# Journal of Inequalities in Pure and Applied Mathematics
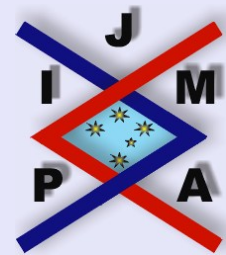
## A GENERALIZATION OF AN INEQUALITY INVOLVING THE GENERALIZED ELEMENTARY SYMMETRIC MEAN

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Abstract

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## Abstract

A generalization of an inequality involving the generalized elementary symmetric mean and its elementary proof are given.

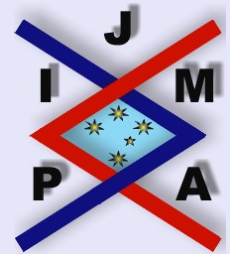# Contents

A Generalization of an
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# 1.  Introduction

Let $a = (a_1, a_2, \cdots, a_n)$ and $r$ be a nonnegative integer, where $a_i$ for $1 \leq i \leq n$ are nonnegative real numbers. Then

$$(1.1) \qquad E_n^{[r]} = E_n^{[r]}(a) = \sum_{\substack{i_1 + i_2 + \cdots + i_n = r, \\ i_1, i_2, \cdots, i_n \geq 0 \text{ are integers}}} \prod_{k=1}^{n} a_k^{i_k}$$

with $E_n^{[0]} = E_n^{[0]}(a) = 1$ for $n \geq 1$ and $E_n^{[r]} = 0$ for $r < 0$ or $n \leq 0$ is called the $r$th generalized elementary symmetric function of $a$.

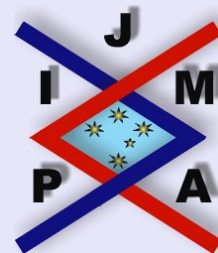The $r$th generalized elementary symmetric mean of $a$ is defined by

$$(1.2) \qquad \sum_n^{[r]} = \sum_n^{[r]}(a) = \frac{E_n^{[r]}(a)}{\binom{n+r-1}{r}}.$$

In 1934, I. Schur [5, p. 182] obtained the following

$$(1.3) \qquad \sum_n^{[r]}(a) = (n-1)! \int \cdots \int \left( \sum_{i=1}^{n} a_i x_i \right)^r \mathrm{d}x_1 \cdots \mathrm{d}x_{n-1},$$

where $x_n = 1 - (x_1 + x_2 + \cdots + x_{n-1})$ and the integral is taken over $x_k \geq 0$ for $k = 1, 2, \ldots, n-1$. By using (1.3) and Cauchy integral inequality, he also proved that

$$(1.4) \qquad \sum_n^{[r-1]}(a) \sum_n^{[r+1]}(a) \geq \left[ \sum_n^{[r]}(a) \right]^2.$$

In 1968, K.V. Menon [2] proved that for $n = 2$ or $n \geq 3$ and $r = 1, 2, 3$, inequality (1.4) is valid.

In [1], the generalized symmetric means of two variables was investigated.

In [3] and [4] a problem was posed: Does inequality (1.4) hold for arbitrary $n, r \in \mathbb{N}$?

In 1997, Zh.-H. Zhang generalized (1.3) in [7] and also proved (1.4) by a similar proof as in [5].

In [6], some inequalities of weighted symmetric mean were established.

In this paper, we shall obtain an identity relating $\sum_{n}^{[r]}(a)$ to $E_n^{[r]}(a)$ and give an elementary proof of an inequality which generalizes (1.4). Our main result is as follows.

**Theorem 1.1.** *If $r, s \in \mathbb{N}$ and $r > s$, then*

$$(1.5) \qquad \sum_{n}^{[s]}(a) \sum_{n}^{[r+1]}(a) \geq \sum_{n}^{[r]}(a) \sum_{n}^{[s+1]}(a).$$

*The equality in (1.5) holds if and only if $a_1 = a_2 = \cdots = a_n$.*

Letting $s = r - 1$ in inequality (1.5) leads to inequality (1.4).
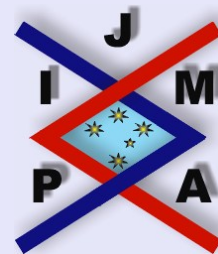
## 2.   Proof of Theorem 1.1

To prove inequality (1.5), the following properties for $E_n^{[r]}$ are necessary.

**Property 1.** *If* $n, r \in \mathbb{N}$, *then*

$$(2.1) \qquad E_n^{[r]} = E_{n-1}^{[r]} + a_n E_n^{[r-1]}$$

*and*

$$(2.2) \qquad E_n^{[r]} = \sum_{j=0}^{r} a_n^j E_{n-1}^{[r-j]}.$$

*Proof.* If $n = 1$ or $r = 0$, (2.1) holds trivially.
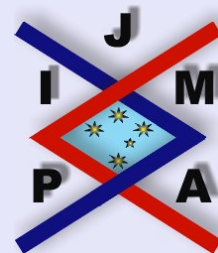
When $n > 1$ and $r \geq 1$, we have

$$(2.3) \quad \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\cdots,i_n \geq 0}} \prod_{k=1}^{n} a_k^{i_k} = \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1,i_2,\cdots,i_{n-1} \geq 0 \\ i_n=0}} \prod_{k=1}^{n} a_k^{i_k} + a_n \sum_{\substack{i_1+i_2+\cdots+i_n=r-1 \\ i_1,i_2,\cdots,i_n \geq 0}} \prod_{k=1}^{n} a_k^{i_k}.$$

Combining the definition of $E_n^{[r]}$ and (2.3), identity (2.1) follows.
Identity (2.2) can be deduced from the recurrence of (2.1). □

**Property 2.** *If* $r$ *is an integer, then*

$$(2.4) \qquad (r+1)E_n^{[r+1]} = \sum_{k=0}^{r} \left( \sum_{i=1}^{n} a_i^{k+1} \right) E_n^{[r-k]}.$$

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*Proof.* It will be verified by induction. It is clear that identity (2.4) holds trivially for $n = 1$. Suppose identity (2.4) is true for $n - 1$ and nonnegative integers $r$.
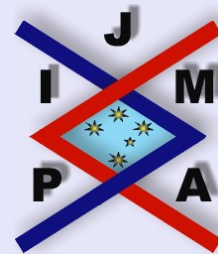
By (2.2), for $0 \le k \le r$, we have

$$E_n^{[r-k]} = \sum_{j=0}^{r-k} a_n^j E_{n-1}^{[r-k-j]},$$

and

$$\sum_{j=0}^{r} (j+1) a_n^{j+1} E_{n-1}^{[r-j]} = a_n E_{n-1}^{[r]} + a_n^2 E_{n-1}^{[r-1]} + \cdots + a_n^r E_{n-1}^{[1]} + a_n^{r+1} E_{n-1}^{[0]}$$

$$+ a_n^2 E_{n-1}^{[r-1]} + \cdots + a_n^r E_{n-1}^{[1]} + a_n^{r+1} E_{n-1}^{[0]}$$

$$\cdots \cdots$$

$$+ a_n^r E_{n-1}^{[1]} + a_n^{r+1} E_{n-1}^{[0]}$$

$$+ a_n^{r+1} E_{n-1}^{[0]}$$

$$= \sum_{k=0}^{r} \sum_{j=0}^{r-k} a_n^{j+k+1} E_{n-1}^{[r-k-j]}.$$

According to the inductive hypothesis, for nonnegative integers $r$ and $0 \le j \le r$, we have

$$(2.5) \qquad (r - j + 1) E_{n-1}^{[r+1-j]} = \sum_{k=0}^{r-j} \left( \sum_{i=1}^{n-1} a_i^{k+1} \right) E_{n-1}^{[r-k-j]}.$$

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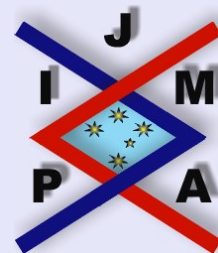
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From Property 1 and the above formula, we have

$$(2.6) \quad (r+1)E_n^{[r+1]}$$

$$= (r+1)\sum_{j=0}^{r+1} a_n^j E_{n-1}^{[r+1-j]}$$

$$= \sum_{j=0}^{r}(r-j+1)a_n^j E_{n-1}^{[r-j+1]} + \sum_{j=1}^{r+1} j a_n^j E_{n-1}^{[r-j+1]}$$

$$= \sum_{j=0}^{r} a_n^j \sum_{k=0}^{r-j}\left(\sum_{i=1}^{n-1} a_i^{k+1}\right) E_{n-1}^{[r-j-k]} + \sum_{j=0}^{r}(j+1)a_n^{j+1} E_{n-1}^{[r-j]}$$

$$= \sum_{k=0}^{r}\sum_{j=0}^{r-k} a_n^j \left(\sum_{i=1}^{n-1} a_i^{k+1}\right) E_{n-1}^{[r-j-k]} + \sum_{k=0}^{r}\sum_{j=0}^{r-k} a_n^{j+k+1} E_{n-1}^{[r-k-j]}$$

$$= \sum_{k=0}^{r}\left(\sum_{i=1}^{n-1} a_i^{k+1} + a_n^{k+1}\right)\left(\sum_{j=0}^{r-k} a_n^j E_{n-1}^{[r-k-j]}\right)$$

$$= \sum_{k=0}^{r}\left(\sum_{i=1}^{n} a_i^{k+1}\right) E_n^{[r-k]}.$$

This shows that (2.4) holds for $n$. The proof is complete. $\qquad\square$

**Property 3.** *If $r, s \in \mathbb{N}$ and $r > s$, then*

$$(2.7) \quad (r+1)(s+1)\binom{n+r}{r+1}\binom{n+s}{s+1}\left[\sum_n^{[s]}\sum_n^{[r+1]} - \sum_n^{[r]}\sum_n^{[s+1]}\right]$$
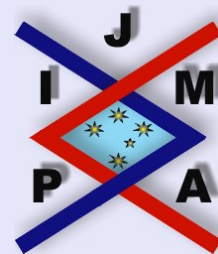
$$= \sum_{j=0}^{r} \sum_{k=0}^{j} \left[ \sum_{1 \leq v < u \leq n} \left( E_n^{[s-k]} E_n^{[r-j]} - E_n^{[s-j]} E_n^{[r-k]} \right) \right.$$

$$\left. \times \left( \sum_{t=0}^{j-k-1} a_v^{j-1-t} a_u^{k+t} \right) (a_v - a_u)^2 \right].$$

*Proof.* When $j > k$, we have

$$(2.8) \quad \sum_{i=1}^{n} a_i^k \sum_{i=1}^{n} a_i^{j+1} - \sum_{i=1}^{n} a_i^j \sum_{i=1}^{n} a_i^{k+1}$$

$$= \frac{1}{2} \sum_{v=1}^{n} \sum_{u=1}^{n} \left( a_v^k a_u^{j+1} + a_v^{j+1} a_u^k - a_v^j a_u^{k+1} - a_v^{k+1} a_u^j \right)$$

$$= \frac{1}{2} \sum_{v=1}^{n} \sum_{u=1}^{n} \left[ a_v^k a_u^k \left( a_v^{j-k+1} + a_u^{j-k+1} - a_v^{j-k} a_u - a_v a_u^{j-k} \right) \right]$$

$$= \frac{1}{2} \sum_{v=1}^{n} \sum_{u=1}^{n} \left[ a_v^k a_u^k \left( a_v^{j-k} - a_u^{j-k} \right) (a_v - a_u) \right]$$

$$= \sum_{1 \leq v < u \leq n} \left[ a_v^k a_u^k \left( a_v^{j-k} - a_u^{j-k} \right) (a_v - a_u) \right]$$

and

$$(2.9) \quad \left( a_v^{j-k} - a_u^{j-k} \right) = (a_v - a_u) \sum_{t=0}^{j-k-1} a_v^{j-k-1-t} a_u^t.$$

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Therefore

$$(2.10) \quad \sum_{i=1}^{n} a_i^k \sum_{i=1}^{n} a_i^{j+1} - \sum_{i=1}^{n} a_i^j \sum_{i=1}^{n} a_i^{k+1}$$
$$= \sum_{1 \le v < u \le n} \left[ \left( \sum_{t=0}^{k-j-1} a_v^{j-1-t} a_u^{k+t} \right) (a_v - a_u)^2 \right].$$
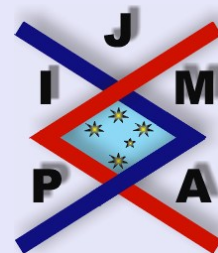
When $k > j$, we have

$$(2.11) \quad \sum_{i=1}^{n} a_i^k \sum_{i=1}^{n} a_i^{j+1} - \sum_{i=1}^{n} a_i^j \sum_{i=1}^{n} a_i^{k+1}$$
$$= - \sum_{1 \le v < u \le n} \left[ \left( \sum_{t=0}^{k-j-1} a_v^{j+t} a_u^{k-1-t} \right) (a_v - a_u)^2 \right].$$

From Property 2, it is deduced that

$$(2.12) \quad (r+1)E_n^{[r+1]} = \sum_{j=0}^{r} \left( \sum_{i=1}^{n} a_i^{j+1} \right) E_n^{[r-j]}$$

and

$$(2.13) \quad (n+r)E_n^{[r]} = nE_n^{[r]} + rE_n^{[r]} = \sum_{j=0}^{r} \left( \sum_{i=1}^{n} a_i^j \right) E_n^{[r-j]}.$$
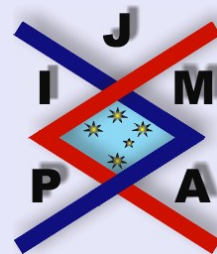
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Hence, using the above formulas and noting $E_n^{[r-k]} = 0$ for $k > r$ yields

$$(2.14) \quad (r+1)(s+1)\binom{n+r}{r+1}\binom{n+s}{s+1}\left[\sum_n^{[s]}\sum_n^{[r+1]} - \sum_n^{[r]}\sum_n^{[s+1]}\right]$$

$$= (n+s)(r+1)E_n^{[s]}E_n^{[r+1]} - (n+r)(s+1)E_n^{[r]}E_n^{[s+1]}$$

$$= \sum_{k=0}^{s}\left(\sum_{i=1}^{n}a_i^k\right)E_n^{[s-k]}\sum_{j=0}^{r}\left(\sum_{i=1}^{n}a_i^{j+1}\right)E_n^{[r-j]}$$

$$- \sum_{j=0}^{r}\left(\sum_{i=1}^{n}a_i^j\right)E_n^{[r-j]}\sum_{k=0}^{s}\left(\sum_{i=1}^{n}a_i^{k+1}\right)E_n^{[s-k]}$$

$$= \sum_{k=0}^{s}\sum_{j=0}^{r}\left(\sum_{i=1}^{n}a_i^k\sum_{i=1}^{n}a_i^{j+1} - \sum_{i=1}^{n}a_i^j\sum_{i=1}^{n}a_i^{k+1}\right)E_n^{[s-k]}\cdot E_n^{[r-j]}$$

$$= \sum_{j=0}^{r}\sum_{k=0}^{j}\left[\sum_{1\le v<u\le n}E_n^{[s-k]}E_n^{[r-j]}\left(\sum_{t=0}^{j-k-1}a_v^{j-1-t}a_u^{k+t}\right)(a_v-a_u)^2\right]$$

$$- \sum_{k=0}^{s}\sum_{j=0}^{k}\left[\sum_{1\le v<u\le n}E_n^{[s-k]}E_n^{[r-j]}\left(\sum_{t=0}^{k-j-1}a_v^{j+t}a_u^{k-1-t}\right)(a_v-a_u)^2\right]$$

$$= \sum_{j=0}^{r}\sum_{k=0}^{j}\left[\sum_{1\le v<u\le n}E_n^{[s-k]}E_n^{[r-j]}\left(\sum_{t=0}^{j-k-1}a_v^{j-1-t}a_u^{k+t}\right)(a_v-a_u)^2\right]$$

$$- \sum_{j=0}^{r}\sum_{k=0}^{j}\left[\sum_{1\le v<u\le n}E_n^{[s-j]}E_n^{[r-k]}\left(\sum_{t=0}^{j-k-1}a_v^{j-1-t}a_u^{k+t}\right)(a_v-a_u)^2\right]$$

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$$= \sum_{j=0}^{r} \sum_{k=0}^{j} \left[ \sum_{1 \le v < u \le n} \left( E_n^{[s-k]} E_n^{[r-j]} - E_n^{[s-j]} E_n^{[r-k]} \right) \right.$$

$$\left. \times \left( \sum_{t=0}^{j-k-1} a_v^{j-1-t} a_u^{k+t} \right) (a_v - a_u)^2 \right],$$
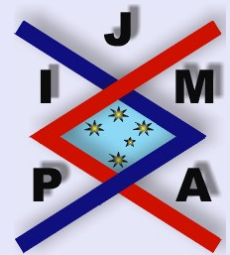
which implies the expression (2.7). □

**Property 4.** *If $r, s \in \mathbb{N}$ and $r > s$, then*

$$(2.15) \qquad E_n^{[r-1]} E_n^{[s]} \ge E_n^{[r]} E_n^{[s-1]}.$$

*The equality in (2.15) holds if and only if at least $n-1$ numbers equal zero among $\{a_1, a_2, \ldots, a_n\}$.*

*Proof.* From Property 1, we have

$$(2.16) \quad E_n^{[r-1]} E_n^{[s]} - E_n^{[r]} E_n^{[s-1]}$$

$$= E_n^{[r-1]} \left( E_{n-1}^{[s]} + a_n E_n^{[s-1]} \right) - \left( E_{n-1}^{[r]} + a_n E_n^{[r-1]} \right) E_n^{[s-1]}$$

$$= E_n^{[r-1]} E_{n-1}^{[s]} - E_{n-1}^{[r]} E_n^{[s-1]}$$

$$= \left( \sum_{j=0}^{r-1} a_n^j E_{n-1}^{[r-1-j]} \right) E_{n-1}^{[s]} - E_{n-1}^{[r]} \left( \sum_{j=0}^{s-1} a_n^j E_{n-1}^{[s-1-j]} \right)$$

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$$= \sum_{j=0}^{s-1} a_n^j \left( E_{n-1}^{[r-1-j]} E_{n-1}^{[s]} - E_{n-1}^{[r]} E_{n-1}^{[s-1-j]} \right)$$

$$+ E_{n-1}^{[s]} \left( \sum_{j=s}^{r-1} a_n^j E_{n-1}^{[r-1-j]} \right).$$

Since (2.15) holds for $n = 1$, it follows by induction that (2.15) holds for $n$. $\qquad \square$

**Property 5.** *If $r, s, j, k \in \mathbb{N}$ and $r > s > j > k$, then*

(2.17) $$E_n^{[s-k]} E_n^{[r-j]} \geq E_n^{[s-j]} E_n^{[r-k]}.$$

*The equality in (2.17) is valid if and only if at least $n - 1$ numbers equal zero among $\{a_1, a_2, \ldots, a_n\}$*

*Proof.* From Property 4, if $r - (k+1) > s - (k+1), r - (k+2) > s - (k+2), \ldots, r - j > s - j$, then

(2.18) $$\prod_{m=k+1}^{j} \left( E_n^{[r-m]} E_n^{[s-m+1]} \right) \geq \prod_{m=k+1}^{j} \left( E_n^{[r-m+1]} E_n^{[s-m]} \right).$$

This implies (2.17).

It is easy to see that the equality in (2.17) is valid. The proof is completed. $\qquad \square$

*Proof of Theorem 1.1.* Combination of Property 3 and Property 5 easily leads to Theorem 1.1. $\qquad \square$

**Remark 2.1.** *Finally, we pose an open problem: Give an explicit expression of $E_n^{[r-1]} E_n^{[s]} - E_n^{[r]} E_n^{[s-1]}$ in terms of $a_1, a_2, \ldots, a_n$.*
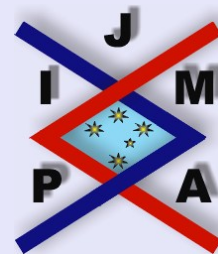
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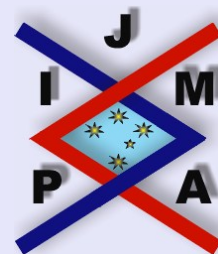
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