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UPPER AND LOWER SOLUTIONS METHOD FOR DISCRETE INCLUSIONS WITH NONLINEAR BOUNDARY CONDITIONS

M. BENCHOHRA, S.K. NTOUYAS AND A. OUAHAB

Département de Mathématiques
Université de Sidi Bel Abbès
BP 89, 22000 Sidi Bel Abbès
Algérie
EMail: benchohra@yahoo.com

Department of Mathematics
University of Ioannina
451 10 Ioannina, Greece
EMail: sntouyas@cc.uoi.gr

Département de Mathématiques
Université de Sidi Bel Abbès
BP 89, 22000 Sidi Bel Abbès
Algérie
EMail: agh_ouahab@yahoo.fr

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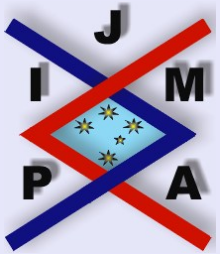
Abstract

In this note the concept of lower and upper solutions combined with the nonlinear alternative of Leray-Schauder type is used to investigate the existence of solutions for first order discrete inclusions with nonlinear boundary conditions.

2000 Mathematics Subject Classification: 39A10.
Key words: Discrete Inclusions, Convex valued multivalued map, Fixed point, Upper and lower solutions, Nonlinear boundary conditions.

Contents



Upper and Lower Solutions Method for Discrete Inclusions with Nonlinear Boundary Conditions

M. Benchohra, S.K. Ntouyas and A. Ouahab

1. Introduction

This note is concerned with the existence of solutions for the discrete boundary multivalued problem

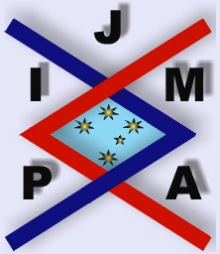
$$\Delta y(i-1) \in F(i, y(i)), \qquad i \in [1, T] = \{1, 2, \ldots, T\}, \tag{1.1}$$

$$L(y(0), y(T+1)) = 0, \tag{1.2}$$

where $F : \mathbb{N} \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a compact convex valued multivalued map and $L : \mathbb{N}^2 \to \mathbb{R}$ is a nonlinear single-valued map.

Very recently Agarwal *et al* [3] applied the concept of upper and lower solutions combined with the Leray-Schauder nonlinear alternative to a class of second order discrete inclusions subjected to Dirichlet conditions. For more details on recent results and applications of difference equations we recommend for instance the monographs by Agarwal *et al* [1], [2], Pachpatte [9] and the references cited therein.

In this note we shall apply the same tool as in [3] to first order discrete inclusions with nonlinear boundary conditions which include the initial, terminal and periodic conditions. The corresponding problem for differential inclusions was studied by Benchohra and Ntouyas in [4].



2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts which are used throughout the note. $C([0,T],\mathbb{R})$ is the Banach space of all continuous functions from $[0,T]$ (discrete topology) into $\mathbb{R}$ with the norm $\|y\| = \sup_{k\in[0,T]} |y(k)|$. Let $(X,|\cdot|)$ be a Banach space. A multivalued map $G : X \longrightarrow \mathcal{P}(X)$ has convex (closed) values if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B)$ is bounded in X for each bounded set B of X (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

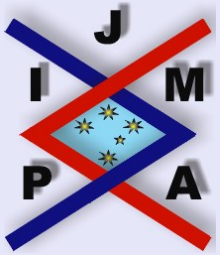
G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighbourhood M of x_0 such that $G(M) \subseteq N$. G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subseteq X$.

If the multivalued G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

For more details on multivalued maps see the books of Deimling [5] and Hu and Papageorgiou [7].

Let us start by defining what we mean by a solution of problem (1.1) – (1.2).

Definition 2.1. *A function $y \in C([0,T],\mathbb{R})$, is said to be a solution of (1.1) – (1.2) if y satisfies the inclusion $\Delta y(i-1) \in F(i,y(i))$ on $\{1,\dots,T\}$ and the condition $L(y(0),y(T+1)) = 0$.*



For any $y \in C([0,T],\mathbb{R})$ we define the set

$$S_{F,y} = \{v \in C([0,T],\mathbb{R}) : \ v(i) \in F(i,y(i)) \text{ for } \ i \in \{1,\dots,T\}\}.$$

Definition 2.2. *A function $\alpha \in C([0,T+1],\mathbb{R})$ is said to be a lower solution of (1.1) – (1.2) if for each $i \in [0,T+1]$ there exists $v_1(i) \in F(i,\alpha(i))$ with $\Delta\alpha(i-1) \le v_1(i)$ and $L(\alpha(0),\alpha(T+1)) \le 0$.*

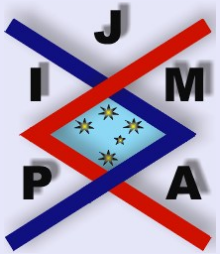
Similarly a function $\beta \in C([0,T+1],\mathbb{R})$ is said to be an upper solution of (1.1) – (1.2) if for each $i \in [0,T+1]$ there exists $v_2(i) \in F(i,\beta(i))$ with $\Delta\beta(i-1) \ge v_2(i)$ and $L(\beta(0),\beta(T+1)) \ge 0$.

Our existence result in the next section relies on the following fixed point principle.

Lemma 2.1 (Nonlinear Alternative [6]). *Let X be a Banach space with $C \subset X$ convex. Assume U is an open subset of C with $0 \in U$ and $G : \overline{U} \to \mathcal{P}(C)$ is a compact multivalued map, u.s.c. with convex closed values. Then either,*

(i) G has a fixed point in $\overline{U}$; or

(ii) there is a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$.



3. Main Result

We are now in a position to state and prove our existence result for the problem (1.1) – (1.2). We first list the following hypotheses:

(H1) $y \longmapsto F(i,y)$ is upper semicontinuous for all $i \in [1,T]$;

(H2) for each $q > 0$, there exists $\phi_q \in C([1,T],\mathbb{R}_+)$ such that

$$\|F(i,y)\| = \sup\{|v| : v \in F(i,y)\} \leq \phi_q(i) \quad \text{for all } |y| \leq q \text{ and } i \in [1,T];$$

(H3) there exist α and $\beta \in C([0,T+1],\mathbb{R})$, lower and upper solutions for the problem (1.1) – (1.2) such that $\alpha \leq \beta$;

(H4) L is a continuous single-valued map in $(x,y) \in [\alpha(0),\beta(0)] \times [\alpha(T+1),\beta(T+1)]$ and nonincreasing in $y \in [\alpha(T+1),\beta(T+1)]$.

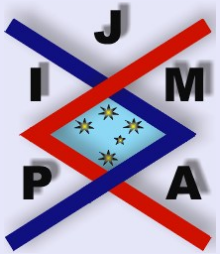
Theorem 3.1. *Assume that hypotheses (H1) – (H4) hold. Then the problem (1.1) – (1.2) has at least one solution y such that*

$$\alpha(i) \leq y(i) \leq \beta(i) \text{ for all } i \in [1,T].$$

Proof. Transform the problem (1.1) – (1.2) into a fixed point problem. Consider the following modified problem

$$\Delta y(i-1) + y(i) \in F_1(i,y(i)), \text{ on } [1,T] \tag{3.1}$$

$$y(0) = \tau(0,y(0) - L(\overline{y}(0),\overline{y}(T+1)), \tag{3.2}$$



where

$$F_1(i,y) = F(i,\tau(i,y)) + \tau(i,y),$$
$$\tau(i,y) = \max(\alpha(i), \min(y,\beta(i))$$

and

$$\overline{y}(i) = \tau(i,y).$$

A solution to (3.1) – (3.2) is a fixed point of the operator $N : C([1,T],\mathbb{R}) \longrightarrow \mathcal{P}(C([1,T],\mathbb{R}))$ defined by:

$$N(y) = \left\{ h \in C([1,T]) : h(k) = y(0) + \sum_{0<l<k} [g(l) + \overline{y}(l)] - \sum_{0<l<k} y(l), \;\; g \in \tilde{S}^1_{F,\overline{y}} \right\},$$

where

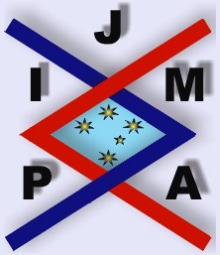
$$\tilde{S}^1_{F,\overline{y}} = \{v \in S^1_{F,\overline{y}} : v(i) \geq v_1(i) \text{ a.e. on } A_1 \text{ and } v(i) \leq v_2(i) \text{ on } A_2\},$$

$$S^1_{F,\overline{y}} = \{v \in C([1,T]) : v(i) \in F(i,(\overline{y})(i)) \text{ for } i \in [1,T]\},$$

$$A_1 = \{i \in [1,T] : y(i) < \alpha(i) \leq \beta(i)\}, \;\; A_2 = \{i \in [1,T] : \alpha(i) \leq \beta(i) < y(i)\}.$$

Remark 1. *Notice that F_1 is an upper semicontinuous multivalued map with compact convex values, and there exists $\phi \in C([1,T],\mathbb{R}^+)$ such that*

$$\|F_1(i,y)\| \leq \phi(i) + \max\left(\sup_{i\in[1,T]} |\alpha(i)|, \sup_{i\in[1,T]} |\beta(i)| \right).$$



We shall show that N satisfies the assumptions of Lemma 2.1. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y \in C([1,T],\mathbb{R})$.*

Indeed, if $h_1,\ h_2$ belong to $N(y)$, then there exist $g_1, g_2 \in \tilde{S}^1_{F,\overline{y}}$ such that for each $k \in [1,T]$ we have

$$h_i(k) = y(0) + \sum_{0<l<k} [g_i(l) + \overline{y}(l)] - \sum_{0<l<k} y(l), \quad i = 1,2.$$

Let $0 \leq d \leq 1$. Then for each $k \in [1,T]$ we have

$$(dh_1 + (1-d)h_2)(k) = y(0) + \sum_{0<l<k} [dg_1(l) + (1-d)g_2(l) + \overline{y}(l)] - \sum_{0<l<k} y(l).$$

Since $\tilde{S}^1_{F_1,\overline{y}}$ is convex (because F_1 has convex values) then

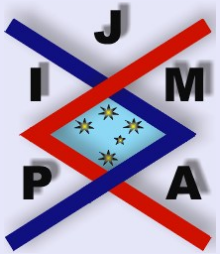
$$dh_1 + (1-d)h_2 \in N(y).$$

Step 2: *N maps bounded sets into bounded sets in $C([1,T],\mathbb{R})$.*

Indeed, it is enough to show that for each $q > 0$ there exists a positive constant ℓ^* such that for each $y \in B_q = \{y \in C([1,T],\mathbb{R}) : \|y\|_\infty \leq q\}$ one has $\|N(y)\|_\infty \leq \ell^*$.

Let $y \in B_q$ and $h \in N(y)$ then there exists $g \in \tilde{S}^1_{F,\overline{y}}$ such that for each $k \in [1,T]$ we have

$$h(k) = y(0) + \sum_{0<l<k} [g(l) + \overline{y}(l)] - \sum_{0<l<k} y(l).$$



By (H2) we have for each $i \in [1,T]$

$$|h(k)| \leq |y(0)| + \sum_{l=1}^{k} |g(l)| + \sum_{l=1}^{k} |\bar{y}(l)| + \sum_{l=1}^{k} |y(l)|$$
$$\leq \max(|\alpha(0)|, |\beta(0)|) + k\|\phi_q\|_\infty$$
$$+ k \max\left(q, \sup_{i\in[1,T]} |\alpha(i)|, \sup_{i\in[1,T]} |\beta(i)|\right) + kq := \ell^*.$$

Step 3: *N maps bounded set into equicontinuous sets of $C([1,T],\mathbb{R})$.*

Let $k_1, k_2 \in [1,T]$, $k_1 < k_2$ and B_q be a bounded set of $C([1,T])$ as in Step 2. Let $y \in B_q$ and $h \in N(y)$ then there exists $g \in \tilde{S}^1_{F,\overline{y}}$ such that for each $k \in [1,T]$ we have

$$h(k) = y(0) + \sum_{0<l<k} [g(l) + \overline{y}(l)] - \sum_{0<l<k} y(l).$$

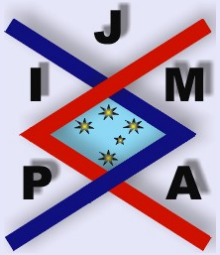
Then

$$|h(k_2) - h(k_1)| \leq \sum_{k_1<l<k_2} [|g(l)| + |\overline{y}(l)|] + \sum_{k_1<l<k_2} |y(l)|.$$

As $k_2 \longrightarrow k_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem we can conclude that $N : C([1,T],\mathbb{R}) \longrightarrow \mathcal{P}(C([1,T],\mathbb{R}))$ is a completely continuous multivalued map.

Step 4: *A priori bounds on solutions exist.*



Let $y \in C([1,T],\mathbb{R})$ and $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. Then

$$y(k) = \lambda \left(y(0) - \sum_{0<l<k} y(l) + \sum_{0<l<k} [g(l) + \overline{y}(l)] \right).$$

Hence

$$\begin{aligned} |y(k)| &\leq |y(0)| + \sum_{l=1}^{k} |g(l)| + \sum_{l=1}^{k} |\overline{y}(l)| + \sum_{l=1}^{k} |y(l)| \\ &\leq \max(|\alpha(0)|, |\beta(0)|) + T\|\phi\|_\infty \\ &\quad + T \max\left(\sup_{i\in[1,T]} |\alpha(i)|, \sup_{i\in[1,T]} |\beta(i)| \right) + 2\sum_{l=1}^{k} |y(l)|. \end{aligned}$$

Using the Pachpatte inequality (see [9, Theorem 2.5]) we get for each $k \in [1,T]$

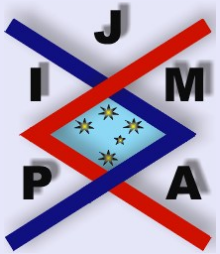
$$|y(k)| \leq c_* \left[1 + 2 \sum_{l=1}^{T} \prod_{s=1}^{l-1} 2 \right],$$

where

$$c_* = \max(|\alpha(0)|, |\beta(0)|) + T\|\phi\|_\infty + T \max\left(\sup_{i\in[1,T]} |\alpha(i)|, \sup_{i\in[1,T]} |\beta(i)| \right).$$

Thus

$$\|y\|_\infty \leq c_*(1 + T2^{T+1}) := M.$$



Set

$$U = \{y \in C([1,T],\mathbb{R}) : \|y\|_\infty < M+1\}.$$

As in Step 3 the operator $N : \overline{U} \longrightarrow \mathcal{P}(C([1,T],\mathbb{R}))$ is continuous and completely continuous.

Step 5: *N has a closed graph.*

Let $y_n \in U \longrightarrow y_*$, $h_n \in N(y_n)$, and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$.

$h_n \in N(y_n)$ means that there exists $g_n \in \tilde{S}^1_{F,\overline{y}_n}$ such that for each $t \in J$

$$h_n(i) = y_n(0) + \sum_{0<l<i} [g_n(l) + \overline{y}_n(l)] - \sum_{0<l<i} y_n(l).$$

We must prove that there exists $g_* \in \tilde{S}^1_{F,\overline{y}_*}$ such that for each $k \in [1,T]$

$$h_*(i) = y_*(0) + \sum_{0<l<i} g_*(l) + \overline{y}_*(l)] - \sum_{0<l<i} y_*(l).$$

Since $y_n \in \overline{U},\ k \in \mathbb{N}$, then (H2) guarantees (see [2, p. 262]) that there exists a compact set Ω of $C([1,T],\mathbb{R})$ with $\{g_n\} \in \Omega$. Thus there exists a subsequence $\{y_{n_m}\}$ with $y_{n_m} \to y_*$ as $k \to \infty$ and $y_{n_m}(i) \in F(i, y_m(i))$ together with the map $y \to F(i,y)$ upper semicontinuous for each $i \in \mathbb{N}$. Since τ and y are continuous, we have

$$\left\| \left(h_n - y_n(0) - \sum_{0<l<i} [\overline{y}_n(l) - y_n(l)] \right) \right.$$

$$- \left(h_* - y_*(0) \sum_{0<l<i} [\overline{y}_*(l) - y_*(l)] \right) \Big\|_\infty \longrightarrow 0, \quad \text{as} \quad n \to \infty.$$

Consider the linear continuous operator (topology on $\mathbb{N}$ is the discrete topology)

$$\Gamma : C([1,T],\mathbb{R}) \longrightarrow C([1,T],\mathbb{R})$$

$$g \longmapsto (\Gamma g)(i) = \sum_{0<l<i} g(l).$$

Moreover, we have that

$$\left(h_n(i) - y_n(0) - \sum_{0<l<i} [\overline{y}_n(l) - y_n(l)] \right) = \Gamma(g_n)(i) \in F_1(i, y_n(i)).$$

Since $y_n \longrightarrow y_*$, it that

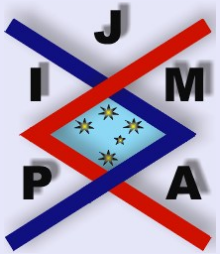
$$\left(h_*(i) - y_*(0) - \sum_{0<l<i} [\overline{y}_*(l) - y_*(l) \right) = \sum_{0<l<i} g_*(l)$$

for some $g_* \in \tilde{S}^1_{F,y_*}$.

Lemma 2.1 guarantees that N has a fixed point which is a solution to problem (3.1) – (3.2).

Step 6: *The solution y of (3.1) – (3.2) satisfies*

$$\alpha(i) \leq y(i) \leq \beta(i) \quad \text{for all} \quad i \in J.$$



Let y be a solution to (3.1) – (3.2). We prove that

$$y(i) \leq \beta(i) \text{ for all } i \in [1, T].$$

Assume that $y - \beta$ attains a positive maximum on $[1, T]$ at $\overline{k} - 1 \in [1, T]$ that is,

$$(y - \beta)(\overline{k}) = \max\{y(k) - \beta(k) : k \in [1, T]\} > 0.$$

By the definition of τ one has

$$\Delta y(\overline{k}) + y(\overline{k}) \in F(t, \beta(\overline{k})) + \beta(\overline{k}).$$

Thus there exists $v(i) \in F(\overline{k}, \beta(\overline{k})),$ with $v(\overline{k}) \leq v_2(\overline{k})$ such that

$$\Delta y(\overline{k} - 1) = v(\overline{k}) + \beta(\overline{k} - 1) - y(\overline{k}),$$

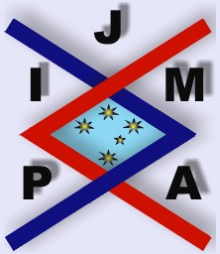
$$\begin{aligned} \Delta y(\overline{k} - 1) &= v(\overline{k}) - y(\overline{k}) + \beta(\overline{k}) \\ &\leq v_2(\overline{k}) - (y(\overline{k}) - \beta(\overline{k})) < v_2(\overline{k}). \end{aligned}$$

Using the fact that β is an upper solution to (1.1) – (1.2) the above inequality yields

$$\begin{aligned} \beta(\overline{k}) - \beta(\overline{k} - 1) &\geq v_2(\overline{k}) \\ &> y(\overline{k}) - y(\overline{k} - 1). \end{aligned}$$

Thus we obtain the contradiction

$$y(\overline{k} - 1) - \beta(\overline{k} - 1) > y(\overline{k}) - \beta(\overline{k}).$$



Thus

$$y(i) \leq \beta(i) \ \text{ forall } \ i \in [1, T].$$

Analogously, we can prove that

$$y(i) \geq \alpha(i) \ \text{ for all } \ i \in [1, T].$$

This shows that the problem (3.1) – (3.2) has a solution in the interval $[\alpha, \beta]$.

Finally, we prove that every solution of (3.1) – (3.2) is also a solution to (1.1) – (1.2). We only need to show that

$$\alpha(0) \leq y(0) - L(\overline{y}(0), \overline{y}(T+1)) \leq \beta(0).$$

Notice first that we can prove

$$\alpha(T+1) \leq y(T+1) \leq \beta(T+1).$$

Suppose now that $y(0) - L(\overline{y}(0), \overline{y}(T+1)) < \alpha(0)$. Then $y(0) = \alpha(0)$ and

$$y(0) - L(\alpha(0), \overline{y}(T)) \leq \alpha(0).$$

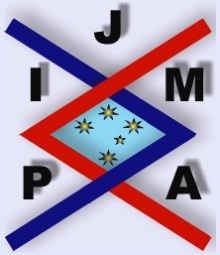
Since L is nonincreasing in y, we have

$$\alpha(0) \leq \alpha(0) - L(\alpha(0), \alpha(T+1)) \leq \alpha(0) - L(\alpha(0), \overline{y}(T+1)) < \alpha(0),$$

which is a contradiction. Analogously, we can prove that

$$y(0) - L(\overline{y}(0), \overline{y}(T+1)) \leq \beta(0).$$

Then y is a solution to (1.1) – (1.2). □

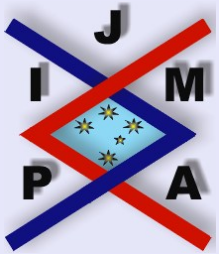


Remark 2. *Observe that if* $L(x, y) = ax - by - c$, *then Theorem 3.1 gives an existence result for the problem*

$$\Delta y(i) \in F(i, y(i)), \qquad i \in [1, T] = \{1, 2, \ldots, T\},$$

$$ay(0) - by(T) = c,$$

with $a,\ b \geq 0,\ a + b > 0$, *which includes the periodic case* $(a = b = 1,\ c = 0)$ *and the initial and the terminal problem.*



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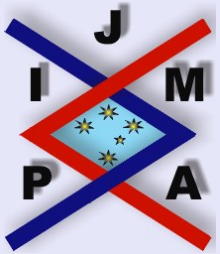

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