m\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right] \\ & \leq \beta_f(r, R) - C_f(P||Q) \\ & \leq M\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right], \end{aligned} \tag{3.30}$$

where $C_f(P||Q), \beta_f(r, R)$ *and* $K(P||Q)$ *as given by (2.1), (2.8) and (1.1) respectively.*

In view of the above proposition, we have the following result.

Result 5. Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exist r, R $(0 < r \leq 1 \leq R < \infty)$ such that

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

then in view of Proposition 3.7, we have

$$\begin{aligned} & r^{s-1}\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right] \\ & \leq \phi_s(r, R) - \Phi_s(P||Q) \\ & \leq R^{s-1}\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right], \quad s \geq 1 \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} & R^{s-1}\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right] \\ & \leq \phi_s(r, R) - \Phi_s(P||Q) \\ & \leq r^{s-1}\left[\frac{(R-1)r\ln r + (1-r)R\ln R}{R-r} - K(P||Q)\right], \quad s \leq 1. \end{aligned} \tag{3.32}$$

Proof. Let us consider $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then according to expression (2.4), we have

$$\phi_s''(u) = u^{s-2}.$$

Let us define the function $g : [r, R] \to \mathbb{R}$ such that $g(u) = \phi_s''(u) = u^{s-1}$, then we have

$$\sup_{u \in [r,R]} g(u) = \begin{cases} R^{s-1}, & s \geq 1; \\ r^{s-1}, & s \leq 1 \end{cases} \tag{3.33}$$

and

$$\inf_{u\in[r,R]} g(u) = \begin{cases} r^{s-1}, & s \geq 1; \\ R^{s-1}, & s \leq 1. \end{cases} \tag{3.34}$$

In view of (3.33), (3.34) and Proposition 3.7 we have the proof of the result. □

In view of Result 5, we have the following corollary.

Corollary 3.8. *Under the conditions of Result 5, we have*

$$\begin{aligned} &\frac{2}{R^2}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right] \\ &\leq \frac{(R-1)(1-r)}{rR}-\chi^2(Q||P) \\ &\leq \frac{2}{r^2}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right], \end{aligned} \tag{3.35}$$

$$\begin{aligned} &\frac{1}{R}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right] \\ &\leq \frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P) \\ &\leq \frac{1}{r}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right], \end{aligned} \tag{3.36}$$

$$\begin{aligned} &\frac{1}{4\sqrt{R}}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right] \\ &\leq \frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q) \\ &\leq \frac{1}{4\sqrt{r}}\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right] \end{aligned} \tag{3.37}$$

and

$$\begin{aligned} &2r\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right] \\ &\leq (R-1)(1-r)-\chi^2(P||Q) \\ &\leq 2R\left[\frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)\right]. \end{aligned} \tag{3.38}$$

Proof. (3.35) follows by taking $s=-1$, (3.36) follows by taking $s=0$, (3.37) follows by taking $s=\frac{1}{2}$ and (3.38) follows by taking $s=2$ in Result 5. For $s=1$, we have equality sign. □

In particular, for $s=0$ in Theorem 3.1, we have the following proposition:

Proposition 3.9. *Let $f: I \subset \mathbb{R}_+ \to \mathbb{R}$ the generating mapping be normalized, i.e., $f(1)=0$ and satisfy the assumptions:*

(i) *f is twice differentiable on (r,R), where $0<r\leq 1\leq R<\infty$;*

(ii) *there exists the real constants* m, M *with* $m < M$ *such that*

$$m \leq x^2 f''(x) \leq M, \quad \forall x \in (r, R). \tag{3.39}$$

If $P, Q \in \Delta_n$ *are discrete probability distributions satisfying the assumption*

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty,$$

then we have the inequalities:

$$\begin{aligned} & m\left[\frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right] \\ & \leq \beta_f(r,R) - C_f(P||Q) \\ & \leq M\left[\frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right], \end{aligned} \tag{3.40}$$

where $C_f(P||Q), \beta_f(r, R)$ *and* $K(Q||P)$ *as given by (2.1), (2.8) and (1.1) respectively.*

In view of Proposition 3.9, we have the following result.

Result 6. Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exist r, R $(0 < r \leq 1 \leq R < \infty)$ such that

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

then in view of Proposition 3.9, we have

$$\begin{aligned} & r^s\left[\frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right] \\ & \leq \phi_s(r,R) - \Phi_s(P||Q) \\ & \leq R^s\left[\frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right], \quad s \geq 0 \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} & R^s\left[\frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right] \\ & \leq \phi_s(r,R) - \Phi_s(P||Q) \\ & \leq r^s\left[\frac{(R-1)\ln\frac{1}{r} + (1-r)\ln\frac{1}{R}}{R-r} - K(Q||P)\right], \quad s \leq 0. \end{aligned} \tag{3.42}$$

Proof. Let us consider $f(u) = \phi_s(u)$, where $\phi_s(u)$ is as given by (2.2), then according to expression (2.4), we have

$$\phi_s''(u) = u^{s-2}.$$

Let us define the function $g : [r, R] \to \mathbb{R}$ such that $g(u) = u^2\phi_s''(u) = u^s$, then we have

$$\sup_{u \in [r,R]} g(u) = \begin{cases} R^s, & s \geq 0; \\ r^s, & s \leq 0 \end{cases} \tag{3.43}$$

and

$$\inf_{u \in [r,R]} g(u) = \begin{cases} r^s, & s \geq 0; \\ R^s, & s \leq 0. \end{cases} \tag{3.44}$$

In view of (3.43), (3.44) and Proposition 3.9, we have the proof of the result. □

In view of Result 6, we have the following corollary.

Corollary 3.10. *Under the conditions of Result 6, we have*

$$\frac{2}{R}\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right] \leq \frac{(R-1)(1-r)}{rR}-\chi^2(Q||P) \leq \frac{2}{r}\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right], \tag{3.45}$$

$$r\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right] \leq \frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q) \leq R\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right], \tag{3.46}$$

$$\frac{\sqrt{r}}{4}\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right] \leq \frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q) \leq \frac{\sqrt{R}}{4}\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right] \tag{3.47}$$

and

$$2r^2\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right] \leq (R-1)(1-r)-\chi^2(P||Q) \leq 2R^2\left[\frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)\right]. \tag{3.48}$$

Proof. (3.45) follows by taking $s=-1$, (3.46) follows by taking $s=1$, (3.47) follows by taking $s=\frac{1}{2}$ and (3.48) follows by taking $s=2$ in Result 6. For $s=0$, we have equality sign. □

3.3. Information Bounds in Terms of Hellinger's Discrimination.

In particular, for $s=\frac{1}{2}$ in Theorem 3.1, we have the following proposition (see also Dragomir [6]).

Proposition 3.11. *Let $f: I\subset\mathbb{R}_+\to\mathbb{R}$ the generating mapping be normalized, i.e., $f(1)=0$ and satisfy the assumptions:*

(i) *f is twice differentiable on (r,R), where $0<r\leq 1\leq R<\infty$;*
(ii) *there exists the real constants m,M with $m<M$ such that*

$$m\leq x^{3/2}f''(x)\leq M,\quad \forall x\in(r,R). \tag{3.49}$$

If $P,Q\in\Delta_n$ are discrete probability distributions satisfying the assumption

$$0<r\leq\frac{p_i}{q_i}\leq R<\infty,$$

then we have the inequalities:

$$4m\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right] \leq \beta_f(r,R)-C_f(P||Q) \leq 4M\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right], \tag{3.50}$$

where $C_f(P||Q), \beta_f(r,R)$ and $h(P||Q)$ as given by (2.1), (2.8) and (1.5) respectively.

In view of Proposition 3.11, we have the following result.

Result 7. Let $P, Q \in \Delta_n$ and $s \in \mathbb{R}$. Let there exist r, R $(0 < r \leq 1 \leq R < \infty)$ such that

$$0 < r \leq \frac{p_i}{q_i} \leq R < \infty, \quad \forall i \in \{1, 2, \ldots, n\},$$

then in view of Proposition 3.11, we have

$$4r^{\frac{2s-1}{2}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right] \leq \phi_s(r,R)-\Phi_s(P||Q) \leq 4R^{\frac{2s-1}{2}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right], \quad s \geq \frac{1}{2} \tag{3.51}$$

and

$$4R^{\frac{2s-1}{2}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right] \leq \phi_s(r,R)-\Phi_s(P||Q) \leq 4r^{\frac{2s-1}{2}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right], \quad s \leq \frac{1}{2}. \tag{3.52}$$

Proof. Let the function $\phi_s(u)$ given by (3.29) be defined over $[r, R]$. Defining $g(u) = u^{3/2}\phi_s''(u) = u^{\frac{2s-1}{2}}$, obviously we have

$$\sup_{u\in[r,R]} g(u) = \begin{cases} R^{\frac{2s-1}{2}}, & s \geq \frac{1}{2}; \\ r^{\frac{2s-1}{2}}, & s \leq \frac{1}{2} \end{cases} \tag{3.53}$$

and

$$\inf_{u\in[r,R]} g(u) = \begin{cases} r^{\frac{2s-1}{2}}, & s \geq \frac{1}{2}; \\ R^{\frac{2s-1}{2}}, & s \leq \frac{1}{2}. \end{cases} \tag{3.54}$$

In view of (3.53), (3.54) and Proposition 3.11, we get the proof of the result. □

In view of Result 7, we have the following corollary.

Corollary 3.12. *Under the conditions of Result 7, we have*

$$\frac{8}{\sqrt{R^3}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right] \tag{3.55}$$
$$\leq \frac{(R-1)(1-r)}{rR}-\chi^2(Q||P)$$
$$\leq \frac{8}{\sqrt{r^3}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right],$$

$$\frac{4}{\sqrt{R}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right] \tag{3.56}$$
$$\leq \frac{(R-1)\ln\frac{1}{r}+(1-r)\ln\frac{1}{R}}{R-r}-K(Q||P)$$
$$\leq \frac{4}{\sqrt{r}}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right],$$

$$4\sqrt{r}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right] \tag{3.57}$$
$$\leq \frac{(R-1)r\ln r+(1-r)R\ln R}{R-r}-K(P||Q)$$
$$\leq 4\sqrt{R}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right]$$

and

$$8\sqrt{r^3}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right] \tag{3.58}$$
$$\leq (R-1)(1-r)-\chi^2(P||Q)$$
$$\leq 8\sqrt{R^3}\left[\frac{\left(\sqrt{R}-1\right)(1-\sqrt{r})}{\sqrt{R}+\sqrt{r}}-h(P||Q)\right].$$

Proof. (3.55) follows by taking $s = -1$, (3.56) follows by taking $s = 0$, (3.57) follows by taking $s = 1$ and (3.58) follows by taking $s = 2$ in Result 7. For $s = \frac{1}{2}$, we have equality sign. □

REFERENCES

[1] A. BHATTACHARYYA, Some analogues to the amount of information and their uses in statistical estimation, *Sankhya*, **8** (1946), 1–14.

[2] P.S. BULLEN, D.S. MITRINOVIĆ AND P.M. VASIĆ, *Means and Their Inequalities*, Kluwer Academic Publishers, 1988.

[3] P. CERONE, S.S. DRAGOMIR AND F. ÖSTERREICHER, Bounds on extended f-divergences for a variety of classes, *RGMIA Research Report Collection*, **6**(1) (2003), Article 7.

[4] I. CSISZÁR, Information type measures of differences of probability distribution and indirect observations, *Studia Math. Hungarica*, **2** (1967), 299–318.

[5] I. CSISZÁR AND J. KÖRNER, *Information Theory: Coding Theorems for Discrete Memoryless Systems*, Academic Press, New York, 1981.

[6] S.S. DRAGOMIR, Upper and lower bounds for Csiszár $f-$divergence in terms of Hellinger discrimination and applications, *Nonlinear Analysis Forum*, **7**(1) (2002), 1–13

[7] S.S. DRAGOMIR, Some inequalities for the Csiszár Φ-divergence - *Inequalities for Csiszár f-Divergence in Information Theory*, `http://rgmia.vu.edu.au/monographs/csiszar.htm`

[8] S.S. DRAGOMIR, A converse inequality for the Csiszár Φ-Divergence- *Inequalities for Csiszár f-Divergence in Information Theory*, `http://rgmia.vu.edu.au/monographs/csiszar.htm`

[9] S.S. DRAGOMIR, Other inequalities for Csiszár divergence and applications - *Inequalities for Csiszár f-Divergence in Information Theory*, `http://rgmia.vu.edu.au/monographs/csiszar.htm`

[10] S.S. DRAGOMIR, Upper and lower bounds Csiszár $f-$divergence in terms of Kullback-Leibler distance and applications - *Inequalities for Csiszár f-Divergence in Information Theory*, `http://rgmia.vu.edu.au/monographs/csiszar.htm`

[11] E. HELLINGER, Neue Begründung der Theorie der quadratischen Formen von unendlichen vielen Veränderlichen, *J. Reine Aug. Math.*, **136** (1909), 210–271.

[12] H. JEFFREYS, An invariant form for the prior probability in estimation problems, *Proc. Roy. Soc. Lon., Ser. A*, **186** (1946), 453–461.

[13] S. KULLBACK AND R.A. LEIBLER, On information and sufficiency, *Ann. Math. Statist.*, **22** (1951), 79–86.

[14] L. LECAM, *Asymptotic Methods in Statistical Decision Theory*, New York, Springer, 1978.

[15] F. LIESE AND I. VAJDA, *Convex Statistical Decision Rules*, Teubner-Texte zur Mathematick, Band 95, Leipzig, 1987.

[16] K. PEARSON, On the criterion that a given system of deviations from the probable in the case of correlated system of variables is such that it can be reasonable supposed to have arisen from random sampling, *Phil. Mag.*, **50** (1900), 157–172.

[17] F. ÖSTERREICHER, Csiszár's f-Divergence – Basic Properties – pre-print, 2002. `http://rgmia.vu.edu.au`

[18] A. RÉNYI, On measures of entropy and information, *Proc. 4th Berk. Symp. Math. Statist. and Probl.*, University of California Press, **Vol. 1** (1961), 547–461.

[19] R. SIBSON, Information radius, *Z. Wahrs. und verw Geb.*, **14** (1969), 149–160.

[20] I.J. TANEJA, New developments in generalized information measures, Chapter in: *Advances in Imaging and Electron Physics*, Ed. P.W. Hawkes, **91**(1995), 37-135.

[21] I.J. TANEJA, *Generalized Information Measures and their Applications*, 2001, [ONLINE `http://www.mtm.ufsc.br/~taneja/book/book.html`.]

[22] I. VAJDA, *Theory of Statistical Inference and Information*, Kluwer Academic Press, London, 1989.