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ON A CERTAIN CLASS OF $p-$VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. In this paper, we introduce the class $A_o^*(p, A, B, \alpha)$ of p-valent functions in the unit disc $U = \{z : |z| < 1\}$. We obtain coefficient estimate, distortion and closure theorems, radii of close-to convexity, starlikeness and convexity of order δ $(0 \leqslant \delta < 1)$ for this class. We also obtain class preserving integral operators for this class. Furthermore, various distortion inequalities for fractional calculus of functions in this class are also given.

Key words and phrases: p-valent, Coefficient, Distortion, Closure, Starlike, Convex, Fractional calculus, Integral operators.

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1. INTRODUCTION

Let $A(n)$ be the class of functions f, analytic and $p-$valent in $U = \{z : |z| < 1\}$ given by

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \quad a_{p+n} > 0. \tag{1.1}$$

A function f belonging to the class $A(n)$ is said to be in the class $A_m^*(p, A, B, \alpha)$ if and only if

$$(p-1) + \operatorname{Re}\left\{\frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right\} > 0 \quad \text{for } z \in U.$$

In the other words, $f \in A_m^*(p, A, B, \alpha)$ if and only if it satisfies the condition

$$\left|\frac{(p-1) + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - p}{(A-B)(p-\alpha) + pB - B\left[(p-1) + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right]}\right| < 1$$

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where $-1 \leq B < A \leq 1$, $-1 \leq B < 0$ and $0 \leq \alpha < p$. Let A_m denote the subclass of $A(n)$ consisting of functions analytic and $p-$valent which can be expressed in the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}; \quad a_{p+n} \geq 0. \tag{1.2}$$

Let us define

$$A_o^*(p, A, B, \alpha) = A_m^*(p, A, B, \alpha) \bigcap A_m.$$

In this paper, we obtain a coefficient estimate, distortion theorems, integral operators and radii of close-to-convexity, starlikeness and convexity, closure properties and distortion inequalities for fractional calculus. This paper is motivated by an earlier work of Nunokawa [1].

2. Coefficient Estimates

Theorem 2.1. *If the function f is defined by (1.1), then $f \in A_o^*(p, A, B, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \frac{(p+n)!\,[n(1-B) + (A-B)(p-\alpha)]}{(n+1)!} a_{p+n} \leq (A-B)(p-\alpha)p!. \tag{2.1}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds true and let $|z| = 1$. Then we obtain

$$\begin{aligned}
&\left| z f^{(p)}(z) - f^{(p-1)}(z) \right| - \left| (A-B)(p-\alpha) f^{(p-1)} - Bzf^{(p)} + Bf^{(p-1)} \right| \\
&= \left| -\sum_{n=1}^{\infty} \frac{n(p+n)!}{(n+1)!} a_{p+n} z^{n+1} \right| - \Bigg| (A-B)(p-\alpha)p!z \\
&\qquad - \left[(A-B)(p-\alpha) \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} a_{p+n} z^{n+1} - B \sum_{n=1}^{\infty} \frac{n(p+n)!}{(n+1)!} a_{p+n} z^{n+1} \right] \Bigg| \\
&\leq \sum_{n=1}^{\infty} \frac{(p+n)!\,[n(1-B) + (A-B)(p-\alpha)]}{(n+1)!} a_{p+n} - (A-B)(p-\alpha)p! \leq 0
\end{aligned}$$

by hypothesis. Hence, by the maximum modulus theorem, we have $f \in A_o^*(p, A, B, \alpha)$. To prove the converse, assume that

$$\begin{aligned}
&\left| \frac{(p-1) + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - p}{(A-B)(p-\alpha) + pB - B\left[(p-1) + \frac{zf^{(p)}(z)}{f^{(p-1)}(z)}\right]} \right| \\
&\qquad = \left| \frac{-\sum_{n=1}^{\infty} \frac{n(p+n)!}{(n+1)!} a_{p+n} z^{n+1}}{(A-B)(p-\alpha)\left(p!z - \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} a_{p+n} z^{n+1}\right) + B \sum_{n=1}^{\infty} \frac{n(p+n)!}{(n+1)!} a_{p+n} z^{n+1}} \right| < 1.
\end{aligned}$$

Since $\mathrm{Re}(z) \leq |z|$ for all z, we have

$$\mathrm{Re}\left\{ \frac{-\sum_{n=1}^{\infty} \frac{n(p+n)!}{(n+1)!} a_{p+n} z^{n+1}}{(A-B)(p-\alpha)\left(p!z - \sum_{n=1}^{\infty} \frac{(p+n)!}{(n+1)!} a_{p+n} z^{n+1}\right) + B \sum_{n=1}^{\infty} \frac{n(p+n)!}{(n+1)!} a_{p+n} z^{n+1}} \right\} < 1. \tag{2.2}$$

Choosing values of z on the real axis and letting $z \to 1^-$ through real values, we obtain

$$\sum_{n=1}^{\infty} \frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!} a_{p+n} \le (A-B)(p-\alpha)p!, \tag{2.3}$$

which obviously is required assertion (2.1). Finally, sharpness follows if we take

$$f(z) = z^p - \frac{(A-B)(p-\alpha)p!(n+1)!}{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]} z^{p+n}. \tag{2.4}$$

□

Corollary 2.2. *If $f \in A_o^*(p,A,B,\alpha)$, then*

$$a_{p+n} \le \frac{(A-B)(p-\alpha)p!(n+1)!}{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}. \tag{2.5}$$

The equality in (2.5) is attained for the function f given by (2.4).

3. DISTORTION PROPERTIES

Theorem 3.1. *If $f \in A_o^*(p,A,B,\alpha)$, then for $|z| = r < 1$*

$$\begin{aligned} r^p - \frac{2(A-B)(p-\alpha)}{(p+1)\,[(1-B)+(A-B)(p-\alpha)]} r^{p+1} \\ \le |f(z)| \le r^p + \frac{2(A-B)(p-\alpha)}{(p+1)\,[(1-B)+(A-B)(p-\alpha)]} r^{p+1} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} pr^{p-1} - \frac{2(A-B)(p-\alpha)}{(1-B)+(A-B)(p-\alpha)} r^p \\ \le |f'(z)| \le pr^{p-1} + \frac{2(A-B)(p-\alpha)}{(1-B)+(A-B)(p-\alpha)} r^p. \end{aligned} \tag{3.2}$$

All the inequalities are sharp.

Proof. Let

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, a_{p+n} > 0.$$

From Theorem 2.1, we have

$$\begin{aligned} &\frac{(p+1)!\,[(1-B)+(A-B)(p-\alpha)]}{2} \sum_{n=1}^{\infty} a_{p+n} \\ &\le \sum_{n=1}^{\infty} \frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!} a_{p+n} \\ &\le (A-B)(p-\alpha)p! \end{aligned}$$

which

$$\sum_{n=1}^{\infty} a_{p+n} \le \frac{2(A-B)(p-\alpha)}{(p+1)\,[(1-B)+(A-B)(p-\alpha)]} \tag{3.3}$$

and

$$\sum_{n=1}^{\infty} (p+n) a_{p+n} \le \frac{2(A-B)(p-\alpha)}{(1-B)+(A-B)(p-\alpha)}. \tag{3.4}$$

Consequently, for $|z| = r < 1$, we obtain

$$|f(z)| \leq r^p + r^{p+1} \sum_{n=1}^{\infty} a_{p+n} \leq r^p + \frac{2(A-B)(p-\alpha)}{(p+1)\left[(1-B)+(A-B)(p-\alpha)\right]} r^{p+1}$$

and

$$|f(z)| \geq r^p - r^{p+1} \sum_{n=1}^{\infty} a_{p+n} \geq r^p - \frac{2(A-B)(p-\alpha)}{(p+1)\left[(1-B)+(A-B)(p-\alpha)\right]} r^{p+1}$$

which prove that the assertion (3.1) of Theorem 3.1 holds.

The inequalities in (3.2) can be proved in a similar manner and we omit the details. □

The bounds in (3.1) and (3.2) are attained for the function f given by

$$f(z) = z^p - \frac{2(A-B)(p-\alpha)}{(p+1)\left[(1-B)+(A-B)(p-\alpha)\right]} z^{p+1}. \tag{3.5}$$

Letting $r \to 1^-$ in the left hand side of (3.1), we have the following:

Corollary 3.2. *If $f \in A_o^*(p, A, B, \alpha)$, then the disc $|z| < 1$ is mapped by f onto a domain that contains the disc*

$$|w| < \frac{(p+1)(1-B)+(A-B)(p-\alpha)(p-1)}{(p+1)\left[(1-B)+(A-B)(p-\alpha)\right]}.$$

The result is sharp with the extremal function f being given by (3.5).

Putting $\alpha = 0$ in Theorem 3.1 and Corollary 3.2, we get

Corollary 3.3. *If $f \in A_o^*(p, A, B, 0)$, then for $|z| = r$*

$$r^p - \frac{2p(A-B)}{(p+1)\left[(1-B)+p(A-B)\right]} r^{p+1} \leq |f(z)| \leq r^p + \frac{2p(A-B)}{(p+1)\left[(1-B)+p(A-B)\right]} r^{p+1}$$

and

$$pr^{p-1} - \frac{2p(A-B)}{(1-B)+p(A-B)} r^p \leq |f'(z)| \leq pr^{p-1} + \frac{2p(A-B)}{(1-B)+p(A-B)} r^p.$$

The result is sharp with the extremal function

$$f(z) = z^p - \frac{2p(A-B)}{(p+1)\left[(1-B)+p(A-B)\right]} z^{p+1}; \quad z = \mp r. \tag{3.6}$$

Corollary 3.4. *If $f \in A_o^*(p, A, B, 0)$, then the disc $|z| < 1$ is mapped by f onto a domain that contains the disc*

$$|w| < \frac{(p+1)(1-B)+p(p-1)(A-B)}{(p+1)\left[(1-B)+p(A-B)\right]}.$$

The result is sharp with the extremal function f being given by (3.6).

4. Radii of Close-To-Convexity, Starlikeness and Convexity

Theorem 4.1. *Let $f \in A_o^*(p, A, B, \alpha)$. Then f is $p-$valent close-to-convex of order δ $(0 \leq \delta < p)$ in $|z| < R_1$, where*

$$R_1 = \inf_n \left\{ \left[\frac{(p+n)![n(1-B)+(A-B)(p-\alpha)]}{(A-B)(p-\alpha)(n+1)p!} \left(\frac{p-\delta}{p+n} \right) \right]^{\frac{1}{n}} \right\}. \tag{4.1}$$

Theorem 4.2. *If $f \in A_o^*(p, A, B, \alpha)$, then f is $p-$valent starlike of order δ $(0 \leq \delta < p)$ in $|z| < R_2$, where*

$$R_2 = \inf_n \left\{ \left[\frac{(p+n)![n(1-B)+(A-B)(p-\alpha)]}{(A-B)(p-\alpha)(n+1)!p!} \left(\frac{p-\delta}{p+n-\delta} \right) \right]^{\frac{1}{n}} \right\}. \tag{4.2}$$

Theorem 4.3. *If $f \in A_o^*(p, A, B, \alpha)$, then f is a $p-$valent convex function of order δ $(0 \leq \delta < p)$ in $|z| < R_3$, where*

$$R_3 = \inf_n \left\{ \left[\frac{[n(1-B)+(A-B)(p-\alpha)](p+n-1)!}{(A-B)(p-\alpha)(n+1)!(p-1)!} \left(\frac{p-\delta}{p+n-\delta} \right) \right]^{\frac{1}{n}} \right\}. \tag{4.3}$$

In order to establish the required results in Theorems 4.1, 4.2 and 4.3, it is sufficient to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \delta \quad \text{for} \quad |z| < R_1,$$

$$\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \delta \quad \text{for} \quad |z| < R_2 \quad \text{and}$$

$$\left| \left[1 + \frac{zf''(z)}{f'(z)} \right] - p \right| \leq p - \delta \quad \text{for} \quad |z| < R_3,$$

respectively.

Remark 4.4. The results in Theorems 4.1, 4.2 and 4.3 are sharp with the extremal function f given by (2.4). Furthermore, taking $\delta = 0$ in Theorems 4.1, 4.2 and 4.3, we obtain radius of close-to-convexity, starlikeness and convexity, respectively.

5. Integral Operators

Theorem 5.1. *Let c be a real number such that $c > -p$. If $f \in A_o^*(p, A, B, \alpha)$, then the function F defined by*

$$F(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt \tag{5.1}$$

also belongs to $A_o^*(p, A, B, \alpha)$.

Proof. Let

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n}.$$

Then from the representation of F, it follows that

$$F(z) = z^p - \sum_{n=1}^{\infty} b_{p+n} z^{p+n},$$

where $b_{p+n}=\left(\frac{c+p}{c+p+n}\right)a_{p+n}$. Therefore using Theorem 2.1 for the coefficients of F, we have

$$\begin{aligned}\sum_{n=1}^{\infty}&\frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!}b_{p+n}\\&=\sum_{n=1}^{\infty}\frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!}\left(\frac{c+p}{c+p+n}\right)a_{p+n}\\&\le(A-B)(p-\alpha)p!\end{aligned}$$

since $\frac{c+p}{c+p+n}<1$ and $f\in A_o^*(p,A,B,\alpha)$. Hence $F\in A_o^*(p,A,B,\alpha)$. □

Theorem 5.2. *Let c be a real number such that $c>-p$. If $F\in A_o^*(p,A,B,\alpha)$, then the function f defined by (5.1) is $p-$valent in $|z|<R^*$, where*

$$R^*=\inf_n\left\{\left[\left(\frac{c+p}{c+p+n}\right)\frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!(A-B)(p-\alpha)p!}\left(\frac{p}{p+n}\right)\right]^{\frac{1}{n}}\right\}. \tag{5.2}$$

The result is sharp. Sharpness follows if we take

$$f(z)=z^p-\left(\frac{c+p+n}{c+p}\right)\frac{(n+1)!(A-B)(p-\alpha)p!}{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}z^{p+n}.$$

6. Closure Properties

In this section we show that the class $A_o^*(p,A,B,\alpha)$ is closed under "arithmetic mean" and "convex linear combinations".

Theorem 6.1. *Let*

$$f_j(z)=z^p-\sum_{n=1}^{\infty}a_{p+n,j}z^{p+n},\quad j=1,2,\ldots$$

and

$$h(z)=z^p-\sum_{n=1}^{\infty}b_{p+n}z^{p+n},$$

where

$$b_{p+n}=\sum_{j=1}^{\infty}\lambda_j a_{p+n,j},\quad \lambda_j>0$$

and $\sum_{j=1}^{\infty}\lambda_j=1$. If $f_j\in A_o^*(p,A,B,\alpha)$ for each $j=1,2,\ldots$, then $h\in A_o^*(p,A,B,\alpha)$.

Proof. If $f_j\in A_o^*(p,A,B,\alpha)$, then we have from Theorem 2.1 that

$$\sum_{n=1}^{\infty}\frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!}a_{p+n,j}\le(A-B)(p-\alpha)p!,\quad j=1,2,\ldots.$$

Therefore

$$\begin{aligned}\sum_{n=1}^{\infty}&\frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!}b_{p+n}\\&=\sum_{n=1}^{\infty}\left[\frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!}\left(\sum_{j=1}^{\infty}\lambda_j a_{p+n,j}\right)\right]\\&\le(A-B)(p-\alpha)p!.\end{aligned}$$

Hence, by Theorem 2.1, $h\in A_o^*(p,A,B,\alpha)$. □

Theorem 6.2. *The class $A_o^*(p, A, B, \alpha)$ is closed under convex linear combinations.*

Theorem 6.3. *Let $f_p(z) = z^p$ and*

$$f_{p+n} = z^p - \frac{(A-B)(p-\alpha)(n+1)!p!}{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]} z^{p+n} \quad (n \geq 1).$$

Then $f \in A_o^*(p, A, B, \alpha)$ if and only if it can be expressed in the form

$$f(z) = \lambda_p f_p(z) + \sum_{n=1}^{\infty} \lambda_n f_{p+n}(z), \quad z \in U,$$

where $\lambda_n \geq 0$ and $\lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_n$.

Proof. Let us assume that

$$\begin{aligned} f(z) &= \lambda_p f_p(z) + \sum_{n=1}^{\infty} \lambda_n f_{p+n}(z) \\ &= z^p - \sum_{n=1}^{\infty} \frac{(A-B)(p-\alpha)(n+1)!p!}{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]} \lambda_n z^{p+n}. \end{aligned}$$

Then from Theorem 2.1 we have

$$\begin{aligned} \sum_{n=1}^{\infty} & \frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!} \\ & \qquad \times \frac{(A-B)(p-\alpha)(n+1)!p!}{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]} \lambda_n \\ & \qquad\qquad \leq (A-B)(p-\alpha)p!. \end{aligned}$$

Hence $f \in A_o^*(p, A, B, \alpha)$. Conversely, let $f \in A_o^*(p, A, B, \alpha)$. It follows from Corollary 2.2 that

$$a_{p+n} \leq \frac{(A-B)(p-\alpha)(n+1)!p!}{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}.$$

Setting

$$\lambda_n = \frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(A-B)(p-\alpha)(n+1)!p!} a_{p+n}, \quad n = 1, 2, \dots$$

and $\lambda_p = 1 - \sum_{n=1}^{\infty} \lambda_n$, we have

$$\begin{aligned} f(z) &= z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \\ &= z^p - \sum_{n=1}^{\infty} \lambda_n z^p + \sum_{n=1}^{\infty} \lambda_n z^p - \sum_{n=1}^{\infty} \lambda_n \frac{(A-B)(p-\alpha)(n+1)!p!}{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]} z^{p+n} \\ &= \lambda_p f_p(z) + \sum_{n=1}^{\infty} \lambda_n f_{p+n}(z). \end{aligned}$$

This completes the proof of Theorem 6.3. □

7. DEFINITIONS AND APPLICATIONS OF FRACTIONAL CALCULUS

In this section, we shall prove several distortion theorems for functions to general class $A_o^*(p,A,B,\alpha)$. Each of these theorems would involve certain operators of fractional calculus we find it to be convenient to recall here the following definition which were used recently by Owa [2] (and more recently, by Owa and Srivastava [3], and Srivastava and Owa [4] ; see also Srivastava et al. [5]).

Definition 7.1. The fractional integral of order λ is defined, for a function f, by

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\lambda}} d\zeta \quad (\lambda > 0), \tag{7.1}$$

where f is an analytic function in a simply – connected region of the z -plane containing the origin, and the multiplicity of $(z-\zeta)^{\lambda-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta>0$.

Definition 7.2. The fractional derivative of order λ is defined, for a function f, by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \le \lambda < 1), \tag{7.2}$$

where f is constrained, and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed, as in Definition 7.1.

Definition 7.3. Under the hypotheses of Definition 7.2, the fractional derivative of order $(n+\lambda)$ is defined by

$$D_z^{n+\lambda} f(z) = \frac{d^n}{dz^n} D_z^{\lambda} f(z) \quad (0 \le \lambda < 1), \tag{7.3}$$

where $0 \le \lambda < 1$ and $n \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$. From Definition 7.2, we have

$$D_z^0 f(z) = f(z) \tag{7.4}$$

which, in view of Definition 7.3 yields,

$$D_z^{n+0} f(z) = \frac{d^n}{dz^n} D_z^0 f(z) = f^n(z). \tag{7.5}$$

Thus, it follows from (7.4) and (7.5) that

$$\lim_{\lambda \to 0} D_z^{-\lambda} f(z) = f(z) \quad \text{and} \quad \lim_{\lambda \to 0} D_z^{1-\lambda} f(z) = f'(z).$$

Theorem 7.1. *Let the function f defined by (1.2) be in the class $A_o^*(p,A,B,\alpha)$. Then for $z \in U$ and $\lambda > 0$,*

$$\left|D_z^{-\lambda} f(z)\right| \ge |z|^{p+\lambda} \left\{ \frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)} - \frac{2(A-B)(p-\alpha)\Gamma(p+1)}{(\lambda+p+1)\Gamma(\lambda+p+1)\left[(1-B)+(A-B)(p-\alpha)\right]} |z| \right\}$$

and

$$\left|D_z^{-\lambda} f(z)\right| \le |z|^{p+\lambda} \left\{ \frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)} + \frac{2(A-B)(p-\alpha)\Gamma(p+1)}{(\lambda+p+1)\Gamma(\lambda+p+1)\left[(1-B)+(A-B)(p-\alpha)\right]} |z| \right\}.$$

The result is sharp.

Proof. Let

$$\begin{aligned} F(z) &= \frac{\Gamma(p+1+\lambda)}{\Gamma(p+1)} z^{-\lambda} D_z^{-\lambda} f(z) \\ &= z^p - \sum_{n=1}^{\infty} \frac{\Gamma(p+n+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(p+n+\lambda+1)} a_{p+n} z^{p+n} \\ &= z^p - \sum_{n=1}^{\infty} \varphi(n) a_{p+n} z^{p+n}, \end{aligned}$$

where

$$\varphi(n) = \frac{\Gamma(p+n+1)\Gamma(p+\lambda+1)}{\Gamma(p+1)\Gamma(p+n+\lambda+1)}, \quad (\lambda > 0,\ n \in \mathbb{N}).$$

Then by using $0 < \varphi(n) \leq \varphi(1) = \frac{p+1}{p+\lambda+1}$ and Theorem 2.1, we observe that

$$\begin{aligned} &\frac{(p+1)!\,[(1-B)+(A-B)(p-\alpha)]}{2!} \sum_{n=1}^{\infty} a_{p+n} \\ &\qquad \leq \sum_{n=1}^{\infty} \frac{(p+n)!\,[n(1-B)+(A-B)(p-\alpha)]}{(n+1)!} a_{p+n} \\ &\qquad \leq (A-B)(p-\alpha)p!, \end{aligned}$$

which shows that $F(z) \in A_o^*(p,A,B,\alpha)$. Consequently, with the aid of Theorem 3.1, we have

$$\begin{aligned} |F(z)| &\geq |z^p| - \varphi(1)\,|z|^{p+1} \sum_{n=1}^{\infty} a_{p+n} \\ &\geq |z|^p - \frac{2(A-B)(p-\alpha)}{(p+\lambda+1)[(1-B)+(A-B)(p-\alpha)]}\,|z|^{p+1} \end{aligned}$$

and

$$\begin{aligned} |F(z)| &\leq |z^p| + \varphi(1)\,|z|^{p+1} \sum_{n=1}^{\infty} a_{p+n} \\ &\leq |z|^p + \frac{2(A-B)(p-\alpha)}{(p+\lambda+1)[(1-B)+(A-B)(p-\alpha)]}\,|z|^{p+1} \end{aligned}$$

which completes the proof of Theorem 7.1.By letting $\lambda \to 0$, Theorem 7.1 reduces at once to Theorem 3.1. □

Corollary 7.2. *Under the hypotheses of Theorem 7.1, $D_z^{-\lambda} f(z)$ is included in a disk with its center at the origin and radius $R_1^{-\lambda}$ given by*

$$R_1^{-\lambda} = \left\{ \frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)} \right\} \left\{ 1 + \frac{2(A-B)(p-\alpha)}{(p+\lambda+1)[(1-B)+(A-B)(p-\alpha)]} \right\}.$$

Theorem 7.3. *Let the function f defined by (1.2) be in the class $A_o^*(p,A,B,\alpha)$. Then,*

$$\begin{aligned} \left| D_z^{\lambda} f(z) \right| \geq |z|^{p-\lambda} \Bigg\{ &\frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)} \\ &- \frac{2(A-B)(p-\alpha)\Gamma(2-\lambda)\Gamma(p+1)}{\Gamma(p-\lambda+1)\Gamma(p-\lambda+2)[(1-B)+(A-B)(p-\alpha)]}\,|z| \Bigg\} \end{aligned}$$

and

$$\left|D_z^\lambda f(z)\right| \leq |z|^{p-\lambda}\left\{\frac{\Gamma(p+1)}{\Gamma(p-\lambda+1)}\right.$$
$$\left.+\frac{2(A-B)(p-\alpha)\Gamma(2-\lambda)\Gamma(p+1)}{\Gamma(p-\lambda+1)\Gamma(p-\lambda+2)[(1-B)+(A-B)(p-\alpha)]}|z|\right\}$$

for $0 \leq \lambda < 1$.

Proof. Using similar arguments as given by Theorem 7.1, we can get the result. ☐

Corollary 7.4. *Under the hypotheses of Theorem 7.3, $D_z^\lambda f(z)$ is included in the disk with its center at the origin and radius R_2^λ given by*

$$R_2^\lambda = \left\{\frac{\Gamma(p+1)}{\Gamma(\lambda+p+1)}\right\}\left\{1+\frac{2(A-B)(p-\alpha)\Gamma(2-\lambda)}{\Gamma(p-\lambda+1)[(1-B)+(A-B)(p-\alpha)]}\right\}.$$

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