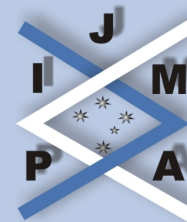


ON A FAMILY OF LINEAR AND POSITIVE OPERATORS IN WEIGHTED SPACES



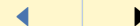
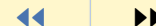
Linear and Positive Operators
in Weighted Spaces

Ayşegül Erençın and Fatma Taşdelen

vol. 8, iss. 2, art. 39, 2007

[Title Page](#)

[Contents](#)



Page 1 of 11

[Go Back](#)

[Full Screen](#)

[Close](#)

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Abstract: In this paper, we present a modification of the sequence of linear operators proposed by Lupaş [6] and studied by Agratini [1]. Some convergence properties of these operators are given in weighted spaces of continuous functions on positive semi-axis by using the same approach as in [4] and [5].

Contents

1	Introduction	3
2	Auxiliary Results	5
3	Main Result	7



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[Title Page](#)

[Contents](#)



Page 2 of 11

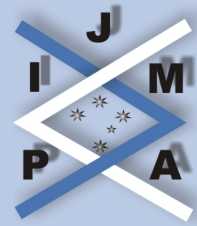
[Go Back](#)

[Full Screen](#)

[Close](#)

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Title Page

Contents



Page 3 of 11

Go Back

Full Screen

Close

1. Introduction

Lupaş in [6] studied the identity

$$\frac{1}{(1-a)^\alpha} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} a^k, \quad |a| < 1$$

and letting $\alpha = nx$ and $x \geq 0$ considered the linear positive operators

$$L_n^*(f; x) = (1-a)^{nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!} a^k f\left(\frac{k}{n}\right)$$

with $f : [0, \infty) \rightarrow \mathbb{R}$. Imposing the condition $L_n(1; x) = 1$ he found that $a = 1/2$. Therefore Lupaş proposed the positive linear operators

$$(1.1) \quad L_n^*(f; x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right).$$

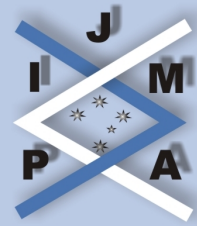
Agratini [1] gave some quantitative estimates for the rate of convergence on the finite interval $[0, b]$ for any $b > 0$ and also established a Voronovskaja-type formula for these operators.

We consider the generalization of the operators (1.1)

$$(1.2) \quad L_n(f; x) = 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)_k}{2^k k!} f\left(\frac{k}{b_n}\right), \quad x \in \mathbb{R}_0, \quad n \in \mathbb{N},$$

where $\mathbb{R}_0 = [0, \infty)$, $\mathbb{N} := \{1, 2, \dots\}$ and $\{a_n\}$, $\{b_n\}$ are increasing and unbounded sequences of positive numbers such that

$$(1.3) \quad \lim_{n \rightarrow \infty} \frac{1}{b_n} = 0, \quad \frac{a_n}{b_n} = 1 + O\left(\frac{1}{b_n}\right).$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 4 of 11

Go Back

Full Screen

Close

In this work, we study the convergence properties of these operators in the weighted spaces of continuous functions on positive semi-axis with the help of a weighted Korovkin type theorem, proved by Gadzhiev in [2, 3]. For this purpose, we now recall the results of [2, 3].

B_ρ : The set of all functions f defined on the real axis satisfying the condition

$$|f(x)| \leq M_f \rho(x),$$

where M_f is a constant depending only on f and $\rho(x) = 1 + x^2$, $-\infty < x < \infty$.

The space B_ρ is normed by

$$\|f\|_\rho = \sup_{x \in \mathbb{R}} \frac{|f(x)|}{\rho(x)}, \quad f \in B_\rho.$$

C_ρ : The subspace of all continuous functions belonging to B_ρ .

C_ρ^* : The subspace of all functions $f \in C_\rho$ for which

$$\lim_{|x| \rightarrow \infty} \frac{f(x)}{\rho(x)} = k,$$

where k is a constant depending on f .

Theorem A ([2, 3]). Let $\{T_n\}$ be the sequence of linear positive operators which are mappings from C_ρ into B_ρ satisfying the conditions

$$\lim_{n \rightarrow \infty} \|T_n(t^\nu, x) - x^\nu\|_\rho = 0 \quad \nu = 0, 1, 2.$$

Then, for any function $f \in C_\rho^*$,

$$\lim_{n \rightarrow \infty} \|T_n f - f\|_\rho = 0,$$

and there exists a function $f^* \in C_\rho \setminus C_\rho^*$ such that

$$\lim_{n \rightarrow \infty} \|T_n f^* - f^*\|_\rho \geq 1.$$



Title Page

Contents



Page 5 of 11

Go Back

Full Screen

Close

2. Auxiliary Results

In this section we shall give some properties of the operators (1.2), which we shall use in the proofs of the main theorems.

Lemma 2.1. *If the operators L_n are defined by (1.2), then for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ the following identities are valid*

$$(2.1) \quad L_n(1; x) = 1,$$

$$(2.2) \quad L_n(t; x) = \frac{a_n}{b_n}x,$$

$$(2.3) \quad L_n(t^2; x) = \frac{a_n^2}{b_n^2}x^2 + 2\frac{a_n}{b_n^2}x,$$

$$(2.4) \quad L_n(t^3; x) = \frac{a_n^3}{b_n^3}x^3 + 6\frac{a_n^2}{b_n^3}x^2 + 6\frac{a_n}{b_n^3}x$$

and

$$(2.5) \quad L_n(t^4; x) = \frac{a_n^4}{b_n^4}x^4 + 12\frac{a_n^3}{b_n^4}x^3 + 36\frac{a_n^2}{b_n^4}x^2 + 26\frac{a_n}{b_n^4}x.$$

Proof. It is clear that (2.1) holds.

By using the recurrence relation $(\alpha)_k = \alpha(\alpha + 1)_{k-1}$, $k \geq 1$ for the function



Title Page

Contents



Page 6 of 11

Go Back

Full Screen

Close

$f(t) = t$ we have

$$\begin{aligned} L_n(t; x) &= \frac{1}{b_n} 2^{-a_n x} \sum_{k=1}^{\infty} \frac{(a_n x)_k}{2^k (k-1)!} \\ &= \frac{a_n}{b_n} x 2^{-a_n x} \sum_{k=1}^{\infty} \frac{(a_n x + 1)_{k-1}}{2^k (k-1)!} \\ &= \frac{a_n}{b_n} x 2^{-(a_n x + 1)} \sum_{k=0}^{\infty} \frac{(a_n x + 1)_k}{2^k k!} \\ &= \frac{a_n}{b_n} x. \end{aligned}$$

In a similar way to that of (2.2), we can prove (2.3) – (2.5). \square

Lemma 2.2. *If the operators L_n are defined by (1.2), then for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$*

$$\begin{aligned} (2.6) \quad L_n((t-x)^4; x) &= \left(\frac{a_n}{b_n} - 1\right)^4 x^4 + \left(12 \frac{a_n^3}{b_n^4} - 24 \frac{a_n^2}{b_n^3} + 12 \frac{a_n}{b_n^2}\right) x^3 \\ &\quad + \left(36 \frac{a_n^2}{b_n^4} - 24 \frac{a_n}{b_n^3}\right) x^2 + 26 \frac{a_n}{b_n^4} x. \end{aligned}$$

Lemma 2.3. *If the operators L_n are defined by (1.2), then for all $x \in \mathbb{R}_0$ and sufficiently large n*

$$(2.7) \quad L_n((t-x)^4; x) = O\left(\frac{1}{b_n}\right) (x^4 + x^3 + x^2 + x).$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 7 of 11

Go Back

Full Screen

Close

3. Main Result

In this part, we firstly prove the following theorem related to the weighted approximation of the operators in (1.2).

Theorem 3.1. *Let L_n be the sequence of linear positive operators (1.2) acting from C_ρ to B_ρ . Then for each function $f \in C_\rho^*$,*

$$\lim_{n \rightarrow \infty} \|L_n(f; x) - f(x)\|_\rho = 0.$$

Proof. It is sufficient to verify the conditions of Theorem A which are

$$\lim_{n \rightarrow \infty} \|L_n(t^\nu, x) - x^\nu\|_\rho = 0 \quad \nu = 0, 1, 2.$$

From (2.1) clearly we have

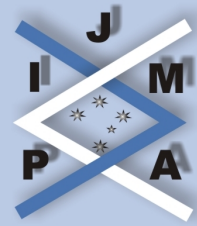
$$\lim_{n \rightarrow \infty} \|L_n(1, x) - 1\|_\rho = 0.$$

By using (1.3) and (2.2) we can write

$$\begin{aligned} \|L_n(t, x) - x\|_\rho &= \sup_{x \in \mathbb{R}_0} \frac{|L_n(t, x) - x|}{1 + x^2} \\ &= \left| \frac{a_n}{b_n} - 1 \right| \sup_{x \in \mathbb{R}_0} \frac{x}{1 + x^2} \\ &= O\left(\frac{1}{b_n}\right) \sup_{x \in \mathbb{R}_0} \frac{x}{1 + x^2}. \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} \|L_n(t; x) - x\|_\rho = 0.$$



Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 8 of 11

Go Back

Full Screen

Close

Similarly, by the equalities (1.3) and (2.3) we find that

$$(3.1) \quad \|L_n(t^2, x) - x^2\|_\rho = \sup_{x \in \mathbb{R}_0} \frac{|L_n(t^2, x) - x^2|}{1 + x^2} \\ \leq \left| \frac{a_n^2}{b_n^2} - 1 \right| \sup_{x \in \mathbb{R}_0} \frac{x^2}{1 + x^2} + 2 \frac{a_n}{b_n^2} \sup_{x \in \mathbb{R}_0} \frac{x}{1 + x^2},$$

which gives

$$\lim_{n \rightarrow \infty} \|L_n(t^2; x) - x^2\|_\rho = 0.$$

Thus all conditions of Theorem A hold and the proof is completed. \square

Now, we find the rate of convergence for the operators (1.2) in the weighted spaces by means of the weighted modulus of continuity $\Omega(f, \delta)$ which tends to zero as $\delta \rightarrow 0$ on an infinite interval, defined in [5]. We now recall the definition of $\Omega(f, \delta)$.

Let $f \in C_\rho^*$. The weighted modulus of continuity of f is denoted by

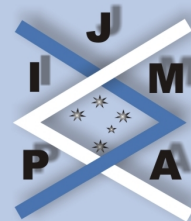
$$\Omega(f, \delta) = \sup_{|h| \leq \delta, x \in \mathbb{R}_0} \frac{|f(x+h) - f(x)|}{(1+h^2)(1+x^2)}.$$

$\Omega(f, \delta)$ has the following properties [4, 5].

Let $f \in C_\rho^*$, then

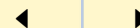
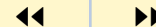
- (i) $\Omega(f, \delta)$ is a monotonically increasing function with respect to δ , $\delta \geq 0$.
- (ii) For every $f \in C_\rho^*$, $\lim_{\delta \rightarrow 0} \Omega(f, \delta) = 0$.
- (iii) For each positive value of λ

$$\Omega(f, \lambda\delta) \leq 2(1 + \lambda)(1 + \delta^2)\Omega(f, \delta).$$



Title Page

Contents



Page 9 of 11

Go Back

Full Screen

Close

(iv) For every $f \in C_\rho^*$ and $x, t \in \mathbb{R}_0$:

$$|f(t) - f(x)| \leq 2 \left(1 + \frac{|t-x|}{\delta} \right) (1 + \delta^2) \Omega(f, \delta) (1 + x^2) (1 + (t-x)^2).$$

Theorem 3.2. Let $f \in C_\rho^*$. Then the inequality

$$\sup_{x \in \mathbb{R}_0} \frac{|L_n(f, x) - f(x)|}{(1+x^2)^3} \leq M \Omega(f, b_n^{-1/4})$$

is valid for sufficiently large n , where M is a constant independent of a_n and b_n .

Proof. By the definition of L_n and the property (iv), we get

$$|L_n(f, x) - f(x)| \leq 2(1 + \delta_n^2) \Omega(f, \delta_n) (1 + x^2) 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)^k}{2^k k!} A_1(x),$$

where

$$A_1(x) = \left(1 + \frac{\left| \frac{k}{b_n} - x \right|}{\delta_n} \right) \left(1 + \left(\frac{k}{b_n} - x \right)^2 \right).$$

Then for all $x, \frac{k}{b_n} \in \mathbb{R}_0$, by using the following inequality (see[5, p. 578])

$$A_1(x) \leq 2(1 + \delta_n^2) \left(1 + \frac{\left(\frac{k}{b_n} - x \right)^4}{\delta_n^4} \right),$$



Title Page

Contents



Page 10 of 11

Go Back

Full Screen

Close

we can write

$$\begin{aligned} |L_n(f, x) - f(x)| &\leq 16\Omega(f, \delta_n)(1 + x^2) \left(1 + \frac{1}{\delta_n^4} 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)^k}{2^k k!} \left(\frac{k}{b_n} - x \right)^4 \right) \\ &= 16\Omega(f, \delta_n)(1 + x^2) \left(1 + \frac{1}{\delta_n^4} L_n((t - x)^4; x) \right). \end{aligned}$$

Thus by means of (2.7), we have

$$|L_n(f, x) - f(x)| \leq 16\Omega(f, \delta_n)(1 + x^2) \left[1 + \frac{1}{\delta_n^4} O\left(\frac{1}{b_n}\right) (x^4 + x^3 + x^2 + x) \right].$$

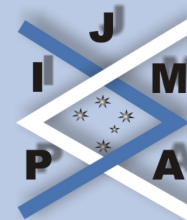
If we choose $\delta_n = b_n^{-1/4}$ for sufficiently large n , then we find

$$\sup_{x \in \mathbb{R}_0} \frac{|L_n(f, x) - f(x)|}{(1 + x^2)^3} \leq M\Omega(f, b_n^{-1/4}),$$

which is the desired result. □

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Title Page

Contents



Page 11 of 11

Go Back

Full Screen

Close