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MEROMORPHIC FUNCTION THAT SHARES ONE SMALL FUNCTION WITH ITS DERIVATIVE

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ABSTRACT. In this paper we study the problem of meromorphic function sharing one small function with its derivative and improve the results of K.-W. Yu and I. Lahiri and answer the open questions posed by K.-W. Yu.

Key words and phrases: Meromorphic function; Shared value; Small function.

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1. INTRODUCTION AND MAIN RESULTS

By a meromorphic function we shall always mean a function that is meromorphic in the open complex plane $\mathbb{C}$. It is assumed that the reader is familiar with the notations of Nevanlinna theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $S(r, f)$ and so on, that can be found, for instance, in [2], [5].

Let f and g be two non-constant meromorphic functions, $a \in \mathbb{C} \cup \{\infty\}$, we say that f and g share the value a **IM** (ignoring multiplicities) if $f - a$ and $g - a$ have the same zeros, they share the value a **CM** (counting multiplicities) if $f - a$ and $g - a$ have the same zeros with the same multiplicities. When $a = \infty$ the zeros of $f - a$ means the poles of f (see [5]).

Let l be a non-negative integer or infinite. For any $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_l(a, f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq l$ and $l + 1$ times if $m > l$. If $E_l(a, f) = E_l(a, g)$, we say f and g share the value a with weight l (see [3], [4]).

f and g share a value a with weight l means that z_0 is a zero of $f - a$ with multiplicity $m(\leq l)$ if and only if it is a zero of $g - a$ with the multiplicity $m(\leq l)$, and z_0 is a zero of $f - a$ with multiplicity $m(> l)$ if and only if it is a zero of $g - a$ with the multiplicity $n(> l)$, where m is not necessarily equal to n.

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We write f and g share (a, l) to mean that f and g share the value a with weight l. Clearly, if f and g share (a, l), then f and g share (a, p) for all integers p, $0 \le p \le l$. Also we note that f and g share a value a IM or CM if and only if f and g share $(a, 0)$ or (a, ∞) respectively (see [3], [4]).

A function $a(z)$ is said to be a small function of f if $a(z)$ is a meromorphic function satisfying $T(r, a) = S(r, f)$, i.e. $T(r, a) = o(T(r, f))$ as $r \to +\infty$ possibly outside a set of finite linear measure. Similarly, we define that f and g share a small function a IM or CM or with weight l by $f - a$ and $g - a$ sharing the value 0 IM or CM or with weight l respectively.

Brück [1] first considered the uniqueness problems of an entire function sharing one value with its derivative and proved the following result.

Theorem A. *Let f be an entire function which is not constant. If f and f' share the value 1 CM and if $N\left(r, \frac{1}{f'}\right) = S(r, f)$, then $\frac{f'-1}{f-1} \equiv c$ for some constant $c \in \mathbb{C}\backslash\{0\}$.*

Brück [1] further posed the following conjecture.

Conjecture 1.1. *Let f be an entire function which is not constant, $\rho_1(f)$ be the first iterated order of f. If $\rho_1(f) < +\infty$ and $\rho_1(f)$ is not a positive integer, and if f and f' share one value a CM, then $\frac{f'-a}{f-a} \equiv c$ for some constant $c \in \mathbb{C}\backslash\{0\}$.*

Yang [7] proved that the conjecture is true if f is an entire function of finite order. Zhang [9] extended Theorem A to meromorphic functions. Yu [8] recently considered the problem of an entire or meromorphic function sharing one small function with its derivative and proved the following two theorems.

Theorem B ([8])**.** *Let f be a non-constant entire function and $a \equiv a(z)$ be a meromorphic function such that $a \not\equiv 0, \infty$ and $T(r, a) = o(T(r, f)$ as $r \to +\infty$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0, f) > \frac{3}{4}$, then $f \equiv f^{(k)}$.*

Theorem C ([8])**.** *Let f be a non-constant, non-entire meromorphic function and $a \equiv a(z)$ be a meromorphic function such that $a \not\equiv 0, \infty$ and $T(r, a) = o(T(r, f)$ as $r \to +\infty$. If*

(i) *f and a have no common poles,*
(ii) *$f - a$ and $f^{(k)} - a$ share the value 0 CM,*
(iii) *$4\delta(0, f) + 2\Theta(\infty, f) > 19 + 2k$,*

then $f \equiv f^{(k)}$, where k is a positive integer.

In the same paper Yu [8] further posed the following open questions:

(1) Can a CM shared be replaced by an IM shared value?
(2) Can the condition $\delta(0, f) > \frac{3}{4}$ of Theorem B be further relaxed?
(3) Can the condition (iii) of Theorem C be further relaxed?
(4) Can, in general, the condition (i) of Theorem C be dropped?

Let p be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. We use $N_{p)}\left(r, \frac{1}{f}\right)$ to denote the counting function of the zeros of $f - a$ (counted with proper multiplicities) whose multiplicities are not greater than p, $N_{(p+1}\left(r, \frac{1}{f}\right)$ to denote the counting function of the zeros of $f - a$ whose multiplicities are not less than $p + 1$. And $\overline{N}_{p)}\left(r, \frac{1}{f}\right)$ and $\overline{N}_{(p+1}\left(r, \frac{1}{f}\right)$ denote their corresponding reduced counting functions (ignoring multiplicities) respectively. We also use $N_p\left(r, \frac{1}{f}\right)$ to denote the counting function of the zeros of $f - a$ where a zero of multiplicity m is counted m

times if $m \leq p$ and p times if $m > p$. Clearly $N_1\left(r, \frac{1}{f}\right) = \overline{N}\left(r, \frac{1}{f}\right)$. Define

$$\delta_p(a, f) = 1 - \limsup_{r \to +\infty} \frac{N_p\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

Obviously $\delta_p(a, f) \geq \delta(a, f)$.

Lahiri [4] improved the results of Zhang [9] with weighted shared value and obtained the following two theorems

Theorem D ([4])**.** *Let f be a non-constant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share $(1, 2)$ and*

$$2\overline{N}(r, f) + N_2\left(r, \frac{1}{f^{(k)}}\right) + N_2\left(r, \frac{1}{f}\right) < (\lambda + o(1))T(r, f^{(k)})$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-1}{f-1} \equiv c$ for some constant $c \in \mathbb{C}\backslash\{0\}$.

Theorem E ([4])**.** *Let f be a non-constant meromorphic function and k be a positive integer. If f and $f^{(k)}$ share $(1, 1)$ and*

$$2\overline{N}(r, f) + N_2\left(r, \frac{1}{f^{(k)}}\right) + 2\overline{N}\left(r, \frac{1}{f}\right) < (\lambda + o(1))T(r, f^{(k)})$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-1}{f-1} \equiv c$ for some constant $c \in \mathbb{C}\backslash\{0\}$.

In the same paper Lahiri [4] also obtained the following result which is an improvement of Theorem C.

Theorem F ([4])**.** *Let f be a non-constant meromorphic function and k be a positive integer. Also, let $a \equiv a(z)(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. If*

(i) *a has no zero (pole) which is also a zero (pole) of f or $f^{(k)}$ with the same multiplicity.*
(ii) *$f - a$ and $f^{(k)} - a$ share $(0, 2)$ CM,*
(iii) *$2\delta_{2+k}(0, f) + (4 + k)\Theta(\infty, f) > 5 + k$,*

then $f \equiv f^{(k)}$.

In this paper, we still study the problem of a meromorphic or entire function sharing one small function with its derivative and obtain the following two results which are the improvement and complement of the results of Yu [8] and Lahiri [4] and answer the four open questions of Yu in [8].

Theorem 1.2. *Let f be a non-constant meromorphic function and $k(\geq 1)$, $l(\geq 0)$ be integers. Also, let $a \equiv a(z)$ $(\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If $l \geq 2$ and*

$$2\overline{N}(r, f) + N_2\left(r, \frac{1}{f^{(k)}}\right) + N_2\left(r, \frac{1}{(f/a)'}\right) < (\lambda + o(1))T(r, f^{(k)}), \tag{1.1}$$

or $l = 1$ and

$$2\overline{N}(r, f) + N_2\left(r, \frac{1}{f^{(k)}}\right) + 2\overline{N}\left(r, \frac{1}{(f/a)'}\right) < (\lambda + o(1))T(r, f^{(k)}), \tag{1.2}$$

or $l = 0$, i.e. $f - a$ and $f^{(k)} - a$ share the value 0 IM and

$$4\overline{N}(r, f) + 3N_2\left(r, \frac{1}{f^{(k)}}\right) + 2\overline{N}\left(r, \frac{1}{(f/a)'}\right) < (\lambda + o(1))T(r, f^{(k)}), \tag{1.3}$$

for $r \in I$, where $0 < \lambda < 1$ and I is a set of infinite linear measure, then $\frac{f^{(k)}-a}{f-a} \equiv c$ for some constant $c \in \mathbb{C}\backslash\{0\}$.

Theorem 1.3. *Let f be a non-constant meromorphic function and $k(\geq 1),\ l(\geq 0)$ be integers. Also, let $a \equiv a(z)\ (\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. Suppose that $f - a$ and $f^{(k)} - a$ share $(0, l)$. If $l \geq 2$ and*

$$(3 + k)\Theta(\infty, f) + 2\delta_{2+k}(0, f) > k + 4, \tag{1.4}$$

or $l = 1$ and

$$(4 + k)\Theta(\infty, f) + 3\delta_{2+k}(0, f) > k + 6, \tag{1.5}$$

or $l = 0$, i.e. $f - a$ and $f^{(k)} - a$ share the value 0 IM and

$$(6 + 2k)\Theta(\infty, f) + 5\delta_{2+k}(0, f) > 2k + 10, \tag{1.6}$$

then $f \equiv f^{(k)}$.

Clearly Theorem 1.2 extends the results of Lahiri (Theorem D and E) to small functions. Theorem 1.3 gives the improvements of Theorem C and F, which removes the restrictions on the zeros (poles) of $a(z)$ and $f(z)$ and relaxes other conditions, which also includes a result of meromorphic function sharing one value or small function IM with its derivative, so it answers the four open questions of Yu [8].

From Theorem 1.2 we have the following corollary which is the improvement of Theorem A.

Corollary 1.4. *Let f be an entire function which is not constant. If f and f' share the value 1 IM and if $N\left(r, \frac{1}{f}\right) = S(r, f)$, then $\frac{f'-1}{f-1} \equiv c$ for some constant $c \in \mathbb{C}\backslash\{0\}$.*

From Theorem 1.3 we have

Corollary 1.5. *Let f be a non-constant entire function and $a \equiv a(z)\ (\not\equiv 0, \infty)$ be a meromorphic function such that $T(r, a) = S(r, f)$. If $f - a$ and $f^{(k)} - a$ share the value 0 CM and $\delta(0, f) > \frac{1}{2}$, or if $f - a$ and $f^{(k)} - a$ share the value 0 IM and $\delta(0, f) > \frac{4}{5}$, then $f \equiv f^{(k)}$.*

Clearly Corollary 1.5 is an improvement and complement of Theorem B.

2. MAIN LEMMAS

Lemma 2.1 (see [4]). *Let f be a non-constant meromorphic function, k be a positive integer, then*

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

This lemma can be obtained immediately from the proof of Lemma 2.3 in [4] which is the special case $p = 2$.

Lemma 2.2 (see [5]). *Let f be a non-constant meromorphic function, n be a positive integer. $P(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f$ where a_i is a meromorphic function such that $T(r, a_i) = S(r, f)\ \ (i = 1, 2, \ldots, n)$. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

3. PROOF OF THEOREM 1.2

Let $F = \frac{f}{a}$, $G = \frac{f^{(k)}}{a}$, then $F - 1 = \frac{f-a}{a}$, $G - 1 = \frac{f^{(k)}-a}{a}$. Since $f - a$ and $f^{(k)} - a$ share $(0, l)$, F and G share $(1, l)$ except the zeros and poles of $a(z)$. Define

(3.1) $$H = \left(\frac{F''}{F'} - 2\frac{F'}{F-1}\right) - \left(\frac{G''}{G'} - 2\frac{G'}{G-1}\right),$$

We have the following two cases to investigate.

Case 1. $H \equiv 0$. Integration yields

(3.2) $$\frac{1}{F-1} \equiv C\frac{1}{G-1} + D,$$

where C and D are constants and $C \neq 0$. If there exists a pole z_0 of f with multiplicity p which is not the pole and zero of $a(z)$, then z_0 is the pole of F with multiplicity p and the pole of G with multiplicity $p + k$. This contradicts with (3.2). So

(3.3) $$\overline{N}(r, f) \leq \overline{N}(r, a) + \overline{N}\left(r, \frac{1}{a}\right) = S(r, f),$$
$$\overline{N}(r, F) = S(r, f), \qquad \overline{N}(r, G) = S(r, f).$$

(3.2) also shows F and G share the value 1 CM. Next we prove $D = 0$. We first assume that $D \neq 0$, then

(3.4) $$\frac{1}{F-1} \equiv \frac{D\left(G - 1 + \frac{C}{D}\right)}{G-1}.$$

So

(3.5) $$\overline{N}\left(r, \frac{1}{G - 1 + \frac{C}{D}}\right) = \overline{N}(r, F) = S(r, f).$$

If $\frac{C}{D} \neq 1$, by the second fundamental theorem and (3.3), (3.5) and $S(r, G) = S(r, f)$, we have

$$T(r, G) \leq \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right) + \overline{N}\left(r, \frac{1}{G - 1 + \frac{C}{D}}\right) + S(r, G)$$
$$\leq \overline{N}\left(r, \frac{1}{G}\right) + S(r, f) \leq T(r, G) + S(r, f).$$

So

(3.6) $$T(r, G) = \overline{N}\left(r, \frac{1}{G}\right) + S(r, f),$$

i.e.

$$T(r, f^{(k)}) = \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r, f),$$

this contradicts with conditions (1.1), (1.2) and (1.3) of this theorem.

If $\frac{C}{D} = 1$, from (3.4) we know

$$\frac{1}{F-1} \equiv C\frac{G}{G-1},$$

then

$$\left(F - 1 - \frac{1}{C}\right)G \equiv -\frac{1}{C}.$$

Noticing that $F = \frac{f}{a}$, $G = \frac{f^{(k)}}{a}$, we have

$$\frac{1}{f(f-(1+\frac{1}{C})a)} \equiv -\frac{C}{a^2}\cdot\frac{f^{(k)}}{f}. \tag{3.7}$$

By Lemma 2.2 and (3.3) and (3.7), then

$$\begin{aligned} 2T(r,f) &= T\left(r, f\left(f-\left(1+\frac{1}{C}\right)a\right)\right) + S(r,f) \\ &= T\left(r, \frac{1}{f(f-(1+\frac{1}{C})a)}\right) + S(r,f) \\ &= T\left(r, \frac{f^{(k)}}{f}\right) + S(r,f) \\ &\le N\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f) \\ &\le T(r,f) + S(r,f). \end{aligned} \tag{3.8}$$

So $T(r,f) = S(r,f)$, this is impossible. Hence $D = 0$, and $\frac{G-1}{F-1} \equiv C$, i.e. $\frac{f^{(k)}-a}{f-a} \equiv C$. This is just the conclusion of this theorem.

Case 2. $H \not\equiv 0$. From (3.1) it is easy to see that $m(r,H) = S(r,f)$.

Subcase 2.1 $l \ge 1$. From (3.1) we have

$$\begin{aligned} N(r,H) \le{}& \overline{N}(r,F) + \overline{N}_{(l+1}\left(r, \frac{1}{F-1}\right) + \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) \\ &+ \overline{N}_0\left(r, \frac{1}{F'}\right) + \overline{N}_0\left(r, \frac{1}{G'}\right) + \overline{N}(r,a) + \overline{N}\left(r, \frac{1}{a}\right), \end{aligned} \tag{3.9}$$

where $N_0\left(r, \frac{1}{F'}\right)$ denotes the counting function of the zeros of F' which are not the zeros of F and $F-1$, and $\overline{N}_0\left(r, \frac{1}{F'}\right)$ denotes its reduced form. In the same way, we can define $N_0\left(r, \frac{1}{G'}\right)$ and $\overline{N}_0\left(r, \frac{1}{G'}\right)$. Let z_0 be a simple zero of $F-1$ but $a(z_0) \ne 0, \infty$, then z_0 is also the simple zero of $G-1$. By calculating z_0 is the zero of H, so

$$N_{1)}\left(r, \frac{1}{F-1}\right) \le N\left(r, \frac{1}{H}\right) + N(r,a) + N\left(r, \frac{1}{a}\right) \le N(r,H) + S(r,f). \tag{3.10}$$

Noticing that $N_{1)}\left(r, \frac{1}{G}\right) = N_{1)}\left(r, \frac{1}{F}\right) + S(r,f)$, we have

$$\begin{aligned} \overline{N}\left(r, \frac{1}{G-1}\right) &= N_{1)}\left(r, \frac{1}{F-1}\right) + \overline{N}_{(2}\left(r, \frac{1}{F-1}\right) \\ &\le \overline{N}(r,F) + \overline{N}_{(l+1}\left(r, \frac{1}{F-1}\right) + \overline{N}_{(2}\left(r, \frac{1}{F-1}\right) \\ &\quad + \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) \\ &\quad + \overline{N}_0\left(r, \frac{1}{F'}\right) + \overline{N}_0\left(r, \frac{1}{G'}\right) + S(r,f). \end{aligned} \tag{3.11}$$

By the second fundamental theorem and (3.11) and noting $\overline{N}(r,F)=\overline{N}(r,G)+S(r,f)$, then

$$(3.12)\qquad T(r,G)\le \overline{N}(r,G)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,G)$$
$$\le 2\overline{N}(r,F)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)$$
$$+\overline{N}_{(l+1}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)+S(r,f).$$

While $l\ge 2$,

$$(3.13)\quad \overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(l+1}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)\le N_2\left(r,\frac{1}{F'}\right),$$

so

$$T(r,G)\le 2\overline{N}(r,F)+N_2\left(r,\frac{1}{G}\right)+N_2\left(r,\frac{1}{F'}\right)+S(r,f),$$

i.e.

$$T(r,f^{(k)})\le 2\overline{N}(r,f)+N_2\left(r,\frac{1}{f^{(k)}}\right)+N_2\left(r,\frac{1}{(f/a)'}\right)+S(r,f).$$

This contradicts with (1.1).

While $l=1$, (3.13) turns into

$$\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(l+1}\left(r,\frac{1}{F-1}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)\le 2\overline{N}\left(r,\frac{1}{F'}\right).$$

Similarly as above, we have

$$T(r,f^{(k)})\le 2\overline{N}(r,f)+N_2\left(r,\frac{1}{f^{(k)}}\right)+2\overline{N}\left(r,\frac{1}{(f/a)'}\right)+S(r,f).$$

This contradicts with (1.2).

Subcase 2.2 $l=0$. In this case, F and G share 1 IM except the zeros and poles of $a(z)$.

Let z_0 be the zero of $F-1$ with multiplicity p and the zero of $G-1$ with multiplicity q. We denote by $N_E^{1)}\left(r,\frac{1}{F}\right)$ the counting function of the zeros of $F-1$ where $p=q=1$; by $\overline{N}_E^{(2}\left(r,\frac{1}{F}\right)$ the counting function of the zeros of $F-1$ where $p=q\ge 2$; by $\overline{N}_L\left(r,\frac{1}{F}\right)$ the counting function of the zeros of $F-1$ where $p>q\ge 1$, each point in these counting functions is counted only once. In the same way, we can define $N_E^{1)}\left(r,\frac{1}{G}\right)$, $\overline{N}_E^{(2}\left(r,\frac{1}{G}\right)$ and $\overline{N}_L\left(r,\frac{1}{G}\right)$. It is easy to see that

$$N_E^{1)}\left(r,\frac{1}{F-1}\right)=N_E^{1)}\left(r,\frac{1}{G-1}\right)+S(r,f),$$
$$\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)=\overline{N}_E^{(2}\left(r,\frac{1}{G-1}\right)+S(r,f),$$

$$(3.14)\qquad \overline{N}\left(r,\frac{1}{F-1}\right)=\overline{N}\left(r,\frac{1}{G-1}\right)+S(r,f)$$
$$=N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)$$
$$+\overline{N}_L\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+S(r,f).$$

From (3.1) we have now

$$N(r,H) \le \overline{N}(r,F) + \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + \overline{N}_{L}\left(r,\frac{1}{F-1}\right) + \overline{N}_{L}\left(r,\frac{1}{G-1}\right) + \overline{N}_{0}\left(r,\frac{1}{F'}\right) + \overline{N}_{0}\left(r,\frac{1}{G'}\right) + S(r,f). \tag{3.15}$$

In this case, (3.10) is replaced by

$$N_E^{1)}\left(r,\frac{1}{F-1}\right) \le N(r,H) + S(r,f). \tag{3.16}$$

From (3.14), (3.15) and (3.16), we have

$$\begin{aligned}
\overline{N}\left(r,\frac{1}{G-1}\right) &\le \overline{N}(r,F) + \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + \overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right) \\
&\quad + 2\overline{N}_L\left(r,\frac{1}{F-1}\right) + 2\overline{N}_L\left(r,\frac{1}{G-1}\right) \\
&\quad + \overline{N}_0\left(r,\frac{1}{F'}\right) + \overline{N}_0\left(r,\frac{1}{G'}\right) + S(r,f) \\
&\le \overline{N}(r,F) + 2\overline{N}\left(r,\frac{1}{F'}\right) + 2\overline{N}_L\left(r,\frac{1}{G-1}\right) \\
&\quad + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + \overline{N}_0\left(r,\frac{1}{G'}\right) + S(r,f).
\end{aligned}$$

By the second fundamental theorem, then

$$\begin{aligned}
T(r,G) &\le \overline{N}(r,G) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) - N_0\left(r,\frac{1}{G'}\right) + S(r,G) \\
&\le 2\overline{N}(r,G) + 2\overline{N}\left(r,\frac{1}{F'}\right) + \overline{N}\left(r,\frac{1}{G}\right) \\
&\quad + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + 2\overline{N}_L\left(r,\frac{1}{G-1}\right) + S(r,f) \\
&\le 2\overline{N}(r,G) + 2\overline{N}\left(r,\frac{1}{F'}\right) + \overline{N}\left(r,\frac{1}{G}\right) + 2\overline{N}\left(r,\frac{1}{G'}\right) + S(r,f).
\end{aligned}$$

From Lemma 2.1 for $p=1, k=1$ we know

$$\overline{N}\left(r,\frac{1}{G'}\right) \le N_2\left(r,\frac{1}{G}\right) + \overline{N}(r,G) + S(r,G).$$

So

$$T(r,G) \le 4\overline{N}(r,F) + 3N_2\left(r,\frac{1}{G}\right) + 2\overline{N}\left(r,\frac{1}{F'}\right) + S(r,f),$$

i.e.

$$T(r,f^{(k)}) \le 4\overline{N}(r,f) + 3N_2\left(r,\frac{1}{f^{(k)}}\right) + 2\overline{N}\left(r,\frac{1}{(f/a)'}\right) + S(r,f).$$

This contradicts with (1.3). The proof is complete.

4. PROOF OF THEOREM 1.3

The proof is similar to that of Theorem 1.2. We define F and G and (3.1) as above, and we also distinguish two cases to discuss.

Case 3. $H \equiv 0$. We also have (3.2). From (3.3) we know that $\Theta(\infty, f) = 1$, and from (1.4), (1.5) and (1.6), we further know $\delta_{2+k}(0, f) > \frac{1}{2}$. Assume that $D \neq 0$, then

$$-\frac{D\left(F-1-\frac{1}{D}\right)}{F-1} \equiv C\frac{1}{G-1},$$

so

$$\overline{N}\left(r, \frac{1}{F-1-\frac{1}{D}}\right) = \overline{N}(r, G) = S(r, f).$$

If $D \neq -1$, using the second fundamental theorem for F, similarly as (3.6) we have

$$T(r, F) = \overline{N}\left(r, \frac{1}{F}\right) + S(r, f),$$

i.e.

$$T(r, f) = \overline{N}\left(r, \frac{1}{f}\right) + S(r, f).$$

Hence $\Theta(0, f) = 0$, this contradicts with $\Theta(0, f) \geq \delta_{2+k}(0, f) > \frac{1}{2}$.

If $D = -1$, then $\overline{N}\left(r, \frac{1}{F}\right) = S(r, f)$, i.e. $\overline{N}\left(r, \frac{1}{f}\right) = S(r, f)$, and

$$\frac{F}{F-1} \equiv C\frac{1}{G-1}.$$

Then

$$F(G-1-C) \equiv -C$$

and thus,

$$f^{(k)}\left(f^{(k)} - (1+C)a\right) \equiv -Ca^2\frac{f^{(k)}}{f}. \tag{4.1}$$

As same as (3.8), by Lemma 2.2 and (3.3) and $\overline{N}\left(r, \frac{1}{f}\right) = S(r, f)$, from (4.1) we have

$$\begin{aligned} 2T(r, f^{(k)}) &= T\left(r, \frac{f^{(k)}}{f}\right) + S(r, f) \\ &= N\left(r, \frac{f^{(k)}}{f}\right) + S(r, f) \\ &\leq k\overline{N}(r, f) + k\overline{N}\left(r, \frac{1}{f}\right) + S(r, f) = S(r, f). \end{aligned}$$

So $T(r, f^{(k)}) = S(r, f)$ and $T\left(r, \frac{f^{(k)}}{f}\right) = S(r, f)$. Hence

$$\begin{aligned} T(r, f) &\leq T\left(r, \frac{f}{f^{(k)}}\right) + T(r, f^{(k)}) + O(1) \\ &= T\left(r, \frac{f^{(k)}}{f}\right) + T(r, f^{(k)}) + O(1) = S(r, f), \end{aligned}$$

this is impossible. Therefore $D = 0$, and from (3.2) then

$$G - 1 \equiv \frac{1}{C}(F-1).$$

If $C \neq 1$, then

$$G \equiv \frac{1}{C}(F - 1 + C),$$

and

$$N\left(r, \frac{1}{G}\right) = N\left(r, \frac{1}{F - 1 + C}\right).$$

By the second fundamental theorem and (3.3) we have

$$\begin{aligned} T(r, F) &\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - 1 + C}\right) + S(r, G) \\ &\leq \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{G}\right) + S(r, f). \end{aligned}$$

By Lemma 2.1 for $p = 1$ and (3.3), we have

$$\begin{aligned} T(r, f) &\leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r, f) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + N_{1+k}\left(r, \frac{1}{f}\right) + \overline{N}(r, f) + S(r, f) \\ &\leq 2N_{1+k}\left(r, \frac{1}{f}\right) + S(r, f). \end{aligned}$$

Hence $\delta_{1+k}(0, f) \leq \frac{1}{2}$. This is a contradiction with $\delta_{1+k}(0, f) \geq \delta_{2+k}(0, f) > \frac{1}{2}$. So $C = 1$ and $F \equiv G$, i.e. $f \equiv f^{(k)}$. This is just the conclusion of this theorem.

Case 4. $H \not\equiv 0$.

Subcase 4.1 $l \geq 1$. As similar as Subcase 2.1, From (3.9) and (3.10) we have

$$\begin{aligned} &\overline{N}\left(r, \frac{1}{F - 1}\right) + \overline{N}\left(r, \frac{1}{G - 1}\right) \\ &\qquad = N_{1)}\left(r, \frac{1}{F - 1}\right) + \overline{N}_{(2}\left(r, \frac{1}{F - 1}\right) + \overline{N}\left(r, \frac{1}{G - 1}\right) \\ &\qquad \leq \overline{N}(r, F) + \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) \\ &\qquad\qquad + \overline{N}_{(l+1}\left(r, \frac{1}{G - 1}\right) + \overline{N}_{(2}\left(r, \frac{1}{G - 1}\right) \\ &\qquad\qquad + \overline{N}\left(r, \frac{1}{G - 1}\right) + \overline{N}_0\left(r, \frac{1}{F'}\right) + \overline{N}_0\left(r, \frac{1}{G'}\right) + S(r, f). \end{aligned}$$

While $l \geq 2$,

$$\begin{aligned} \overline{N}_{(l+1}\left(r, \frac{1}{G - 1}\right) + \overline{N}_{(2}\left(r, \frac{1}{G - 1}\right) + \overline{N}\left(r, \frac{1}{G - 1}\right) &\leq N\left(r, \frac{1}{G - 1}\right) \\ &\leq T(r, G) + O(1), \end{aligned}$$

so

$$\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)$$
$$\leq \overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)$$
$$+\overline{N}_0\left(r,\frac{1}{F'}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)+T(r,G)+S(r,f).$$

By the second fundamental theorem, we have

$$T(r,F)+T(r,G)$$
$$\leq \overline{N}(r,F)+\overline{N}(r,G)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F-1}\right)$$
$$+\overline{N}\left(r,\frac{1}{G-1}\right)-N_0\left(r,\frac{1}{F'}\right)-N_0\left(r,\frac{1}{G'}\right)+S(r,F)+S(r,G)$$
$$\leq 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+T(r,G)+S(r,f),$$

so

$$T(r,F)\leq 3\overline{N}(r,F)+N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+S(r,f),$$

i.e.

$$T(r,f)\leq 3\overline{N}(r,f)+N_2\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{f^{(k)}}\right)+S(r,f).$$

By Lemma 2.1 for $p=2$ we have

$$T(r,f)\leq (3+k)\overline{N}(r,f)+2N_{2+k}\left(r,\frac{1}{f}\right)+S(r,f),$$

so

$$(3+k)\Theta(\infty,f)+2\delta_{2+k}(0,f)\leq k+4.$$

This contradicts with (1.4).

While $l=1$,

$$\overline{N}_{(l+1}\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\leq N\left(r,\frac{1}{G-1}\right)\leq T(r,G)+O(1),$$

so by Lemma 2.1 for $p=1,k=1$, we have

$$\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)$$
$$\leq \overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}_{(2}\left(r,\frac{1}{F-1}\right)$$
$$+\overline{N}_0\left(r,\frac{1}{F'}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)+T(r,G)+S(r,f)$$
$$\leq \overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\overline{N}\left(r,\frac{1}{F'}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)+T(r,G)+S(r,f)$$
$$\leq 2\overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+N_2\left(r,\frac{1}{F}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)+T(r,G)+S(r,f)$$

As same as above, by the second fundamental theorem we have

$$T(r,F)+T(r,G)\le 4\overline{N}(r,F)+2N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+T(r,G)+S(r,f),$$

so

$$T(r,F)\le 4\overline{N}(r,F)+2N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+S(r,f),$$

i.e.

$$T(r,f)\le 4\overline{N}(r,f)+2N_2\left(r,\frac{1}{f}\right)+N_2\left(r,\frac{1}{f^{(k)}}\right)+S(r,f).$$

By Lemma 2.1 for $p=2$ we have

$$T(r,f)\le (4+k)\overline{N}(r,f)+3N_{2+k}\left(r,\frac{1}{f}\right)+S(r,f),$$

so

$$(4+k)\Theta(\infty,f)+3\delta_{2+k}(0,f)\le k+6.$$

This contradicts with (1.5).

Subcase 4.2 $l=0$. From (3.14), (3.15) and (3.16) and Lemma 2.1 for $p=1,k=1$, noticing

$$\begin{aligned}\overline{N}_E^{(2}\left(r,\frac{1}{G-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)&\le N\left(r,\frac{1}{G-1}\right)\\&\le T(r,G)+S(r,f),\end{aligned}$$

then

$$\begin{aligned}&\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\&=N_E^{1)}\left(r,\frac{1}{F-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)\\&\quad+\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\&\le \overline{N}(r,F)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+2\overline{N}_L\left(r,\frac{1}{F-1}\right)\\&\quad+\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{G-1}\right)+\overline{N}_L\left(r,\frac{1}{G-1}\right)\\&\quad+\overline{N}\left(r,\frac{1}{G-1}\right)+\overline{N}_0\left(r,\frac{1}{F'}\right)+\overline{N}_0\left(r,\frac{1}{G'}\right)+S(r,f)\\&\le \overline{N}(r,F)+2\overline{N}\left(r,\frac{1}{F'}\right)+\overline{N}\left(r,\frac{1}{G'}\right)+T(r,G)+S(r,f)\\&\le 4\overline{N}(r,F)+2N_2\left(r,\frac{1}{F}\right)+N_2\left(r,\frac{1}{G}\right)+T(r,G)+S(r,f).\end{aligned}$$

As same as above, by the second fundamental theorem, we can obtain

$$T(r,F)+T(r,G)\le 6\overline{N}(r,F)+3N_2\left(r,\frac{1}{F}\right)+2N_2\left(r,\frac{1}{G}\right)+T(r,G)+S(r,f),$$

so

$$T(r,F)\le 6\overline{N}(r,F)+3N_2\left(r,\frac{1}{F}\right)+2N_2\left(r,\frac{1}{G}\right)+S(r,f),$$

i.e.

$$T(r,f) \leq 6\overline{N}(r,f) + 3N_2\left(r, \frac{1}{f}\right) + 2N_2\left(r, \frac{1}{f^{(k)}}\right) + S(r,f).$$

By Lemma 2.1 for $p = 2$ we have

$$T(r,f) \leq (6+2k)\overline{N}(r,f) + 5N_{2+k}\left(r, \frac{1}{f}\right) + S(r,f),$$

so

$$(6+2k)\Theta(\infty, f) + 5\delta_{2+k}(0,f) \leq 2k + 10.$$

This contradicts with (1.6). Now the proof has been completed.

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