# ON A FAMILY OF LINEAR AND POSITIVE OPERATORS IN WEIGHTED SPACES

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*Abstract:* In this paper, we present a modification of the sequence of linear operators proposed by Lupaş [6] and studied by Agratini [1]. Some convergence properties of these operators are given in weighted spaces of continuous functions on positive semi-axis by using the same approach as in [4] and [5].



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# 1. Introduction

Lupaş in [6] studied the identity

$$\frac{1}{(1-a)^{\alpha}} = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!} a^k, \quad |a| < 1$$

and letting $\alpha = nx$ and $x \geq 0$ considered the linear positive operators

$$L_n^*(f;x) = (1-a)^{nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{k!} a^k f\left(\frac{k}{n}\right)$$

with $f : [0,\infty) \to \mathbb{R}$. Imposing the condition $L_n(1;x) = 1$ he found that $a = 1/2$. Therefore Lupaş proposed the positive linear operators

$$L_n^*(f;x) = 2^{-nx} \sum_{k=0}^{\infty} \frac{(nx)_k}{2^k k!} f\left(\frac{k}{n}\right). \tag{1.1}$$

Agratini [1] gave some quantitative estimates for the rate of convergence on the finite interval $[0,b]$ for any $b > 0$ and also established a Voronovskaja-type formula for these operators.

We consider the generalization of the operators (1.1)

$$L_n(f;x) = 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)_k}{2^k k!} f\left(\frac{k}{b_n}\right), \quad x \in \mathbb{R}_0,\ n \in \mathbb{N}, \tag{1.2}$$

where $\mathbb{R}_0 = [0,\infty)$, $\mathbb{N} := \{1,2,\dots\}$ and $\{a_n\}$, $\{b_n\}$ are increasing and unbounded sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{1}{b_n} = 0, \quad \frac{a_n}{b_n} = 1 + O\left(\frac{1}{b_n}\right). \tag{1.3}$$



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In this work, we study the convergence properties of these operators in the weighted spaces of continuous functions on positive semi-axis with the help of a weighted Korovkin type theorem, proved by Gadzhiev in [2, 3]. For this purpose, we now recall the results of [2, 3].

$B_\rho$: The set of all functions $f$ defined on the real axis satisfying the condition

$$|f(x)| \leq M_f \rho(x),$$

where $M_f$ is a constant depending only on $f$ and $\rho(x) = 1 + x^2$, $-\infty < x < \infty$.

The space $B_\rho$ is normed by

$$\|f\|_\rho = \sup_{x \in \mathbb{R}} \frac{|f(x)|}{\rho(x)}, \quad f \in B_\rho.$$

$C_\rho$: The subspace of all continuous functions belonging to $B_\rho$.
$C_\rho^*$: The subspace of all functions $f \in C_\rho$ for which

$$\lim_{|x| \to \infty} \frac{f(x)}{\rho(x)} = k,$$

where $k$ is a constant depending on $f$.

**Theorem A ([2, 3]).** *Let $\{T_n\}$ be the sequence of linear positive operators which are mappings from $C_\rho$ into $B_\rho$ satisfying the conditions*

$$\lim_{n \to \infty} \|T_n(t^\nu, x) - x^\nu\|_\rho = 0 \qquad \nu = 0, 1, 2.$$

*Then, for any function $f \in C_\rho^*$,*

$$\lim_{n \to \infty} \|T_n f - f\|_\rho = 0,$$

*and there exists a function $f^* \in C_\rho \backslash C_\rho^*$ such that*

$$\lim_{n \to \infty} \|T_n f^* - f^*\|_\rho \geq 1.$$



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## 2. Auxiliary Results

In this section we shall give some properties of the operators (1.2), which we shall use in the proofs of the main theorems.

**Lemma 2.1.** *If the operators $L_n$ are defined by (1.2), then for all $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$ the following identities are valid*

$$L_n(1;x) = 1, \tag{2.1}$$

$$L_n(t;x) = \frac{a_n}{b_n}x, \tag{2.2}$$

$$L_n(t^2;x) = \frac{a_n^2}{b_n^2}x^2 + 2\frac{a_n}{b_n^2}x, \tag{2.3}$$

$$L_n(t^3;x) = \frac{a_n^3}{b_n^3}x^3 + 6\frac{a_n^2}{b_n^3}x^2 + 6\frac{a_n}{b_n^3}x \tag{2.4}$$

*and*

$$L_n(t^4;x) = \frac{a_n^4}{b_n^4}x^4 + 12\frac{a_n^3}{b_n^4}x^3 + 36\frac{a_n^2}{b_n^4}x^2 + 26\frac{a_n}{b_n^4}x. \tag{2.5}$$

*Proof.* It is clear that (2.1) holds.

By using the recurrence relation $(\alpha)_k = \alpha(\alpha+1)_{k-1}$, $k \geq 1$ for the function

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$f(t)=t$ we have

$$\begin{aligned} L_n(t;x) &= \frac{1}{b_n}2^{-a_n x}\sum_{k=1}^{\infty}\frac{(a_n x)_k}{2^k(k-1)!} \\ &= \frac{a_n}{b_n}x2^{-a_n x}\sum_{k=1}^{\infty}\frac{(a_n x+1)_{k-1}}{2^k(k-1)!} \\ &= \frac{a_n}{b_n}x2^{-(a_n x+1)}\sum_{k=0}^{\infty}\frac{(a_n x+1)_k}{2^k k!} \\ &= \frac{a_n}{b_n}x. \end{aligned}$$

In a similar way to that of (2.2), we can prove (2.3) – (2.5). □

**Lemma 2.2.** *If the operators $L_n$ are defined by (1.2), then for all $x\in\mathbb{R}_0$ and $n\in\mathbb{N}$*

$$L_n\left((t-x)^4;x\right)=\left(\frac{a_n}{b_n}-1\right)^4x^4+\left(12\frac{a_n^3}{b_n^4}-24\frac{a_n^2}{b_n^3}+12\frac{a_n}{b_n^2}\right)x^3 + \left(36\frac{a_n^2}{b_n^4}-24\frac{a_n}{b_n^3}\right)x^2+26\frac{a_n}{b_n^4}x. \tag{2.6}$$

**Lemma 2.3.** *If the operators $L_n$ are defined by (1.2), then for all $x\in\mathbb{R}_0$ and sufficiently large $n$*

$$L_n\left((t-x)^4;x\right)=O\left(\frac{1}{b_n}\right)\left(x^4+x^3+x^2+x\right). \tag{2.7}$$



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## 3. Main Result

In this part, we firstly prove the following theorem related to the weighted approximation of the operators in (1.2).

**Theorem 3.1.** *Let $L_n$ be the sequence of linear positive operators (1.2) acting from $C_\rho$ to $B_\rho$. Then for each function $f \in C_\rho^*$,*

$$\lim_{n\to\infty} \left\| L_n\left(f;x\right) - f\left(x\right) \right\|_\rho = 0.$$

*Proof.* It is sufficient to verify the conditions of Theorem A which are

$$\lim_{n\to\infty} \| L_n(t^\nu, x) - x^\nu \|_\rho = 0 \qquad \nu = 0, 1, 2.$$

From (2.1) clearly we have

$$\lim_{n\to\infty} \| L_n(1, x) - 1 \|_\rho = 0.$$

By using (1.3) and (2.2) we can write

$$\begin{aligned} \| L_n(t,x) - x \|_\rho &= \sup_{x\in\mathbb{R}_0} \frac{|L_n(t,x) - x|}{1+x^2} \\ &= \left| \frac{a_n}{b_n} - 1 \right| \sup_{x\in\mathbb{R}_0} \frac{x}{1+x^2} \\ &= O\left(\frac{1}{b_n}\right) \sup_{x\in\mathbb{R}_0} \frac{x}{1+x^2}. \end{aligned}$$

This implies that

$$\lim_{n\to\infty} \left\| L_n\left(t;x\right) - x \right\|_\rho = 0.$$


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Similarly, by the equalities (1.3) and (2.3) we find that

$$\|L_n(t^2,x)-x^2\|_\rho = \sup_{x\in\mathbb{R}_0}\frac{|L_n(t^2,x)-x^2|}{1+x^2} \tag{3.1}$$
$$\le \left|\frac{a_{n^2}}{b_n^2}-1\right|\sup_{x\in\mathbb{R}_0}\frac{x^2}{1+x^2}+2\frac{a_n}{b_n^2}\sup_{x\in\mathbb{R}_0}\frac{x}{1+x^2},$$

which gives

$$\lim_{n\to\infty}\left\|L_n\left(t^2;x\right)-x^2\right\|_\rho=0.$$

Thus all conditions of Theorem A hold and the proof is completed. □

Now, we find the rate of convergence for the operators (1.2) in the weighted spaces by means of the weighted modulus of continuity $\Omega(f,\delta)$ which tends to zero as $\delta\to 0$ on an infinite interval, defined in [5]. We now recall the definition of $\Omega(f,\delta)$.

Let $f\in C_\rho^*$. The weighted modulus of continuity of $f$ is denoted by

$$\Omega(f,\delta)=\sup_{|h|\le\delta,x\in\mathbb{R}_0}\frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}.$$

$\Omega(f,\delta)$ has the following properties [4, 5].
Let $f\in C_\rho^*$, then

$(i)$ $\Omega(f,\delta)$ is a monotonically increasing function with respect to $\delta$, $\delta\ge 0$.

$(ii)$ For every $f\in C_\rho^*$, $\lim_{\delta\to 0}\Omega(f,\delta)=0$.

$(iii)$ For each positive value of $\lambda$

$$\Omega(f,\lambda\delta)\le 2(1+\lambda)(1+\delta^2)\Omega(f,\delta).$$

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(iv) For every $f \in C_\rho^*$ and $x, t \in \mathbb{R}_0$ :

$$|f(t) - f(x)| \leq 2\left(1 + \frac{|t-x|}{\delta}\right)(1+\delta^2)\Omega(f,\delta)(1+x^2)(1+(t-x)^2).$$

**Theorem 3.2.** *Let $f \in C_\rho^*$. Then the inequality*

$$\sup_{x\in\mathbb{R}_0} \frac{|L_n(f,x) - f(x)|}{(1+x^2)^3} \leq M\Omega\left(f, b_n^{-1/4}\right)$$

*is valid for sufficiently large $n$, where $M$ is a constant independent of $a_n$ and $b_n$.*

*Proof.* By the definition of $L_n$ and the property $(iv)$, we get

$$|L_n(f,x) - f(x)| \leq 2(1+\delta_n^2)\Omega(f,\delta_n)(1+x^2)2^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)_k}{2^k k!}A_1(x),$$

where

$$A_1(x) = \left(1 + \frac{\left|\frac{k}{b_n} - x\right|}{\delta_n}\right)\left(1 + \left(\frac{k}{b_n} - x\right)^2\right).$$

Then for all $x, \frac{k}{b_n} \in \mathbb{R}_0$, by using the following inequality (see[5, p. 578])

$$A_1(x) \leq 2(1+\delta_n^2)\left(1 + \frac{\left(\frac{k}{b_n} - x\right)^4}{\delta_n^4}\right),$$


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we can write

$$|L_n(f,x) - f(x)| \leq 16\Omega(f,\delta_n)(1+x^2)\left(1 + \frac{1}{\delta_n^4} 2^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)_k}{2^k k!} \left(\frac{k}{b_n} - x\right)^4\right)$$
$$= 16\Omega(f,\delta_n)(1+x^2)\left(1 + \frac{1}{\delta_n^4} L_n\left((t-x)^4; x\right)\right).$$

Thus by means of (2.7), we have

$$|L_n(f,x) - f(x)| \leq 16\Omega(f,\delta_n)(1+x^2)\left[1 + \frac{1}{\delta_n^4} O\left(\frac{1}{b_n}\right)\left(x^4 + x^3 + x^2 + x\right)\right].$$

If we choose $\delta_n = b_n^{-1/4}$ for sufficiently large $n$, then we find

$$\sup_{x \in \mathbb{R}_0} \frac{|L_n(f,x) - f(x)|}{(1+x^2)^3} \leq M\Omega\left(f, b_n^{-1/4}\right),$$

which is the desired result. □



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