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ESTIMATING THE SEQUENCE OF REAL BINOMIAL COEFFICIENTS

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ABSTRACT. The sequence $n \mapsto \binom{a}{n}$ of real binomial coefficients is studied in two main cases: $a \gg n$ and $n \gg a$. In the first case a uniform approximation with high accuracy is obtained, in contrast to DeMoivre-Laplace approximation, which has essentially local character and is good only for $n \approx \frac{a}{2}$. In the second case, for every $a \in \mathbb{R} \setminus (\,\mathbb{N} \cup \{-1, 0\}\,)$, the functions $A(a,m)$ and $B(a,m)$ are determined, such that $\lim\limits_{m\to\infty} \frac{A(a,m)}{B(a,m)} = 1$, and

$$A(a,m) \cdot (n-a)^{-(a+1)} < \left| \binom{a}{n} \right| < B(a,m) \cdot (n-a)^{-(a+1)},$$

for integers m and n, obeying $n > m > |a|$.

Key words and phrases: Approximation, Binomial, Coefficient, Convergence, Estimate, Error, DeMoivre-Laplace, Euler-Maclaurin, Remainder, Sequence.

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1. INTRODUCTION

Binomial coefficients $\binom{a}{n}$, where $a \in \mathbb{R}$ and $n \in \mathbb{N}$, occur frequently, for example, in analysis [11, 13, 15], in combinatorics and discrete mathematics and computer science [7, 8, 14], and in probability [3, 4]. Computing $\binom{a}{n}$ directly by computer is not problematic as long as a and n stay within reasonable limits. However, for very large a or n the computation of binomial coefficients becomes difficult, even in the case when a and n are positive integers. An interesting discussion on computing the binomial coefficients in this particular case can be found in [6]. In this note we are interested in the estimate of $\binom{a}{n}$ for a or n being very large, where $a \in \mathbb{R}$ and $n \in \mathbb{N}$.

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We observe that the sequence $n \mapsto \binom{a}{n}$ converges in some cases but in the other cases it diverges, depending on a. We want to determine exactly when the sequence of binomial coefficients does converge, i.e. we wish to show that

$$\lim_{\substack{n\in\mathbb{N}\\ n\to\infty}} \left|\binom{a}{n}\right| = \begin{cases} \infty, & \text{if} \quad a < -1 \\ 0, & \text{if} \quad a > -1\,. \end{cases}$$

Moreover, we also want to estimate precisely the rate of divergence/convergence of the sequence of binomial coefficients. For example, we are searching for estimates like

$$0.436 \cdot (n+\pi)^{\pi-1} < \left|\binom{-\pi}{n}\right| < 0.438 \cdot (n+\pi)^{\pi-1}$$

and

$$0.983 \cdot (n-\pi)^{-(\pi+1)} < \left|\binom{\pi}{n}\right| < 0.985 \cdot (n-\pi)^{-(\pi+1)},$$

valid for $n \geq 9000$. In addition, we wish to estimate the binomial coefficients $\binom{a}{n}$ for large a and positive integers $n < a$. It turns out, in this case, that the construction of a binomial coefficient approximation, based on Taylor's formula, similar to the treatment in [4, pp. 174-190], is less accurate than the construction based on the Euler-Maclaurin summation formula. The latter produces two main results, *Theorem 4.1* and *Theorem 4.2*, presented on pages 8 and 12, respectively.

2. PRELIMINARIES

For $a \in \mathbb{R}$ and $n \in \mathbb{N}$ we define the binomial coefficients $C(a,n)$ as

(2.1) $$C(a,n) := \binom{a}{n} = \frac{a\cdot(a-1)\cdot\dots\cdot(a-n+1)}{1\cdot 2\cdot\dots\cdot n}$$

(2.2) $$= \frac{1}{n\,!}\cdot\prod_{i=0}^{n-1}(a-i)$$

(2.3) $$= \frac{a}{a-n}\cdot\prod_{i=1}^{n}\frac{a-i}{i} \quad \text{for} \quad a \neq n.$$

Due to (2.2) we have recursion and addition relations, respectively,

(2.4) $$C(a,n+1) = \frac{a-n}{n+1}\cdot C(a,n),$$

(2.5) $$C(a,n) + C(a,n+1) = C(a+1,n+1).$$

Substituting $n+1 = m$ in (2.4), we obtain the equality

(2.6) $$m\,C(a,m) = (a-m+1)C(a,m-1)\,,$$

which holds for every $a \in \mathbb{R}$ and $m \in \mathbb{N}$, if, in addition, we define $C(a,0) := 1$.

From (2.3) we obtain the relation

$$C(a,n) = C(a,m-1)\cdot\frac{a-m+1}{a-n}\cdot\prod_{i=m}^{n}\frac{a-i}{i}\,.$$

Hence, by (2.6),

$$C(a,n) = m\,C(a,m)\left(\frac{1}{a-n}\cdot\prod_{i=m}^{n}\frac{a-i}{i}\right), \tag{2.7}$$

and, consequently,

$$C(a,n) = (-1)^{n-m}\cdot m\,C(a,m)\cdot\left(\frac{1}{n-a}\cdot\prod_{i=m}^{n}\frac{i-a}{i}\right), \tag{2.8}$$

valid for integers m and n such that $a \neq n \geq m \geq 1$.

For any $n \in \mathbb{N}$ and $a \in \mathbb{R}$, using (2.2), we read the equivalence

$$C(a,n) = 0 \Longleftrightarrow a \in \{0, 1, \ldots, n-1\}\,. \tag{2.9}$$

We find, using induction and (2.5), that $C(a,n)$ are positive integers, provided that a and n are positive integers as well and, $1 \leq n \leq a$. Further, $C(a,n) = 0$ for $a \in \mathbb{N}$ and $n > a$.

3. MONOTONY

If $C(a,n) \neq 0$, we have, according to (2.6), the following equivalences

$$|C(a,n+1)| < |C(a,n)| \Leftrightarrow |a-n| < n+1 \Leftrightarrow -1 < a < 2n+1 \tag{3.1}$$

and

$$|C(a,n+1)| = |C(a,n)| \Leftrightarrow |a-n| = n+1 \Leftrightarrow a = -1 \text{ or } a = 2n+1\,. \tag{3.2}$$

3.1. Referring to (2.9), we have $C(0,n) = 0$ for every positive integer n.
3.2. Considering (2.2), the equality $C(-k,n) = (-1)^n \cdot C(k+n-1,n)$ holds for any $k, n \in \mathbb{N}$.
3.3. Let n and a be integers and $1 \leq n \leq a$. Then, according to (2.9), (3.1), and (3.2), the sequence $n \mapsto |C(a,n)| \equiv C(a,n)$ strictly increases for $n < \frac{a-1}{2}$ and strictly decreases for $n > \frac{a-1}{2}$, while for $n = \frac{a-1}{2}$ the equality $C(a,n+1) = C(a,n)$ holds. This means:

- **(i)** *If a is an even positive integer, then the sequence $n \mapsto C(a,n)$ strictly increases on the set $\left\{1, \ldots, \left\lfloor\frac{a+1}{2}\right\rfloor\right\}$ and strictly decreases on $\left\{\left\lfloor\frac{a+1}{2}\right\rfloor, \ldots, a\right\}$, where $\left\lfloor\frac{a+1}{2}\right\rfloor$ denotes the integer part of $\frac{a+1}{2}$.*
- **(ii)** *If a is an odd positive integer and $a \geq 3$, then the sequence $n \mapsto C(a,n)$ strictly increases on the set $\left\{1, \ldots, \frac{a-1}{2}\right\}$ and strictly decreases on $\left\{\frac{a+1}{2}, \ldots, a\right\}$, where $C\left(a, \frac{a-1}{2}\right) = C\left(a, \frac{a+1}{2}\right)$.*

From the considerations above we conclude that

$$\max_{1\leq n\leq a} C(a,n) = C\left(a, \left\lfloor\tfrac{a+1}{2}\right\rfloor\right), \tag{3.3}$$

if the conditions, quoted in this subsection, are satisfied.
3.4. Let $a \notin \{-1, 0\} \cup \mathbb{N}$. Then, by (2.9), all $C(a,n)$ are different from zero. Consequently, considering (2.9), (3.1), and (3.2), we find for the sequence $n \mapsto |C(a,n)|$ the following result:

- **(i)** $a < -1 \Rightarrow$ *The sequence strictly increases on the entire set* $\mathbb{N}$.
- **(ii)** $a \in (-1, 0) \cup (0, 3) \Rightarrow$ *The sequence strictly decreases on the entire set* $\mathbb{N}$.
- **(iii)** $a > 3 \Rightarrow$ *The sequence strictly increases on the set $\left\{1, \ldots, \left\lfloor\frac{a+1}{2}\right\rfloor\right\}$ and strictly decreases for $n \geq \left\lfloor\frac{a+1}{2}\right\rfloor$. (Here $\lfloor x\rfloor$ means the integer part of x.) Consequently,*

$$\max_{n\in\mathbb{N}} |C(a,n)| = \left|C\left(a, \left\lfloor\tfrac{a+1}{2}\right\rfloor\right)\right|. \tag{3.4}$$

Figures 3.1 – 3.5 illustrate the sequences $n \mapsto |C(a,n)|$ for several values of a.

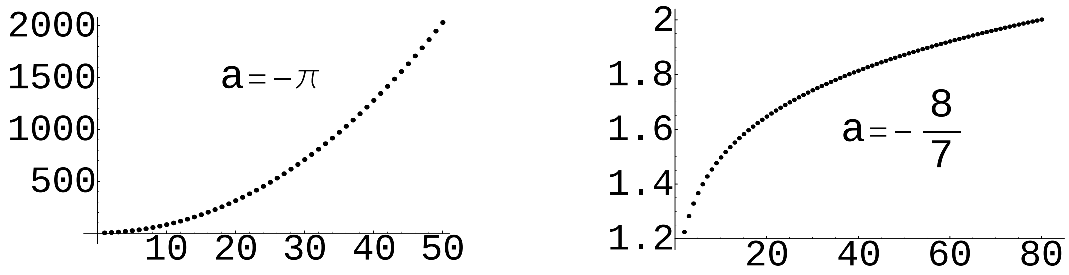

Figure 3.1: The sequences $n \mapsto |C(a,n)|$.

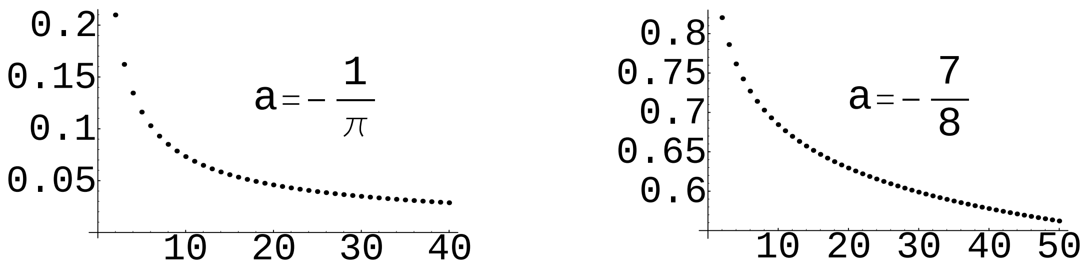

Figure 3.2: The sequences $n \mapsto |C(a,n)|$.

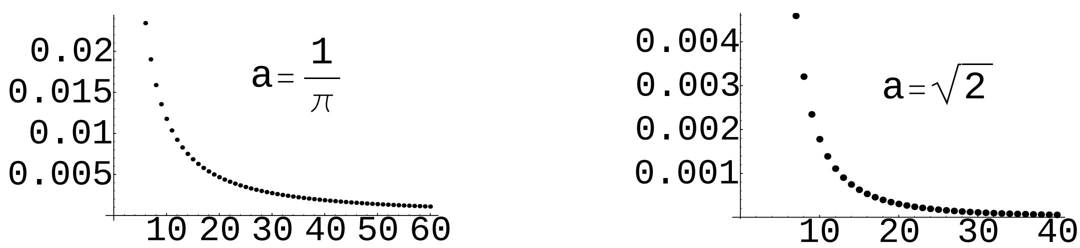

Figure 3.3: The sequences $n \mapsto |C(a,n)|$.

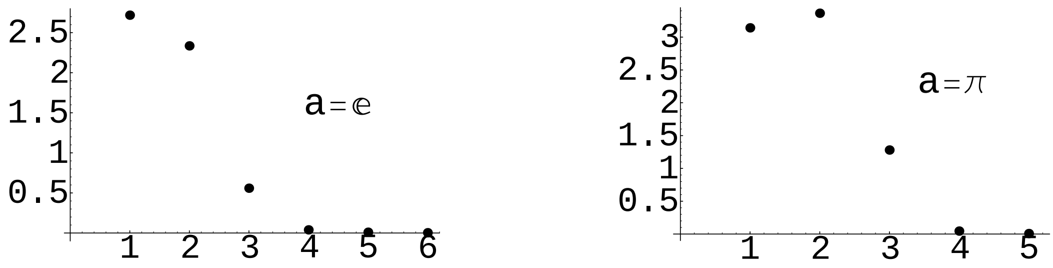

Figure 3.4: The sequences $n \mapsto |C(a,n)|$.

4. Approximating the Binomial Coefficients

Concerning the method of approximating binomial coefficients, we consider two main cases: the case when a is very large and $1 \le n < a$, and the case when a is any real number and integer $n \gg a$. The corresponding results for the first and for the second case, respectively, are presented in *Theorem 4.1* and *Theorem 4.2*, pp. 8 and 12.

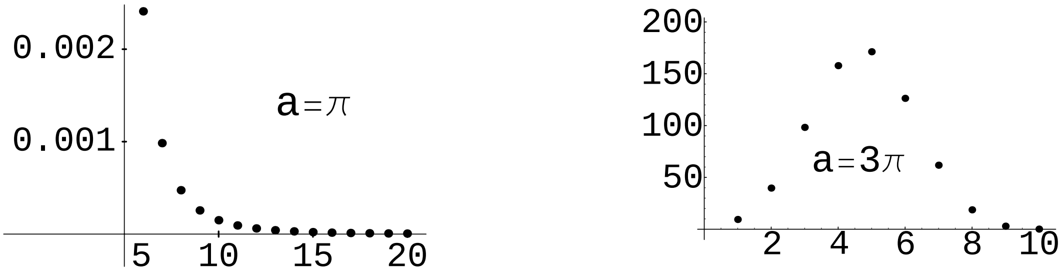

Figure 3.5: The sequences $n \mapsto |C(a,n)|$.

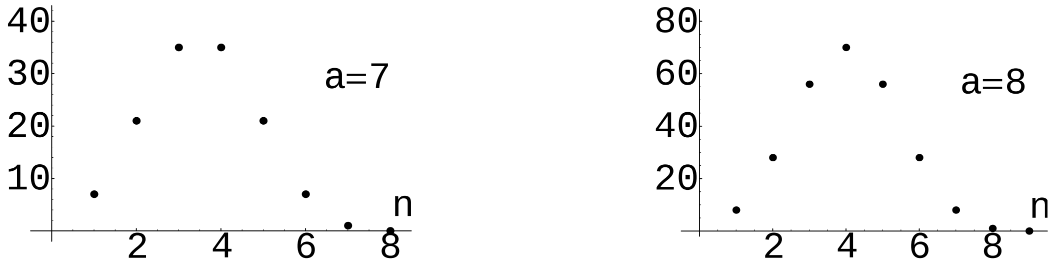

Figure 3.6: The sequences $n \mapsto C(a,n)$.

In the case $a \gg 1$ and $n < a$ we use the expression (2.7), and in the case $n \gg a$, we use the equation (2.8). In both cases we can obtain for the last factor on the right of (2.7) and (2.8), respectively, some product with all factors positive. Surely, this can be achieved in the first case by choosing for m any positive integer and, in the second case, by selecting any positive integer $m > a$. In the sequel we shall see that parameter m plays an important role concerning the accuracy of the obtained approximation; the larger m is, the more accurate the approximation is. Therefore, we demand that at least, $m \geq 2$ and, $m - a \geq 2$, in the first and in the second case, respectively. However, to compute $C(a,m)$ directly, using (2.7) and (2.8), both m and $m - a$ should not be large.

Since all factors in the above mentioned product are positive, we can use the real logarithmic function to transform it into a sum, which could be approximated easily using the Euler-Maclaurin summation formula [2, 7, 9, 11, 12]. For example, from [12, p. 117 - items (21a) and (21b)], setting $p = 3$, we obtain the Euler-Maclaurin formula[1] of the third order

(4.1) $$\sum_{j=m}^{n} f(j) = \int_m^n f(x)\,dx + \frac{f(m)+f(n)}{2} + \frac{f'(n)-f'(m)}{12} + R(m,n),$$

having the remainder

(4.2) $$R(m,n) := \rho_3(m,n) = -\frac{1}{3!}\int_m^n P_3(-x) f^{(3)}(x)\,dx,$$

where m and n are integers, $m < n$, and $f \in C^3[m,n]$. $P_3(x)$ stands for the third Bernoulli 1-periodic function [12, p. 114 - items (13) and (14)]

(4.3) $$\begin{aligned} &\text{(i)} \quad P_3(x) = x\left(x - \tfrac{1}{2}\right)(x-1) \quad &&\text{for } x \in [0,1] \\ &\text{(ii)} \quad P_3(x+1) = P_3(x) &&\text{for } x \in \mathbb{R}\,. \end{aligned}$$

[1] In 2007 we shall be celebrating the third-centenary of Euler's birth: April 15, 1707.

Therefore, due to

$$P_3(-x) \equiv P_3(1-x) \equiv -P_3(x), \tag{4.4}$$

we have, referring to (4.2),

$$R(m,n) = \frac{1}{6}\int_m^n P_3(x) f^{(3)}(x)\,dx. \tag{4.5}$$

We wish to obtain a better estimate of the remainder $R(m,n)$. Indeed, due to (4.3), we have

$$\max_{0\le x\le 1}|P_3(x)| = \max_{-\frac{1}{2}\le t\le\frac{1}{2}}\left|P_3\left(\frac{1}{2}+t\right)\right| = \max_{-\frac{1}{2}\le t\le\frac{1}{2}}\left|\left(t^2-\frac{1}{4}\right)t\right| = \frac{1}{12\sqrt{3}}\,.$$

Hence, using (4.5), we estimate

$$|R(m,n)| \le \frac{1}{72\sqrt{3}}\int_m^n \left|f^{(3)}(x)\right| dx, \tag{4.6}$$

for integers $n > m$. Moreover, if $f^{(3)}(x)$ is a monotonic function, not necessarily keeping its sign, then, for integers $n > m$:

$$\begin{aligned} &\text{(i)}\quad R(m,n) \le 0 \quad \text{if}\ \ f^{(3)}(x)\ \text{grows} \\ &\text{(ii)}\quad R(m,n) \ge 0 \quad \text{if}\ \ f^{(3)}(x)\ \text{decreases}\,. \end{aligned} \tag{4.7}$$

Indeed, substituting $x = i+t$, i being an integer, and considering the periodicity, $P_3(i+t) \equiv P_3(t)$, we have

$$\int_i^{i+1} P_3(x) f^{(3)}(x)\,dx = \int_0^{1/2} P_3(t) f^{(3)}(i+t)\,dt + \int_{1/2}^1 P_3(t) f^{(3)}(i+t)\,dt\,. \tag{4.8}$$

Additionally, substituting $t = 1-\tau$ and referring to the identity (4.4), we obtain

$$\int_{1/2}^1 P_3(t) f^{(3)}(i+t)\,dt = -\int_0^{1/2} P_3(\tau) f^{(3)}(i+1-\tau)\,d\tau\,.$$

Therefore, using (4.8), we find

$$\int_m^n P_3(x) f^{(3)}(x)\,dx = \sum_{i=m}^{n-1}\int_0^{1/2} P_3(t)\left[f^{(3)}(i+t) - f^{(3)}(i+1-t)\right] dt\,. \tag{4.9}$$

Because $i+t \le i+1-t$ for $t \in \left[0, \frac{1}{2}\right]$, the difference $f^{(3)}(i+t) - f^{(3)}(i+1-t)$, occurring in (4.9), is non-positive or non-negative if $f^{(3)}(x)$ grows or decreases, respectively. But, $P_3(t) \ge 0$ for $t \in \left[0, \frac{1}{2}\right]$, due to (4.3). Hence, all the integrands in the right hand side of (4.9) keep their sign over the entire interval $\left[0, \frac{1}{2}\right]$, provided that $f^{(3)}(x)$ is monotonous. According to (4.5), this confirms the assertion (4.7).

4.1. **Case $a \gg 1,\ 2 \le m \le n \le a-m$.** In this section we are supposing that the real number a and the integers m and n satisfy the following conditions:

$$a \gg 1, \quad 2 \le m \le n \le a-m\,. \tag{4.10}$$

Using the logarithmic function we can transform the last product in (2.7) into the sum

$$\ln\prod_{i=m}^n \frac{a-i}{i} = \sum_{i=m}^n f(i),$$

where $f(x) \equiv \ln \frac{a-x}{x}$. Unfortunately the second and the fourth derivatives of the function f do not keep their respective sign for $x \in [m, a-m]$, which has some disadvantage for estimating the remainder in the summation formula. However,

$$\prod_{i=m}^{n} \frac{a-i}{i} = \prod_{i=m}^{n} \frac{a-i}{m+n-i}\,.$$

Thus,

$$\ln \prod_{i=m}^{n} \frac{a-i}{i} = \sum_{i=m}^{n} f_{a,b}(i), \tag{4.11}$$

where

$$f_{a,b}(x) \;:=\; \ln \frac{a-x}{b-x} \;\equiv\; \ln(a-x) - \ln(b-x), \tag{4.12}$$

$\underline{b = m+n}$, and $0 < x < b \le a$.

We have the derivatives

$$f_{a,b}^{(1)}(x) \equiv -\frac{1}{a-x} + \frac{1}{b-x} \ge 0, \tag{4.13}$$

$$f_{a,b}^{(2)}(x) \equiv -\frac{1}{(a-x)^2} + \frac{1}{(b-x)^2} \ge 0, \tag{4.14}$$

$$f_{a,b}^{(3)}(x) \equiv -\frac{2}{(a-x)^3} + \frac{2}{(b-x)^3} \ge 0 \tag{4.15}$$

and

$$f_{a,b}^{(4)}(x) \equiv -\frac{6}{(a-x)^4} + \frac{6}{(b-x)^4} \ge 0, \tag{4.16}$$

for $0 < x < a$.

Formula (4.1), applied to the function $f_{a,b}$, determines the remainder, denoted as $R_a(m,n)$. Referring to (4.6), (4.14) and (4.15), we have

$$\begin{aligned} |R_a(m,n)| &\le \frac{1}{72\sqrt{3}} \cdot \int_m^n f_{a,b}^{(3)}(x)dx \\ &= \frac{1}{72\sqrt{3}} \cdot \left(f_{a,b}^{(2)}(n) - f_{a,b}^{(2)}(m) \right) \\ &\le \frac{1}{72\sqrt{3}} \cdot f_{a,b}^{(2)}(n) \\ &= \frac{1}{72\sqrt{3}} \left(\frac{-1}{(a-n)^2} + \frac{1}{(b-n)^2} \right) \\ &= \frac{1}{72\sqrt{3}} \left(\frac{-1}{(a-n)^2} + \frac{1}{m^2} \right), \end{aligned}$$

that is

$$|R_a(m,n)| \le \frac{1}{72\, m^2 \sqrt{3}}\,. \tag{4.17}$$

Moreover, due to (4.7) and (4.16), we have

$$R_a(m,n) \le 0\,.$$

Therefore, considering (4.17), we conclude that for a, m and n, as determined by (4.10), there exists some $\vartheta \in [0,1]$, depending on a, m and n, such that

$$R_a(m,n) = -\frac{\vartheta}{72\,m^2\sqrt{3}}, \tag{4.18}$$

for $n \in [m, a-m]$.

After the remainder has been uniformly estimated, the summation formula (4.1) can be applied. According to (4.12), we find

$$\int_m^n f_{a,b}(x)\,dx = \left[\ln\frac{(b-x)^{b-x}}{(a-x)^{a-x}}\right]_m^n = \ln\left(\frac{m^m(a-m)^{a-m}}{n^n(a-n)^{a-n}}\right) \tag{4.19}$$

and

$$\frac{1}{2}\left[f_{a,b}(m)+f_{a,b}(n)\right] = \ln\sqrt{\frac{(a-m)(a-n)}{m\cdot n}}\,. \tag{4.20}$$

Referring to (4.13), and recalling that $b = m+n$, defined below (4.12), we have

$$\frac{1}{12}\left[f_{a,b}^{(1)}(n) - f_{a,b}^{(1)}(m)\right] = \frac{a}{12}\left(\frac{1}{m(a-m)} - \frac{1}{n(a-n)}\right). \tag{4.21}$$

From (4.11) and (4.1), using (4.19)–(4.21), we obtain the expression

$$\ln\prod_{i=m}^{n}\frac{a-i}{i} = \ln\left(\sqrt{\frac{(a-m)(a-n)}{m\cdot n}}\cdot\frac{m^m(a-m)^{a-m}}{n^n(a-n)^{a-n}}\right) + \frac{a}{12}\left(\frac{1}{m(a-m)} - \frac{1}{n(a-n)}\right) + R_a(m,n)\,.$$

Consequently, we conclude, according to (2.7) and (4.18), that the following theorem holds.

Theorem 4.1. *For any integers m and n, obeying condition* (4.10)*, there exists some $\vartheta = \vartheta(a,m,n) \in [0,1]$, depending on a, m and n, such that*

$$C(a,n) = B(a,m,n)\cdot\exp\left(-\frac{\vartheta}{72\,m^2\sqrt{3}}\right), \tag{4.22}$$

where

$$B(a,m,n) = m\,C(a,m)\cdot\frac{m^m(a-m)^{a-m}}{n^n(a-n)^{a-n}}\cdot\sqrt{\frac{a-m}{m\,n(a-n)}} \cdot\exp\left[\frac{a}{12}\left(\frac{1}{m(a-m)} - \frac{1}{n(a-n)}\right)\right],$$

i.e.

$$B(a,m,n) = C(a,m)\cdot\frac{m^m(a-m)^{a-m}}{n^n(a-n)^{a-n}}\cdot\sqrt{\frac{m\,(a-m)}{n\,(a-n)}} \cdot\exp\left[\frac{a}{12}\left(\frac{1}{m(a-m)} - \frac{1}{n(a-n)}\right)\right]. \tag{4.23}$$

For every a and m, satisfying (4.10), the function $x \mapsto B(a,m,x)$ has the symmetric property

$$B\left(a,m,\frac{a}{2}-x\right) \equiv B\left(a,m,\frac{a}{2}+x\right)$$

and, moreover, strictly increases/decreases on the intervals $\left(0, \frac{a}{2}\right]$ and $\left[\frac{a}{2}, a\right)$ respectively. Indeed, due to (4.23) we have

$$\frac{d}{dx} B\,(a, m, x) \equiv B(a, m, x) \cdot \varphi(x),$$

where $\varphi(x) \equiv \psi(a-x) - \psi(x)$ and $\psi(x) \equiv 1/(2x) - 1/(12x^2) + \ln x$. Because

$$\psi'(x) \equiv \frac{6x^2 - 3x + 1}{6x^3} > 0$$

for $x > 0$ and consequently $\varphi'(x) \equiv -\psi'(a-x) - \psi'(x) < 0$, function $\varphi(x)$ strictly decreases. Thus, $\varphi(x) > \varphi(\frac{a}{2}) = 0$ for $x \in \left(0, \frac{a}{2}\right)$ and $\varphi(x) < \varphi(\frac{a}{2}) = 0$ for $x \in \left(\frac{a}{2}, a\right)$. Hence $\frac{d}{dx} B\,(a, m, x) > 0$ for $x \in \left(0, \frac{a}{2}\right)$ and $\frac{d}{dx} B\,(a, m, x) < 0$ for $x \in \left(\frac{a}{2}, a\right)$.

The expression for $B(a, m, n)$ can be further simplified. Namely, thanks to the estimate $0 < n < a$, the relative deviation

$$d(a, t) := \frac{t - \frac{a}{2}}{\frac{a}{2}} = 2\frac{t}{a} - 1 \tag{4.24}$$

lies within the open interval $(-1, 1)$ and generates the equalities

$$t(a-t) = \left(\frac{a}{2}\right)^2 \left[1 - d^2(a, t)\right] \tag{4.25}$$

and

$$t^t (a-t)^{a-t} = \left(\frac{a}{2}\right)^a \left[(1+x)^{1+x}(1-x)^{1-x}\right]^{\frac{a}{2}}, \quad x = d(a, t). \tag{4.26}$$

Using (4.25) and (4.26), the equation (4.23) can be written in a more compact form as

$$B(a, m, n) = C(a, m) \cdot \frac{D(a, \mu)}{D(a, \nu)} \tag{4.27}$$

$$= C(a, m) \left[\frac{F(\mu)F(-\mu)}{F(\nu)F(-\nu)}\right]^{a/2} \sqrt{\frac{1-\mu^2}{1-\nu^2}}\ \exp\left[\frac{3}{a}\left(\frac{1}{1-\mu^2} - \frac{1}{1-\nu^2}\right)\right], \tag{4.28}$$

where $\mu = d(a, m)$, $\nu = d(a, n)$, $F(t) \equiv (1+t)^{1+t}$ and

$$D(a, t) = [F(t)\,F(-t)]^{a/2}\sqrt{1-t^2}\exp\left(\frac{3}{a(1-t^2)}\right).$$

The graphs of the functions $x \mapsto 1/D(a, x)$, for $a = 3\pi$ and $a = 33\pi$, are shown in Figure 4.1, and the graphs of the sequences $n \mapsto C(a, n)$ and the functions $x \mapsto B(a, 2, x)$ are illustrated in Figure 4.2.

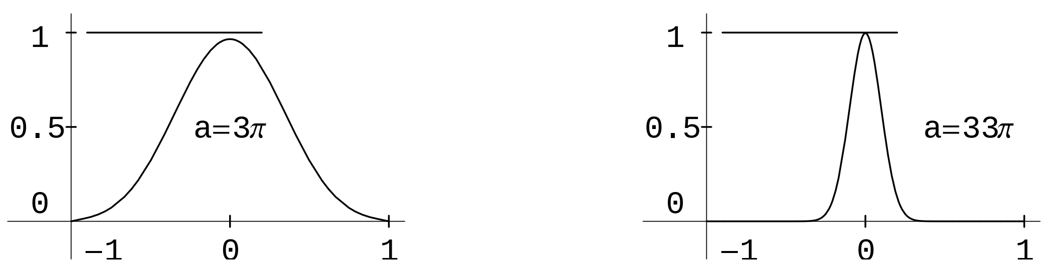

Figure 4.1: Graphs of functions $x \mapsto 1/D(a, x)$.

Figure 4.2 indicates that the approximation $B(a, m, n)$ is very close to $C(a, n)$, even for small m, for example, $m = 2$. Here, "to be close" has its meaning in the absolute sense, i.e.

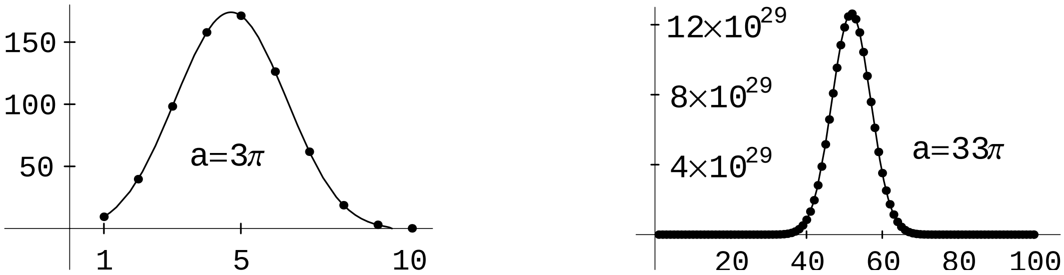

Figure 4.2: Approximations $B(a,2,x)$ to sequences $n \mapsto C(a,n)$.

proportionally to $\max\{C(a,n) : 1 \le n \le a\}$. The curves in these figures are reminiscent of the Gauss (normal) bell-shaped curves arising from the function $x \mapsto \exp(-x^2)$. Indeed, in probability theory we have the well known DeMoivre-Laplace local approximation [4, pp. 174-190]

$$C(a,n) \cdot \left(\frac{1}{2}\right)^n \cdot \left(\frac{1}{2}\right)^{a-n} \approx \frac{1}{\sqrt{2\pi \cdot \frac{a}{4}}} \cdot \exp\left(-\frac{1}{2}\left(\frac{n-\frac{a}{2}}{\sqrt{\frac{a}{4}}}\right)^2\right),$$

from which we obtain a DeMoivre-Laplace approximation to the sequence of binomial coefficients

$$C(a,n) \approx M(a,n) := \frac{2^{\,a+\frac{1}{2}}}{\sqrt{a\pi}} \cdot \exp\left(-\frac{(2n-a)^2}{2a}\right).$$

A figure representing the graphs of the sequences $n \mapsto C(a,n)$ and the function $x \mapsto M(a,x)$ for $a = 3\pi$ and $a = 33\pi$, is not shown because it is indistinguishable from Figure 4.2. Consequently, it seems that the approximation $C(a,n) \approx M(a,n)$ should be very good for a and n obeying (4.10). Unfortunately, this approximation is good relatively to $\max\{C(a,n) : 1 \le n \le a\}$. It is true that the relative error

$$\rho(a,n) := \frac{C(a,n) - M(a,n)}{M(a,n)}$$

is small for $n \approx \frac{a}{2}$. However, it can approach even to the number -1 for $n \approx 1$ or $n \approx a$, as is evident from Figure 4.3, which shows the sequences of errors $\rho(a,n)$ for $a = 3\pi$ and $a = 33\pi$.

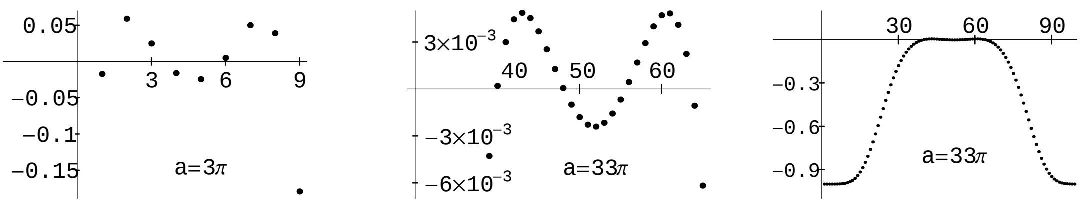

Figure 4.3: The graphs of sequences $n \mapsto \rho(a,n)$.

Fortunately, the situation is quite different concerning the approximation $C(a,n) \approx B(a,m,n)$. Indeed, due to (4.22), the absolute value of the relative error

$$r(a,m,n) := \frac{C(a,n) - B(a,m,n)}{B(a,m,n)} = \exp\left(-\frac{\vartheta}{72\,m^2\sqrt{3}}\right) - 1 \tag{4.29}$$

becomes small uniformly for large m. Because the function $\varphi(x) := e^{-x} - 1 + x$ strictly increases on $[0, \infty)$ and $\varphi(0) = 0$, we have $|\exp(-x) - 1| < x$ for $x > 0$. Thus, considering (4.29), we find the uniform estimate

$$-\frac{1}{124\, m^2} < r(a, m, n) \leq 0, \tag{4.30}$$

for a, m and n, which obey condition (4.10). For example $|r(a, 2, n)| < 2.1{\times}10^{-3}$, $|r(a, 10, n)| < 8.4 \times 10^{-5}$, $|r(a, 100, n)| < 8.4 \times 10^{-7}$, and $|r(a, 1000, n)| < 8.4 \times 10^{-9}$. Direct computations, using [19], give $-3.3 \times 10^{-4} < r(33\pi, 2, 50) < -3.2 \times 10^{-4}$, $-2.8 \times 10^{-6} < r(33\pi, 10, 50) < -2.7 \times 10^{-6}$, and $-2.8 \times 10^{-9} < r(333\pi, 100, 500) < -2.7 \times 10^{-9}$. Hence, a priori estimate (4.30) appears as rather rough.

4.2. **Case $n \gg a$.** Let us suppose that the real number a and the integers m and n satisfy the following conditions:

$$a \in \mathbb{R} \setminus (\, \mathbb{N} \cup \{-1, 0\}\,) \quad \text{and} \quad n > m > |a|\,. \tag{4.31}$$

We relate the last product in (2.8) with a sum

$$\ln \prod_{i=m}^{n} \frac{i-a}{i} = \sum_{i=m}^{n} f_a(i)\,, \tag{4.32}$$

where

$$f_a(x) \equiv \ln\left(\frac{x-a}{x}\right) \equiv \ln(x-a) - \ln x\,. \tag{4.33}$$

We shall use formula (4.1) for the function f_a, which determines the remainder, denoted as $R_a(m, n)$. To this effect we need the derivatives

$$f_a'(x) \equiv \frac{1}{x-a} - \frac{1}{x}\,, \tag{4.34}$$

$$f_a''(x) \equiv \frac{-1}{(x-a)^2} + \frac{1}{x^2}\,, \tag{4.35}$$

$$f_a^{(3)}(x) \equiv \frac{2}{(x-a)^3} - \frac{2}{x^3} \tag{4.36}$$

and

$$f_a^{(4)}(x) \equiv \frac{-6}{(x-a)^4} + \frac{6}{x^4}\,, \tag{4.37}$$

for $x > a$. It is evident from these expressions that, for $a > 0$, all derivatives of odd/even orders are positive/negative and, for $a < 0$, all derivatives of odd/even orders are negative/positive. Thus, using the function signum, $\operatorname{sgn}(a) := -1$ for $a < 0$ and $\operatorname{sgn}(a) = 1$ for $a > 0$, we have

$$\begin{aligned} &\text{(i)} \quad \operatorname{sgn}(a) \cdot f_a''(x) \leq 0 \\ &\text{(ii)} \quad \left| f_a^{(3)}(x) \right| \equiv \operatorname{sgn}(a) \cdot f_a^{(3)}(x) \\ &\text{(iii)} \quad \operatorname{sgn}(a) \cdot f_a^{(4)}(x) \leq 0\,. \end{aligned} \tag{4.38}$$

According to (4.33) we get

$$\int_m^n f_a(x)\, dx = \left[\ln \frac{(x-a)^{x-a}}{x^x}\right]_m^n = \ln\left(\frac{m^m (n-a)^{n-a}}{n^n (m-a)^{m-a}}\right) \tag{4.39}$$

and

$$\frac{1}{2}\left[f_a(m)+f_a(n)\right]=\ln\sqrt{\frac{(m-a)(n-a)}{m\cdot n}}\,. \tag{4.40}$$

Due to (4.34), we have

$$\frac{1}{12}\left[f_a'(n)-f_a'(m)\right]=\frac{a}{12}\left(\frac{1}{n(n-a)}-\frac{1}{m(m-a)}\right). \tag{4.41}$$

Considering (4.6), (4.38)(i) – (ii) and (4.35) we estimate the remainder $R_a(m,n)$ as

$$\begin{aligned}|R_a(,m,n)| &\le \frac{1}{72\sqrt{3}}\cdot\operatorname{sgn}(a)\int_m^n f_a^{(3)}(x)dx\\ &= \frac{1}{72\sqrt{3}}\cdot\Big(\operatorname{sgn}(a)\cdot f_a''(n)-\operatorname{sgn}(a)\cdot f_a''(m)\Big)\\ &\le \frac{-\operatorname{sgn}(a)}{72\sqrt{3}}\cdot f_a''(m)\\ &= \frac{\operatorname{sgn}(a)}{72\sqrt{3}}\left(\frac{1}{(m-a)^2}-\frac{1}{m^2}\right)\\ &= \frac{\operatorname{sgn}(a)}{72\sqrt{3}}\,\frac{2ma-a^2}{(m-a)^2m^2}\,,\end{aligned}$$

i.e.

$$|R_a(m,n)|\le\frac{|a|}{36\sqrt{3}\,m(m-a)^2}\,. \tag{4.42}$$

However, from (4.7) and (4.38)(iii) we conclude that

$$\operatorname{sgn}(a)\cdot R_a(m,n)\ge 0\,.$$

Consequently, according to (4.42), there exists some $\vartheta\in[0,1]$ such that

$$\operatorname{sgn}(a)\cdot R_a(m,n)=\frac{\vartheta\,|a|}{36\sqrt{3}\,m(m-a)^2}\,.$$

Hence, for integers $n>m>|a|$, we have

$$R_a(m,n)=\frac{\vartheta\cdot a}{36\sqrt{3}\,m(m-a)^2} \tag{4.43}$$

for some $\vartheta\in[0,1]$, depending on a, m and n.

Inserting expressions (4.39) – (4.41), and (4.43) into the summation formula (4.1), we obtain the expression

$$\begin{aligned}\ln\prod_{i=m}^{n}\frac{i-a}{i}&=\ln\left(\sqrt{\frac{(m-a)(n-a)}{m\,n}}\cdot\frac{m^m(n-a)^{n-a}}{n^n(m-a)^{m-a}}\right)\\ &\qquad+\frac{a}{12}\left(\frac{1}{n(n-a)}-\frac{1}{m(m-a)}\right)+\frac{\vartheta\cdot a}{36\sqrt{3}\,m(m-a)^2}\,.\end{aligned}$$

Hence, considering (2.8), we conclude that the following theorem was proved true.

Theorem 4.2. *For any integers* m *and* n*, which obey* (4.31)*, there exists some* $\vartheta\in[0,1]$*, depending on* a*,* m *and* n*, such that*

$$C(a,n)=B^*(a,m,n)\cdot\exp\left(\frac{\vartheta\cdot a}{36\sqrt{3}\,m(m-a)^2}\right), \tag{4.44}$$

where

$$\begin{aligned} B^*(a,m,n) &= (-1)^{n-m} \cdot m\, C(a,m) \cdot \frac{1}{n-a} \\ &\quad \cdot \sqrt{\frac{(m-a)(n-a)}{m\,n}} \cdot \frac{m^m (n-a)^{n-a}}{n^n (m-a)^{m-a}} \\ &\quad \cdot \exp\left[\frac{a}{12}\left(\frac{1}{n(n-a)} - \frac{1}{m(m-a)}\right)\right] \end{aligned}$$

i.e.

$$\begin{aligned} B^*(a,m,n) &= (-1)^{n-m} \cdot C(a,m) \cdot \frac{m^{m+1/2}}{(m-a)^{m-a-1/2}} \\ &\quad \cdot \exp\left[\frac{a}{12}\left(\frac{1}{n(n-a)} - \frac{1}{m(m-a)}\right)\right] \\ &\quad \cdot \left(1-\frac{a}{n}\right)^n \cdot \sqrt{1-\frac{a}{n}} \cdot (n-a)^{-(a+1)}\,. \end{aligned} \tag{4.45}$$

The rate of convergence

$$\lim_{n\to\infty}\left(1-\frac{a}{n}\right)^n = e^{-a}, \quad a \in \mathbb{R},$$

of the sequence occurring in (4.45), can be estimated using the following lemma:

Lemma 4.3. *For any positive real x and $t \geq 2x$ there hold the estimates*

$$\text{(i)} \quad \exp\left(-x-\tfrac{x^2}{t}\right) < \left(1-\tfrac{x}{t}\right)^t < \exp\left(-x-\tfrac{x^2}{2t}\right)$$

$$\text{(ii)} \quad \exp\left(x-\tfrac{x^2}{2t}\right) < \left(1+\tfrac{x}{t}\right)^t < \exp\left(x-\tfrac{x^2}{3t}\right).$$

Proof. Indeed, integrating the inequalities

$$1+\tau < \frac{1}{1-\tau} < 1+2\tau$$

and

$$1-\tau < \frac{1}{1+\tau} < 1-\frac{2}{3}\tau\,,$$

valid for $\tau \in \left(0, \frac{1}{2}\right)$, we obtain the relations

$$y + \frac{y^2}{2} < \int_0^y \frac{d\tau}{1-\tau} = -\ln(1-y) < y + y^2$$

and

$$y - \frac{y^2}{2} < \int_0^y \frac{dt}{1+t} = \ln(1+y) < y - \frac{y^2}{3}\,,$$

true for $y \in \left(0, \frac{1}{2}\right]$. Moreover, for $x > 0$ and $t \geq 2x$ the number $y := \frac{x}{t}$ lies in the interval $\left(0, \frac{1}{2}\right]$ and the relations above could be applied for this y and the lemma is thus verified. □

Remark 4.4. From the above lemma we obtain

$$\exp\left(x-\frac{x^2}{t}\right) < \left(1+\frac{x}{t}\right)^t < \exp\left(x-\frac{x^2}{3t}\right) \tag{4.46}$$

for any real $x \neq 0$ and $t \geq 2|x|$.

From (4.44) and (4.45), using our Lemma, and considering the inequalities $1 < \sqrt{1+h} < 1 + h/2$ and $1 - h < \sqrt{1-h} < 1$, true for $h \in (0,1)$, together with the estimate $-\frac{1}{n} > -\frac{1}{m}$, valid for positive integers $n > m$, we find that the next proposition is valid.

Proposition 4.5. *For a, m and n, which follow* (4.31)*, the following estimates hold*

$$\text{(4.47)} \qquad \begin{aligned} &\textit{(i)} \quad |C(a,n)| > C^*(a,m) \cdot I(a,m) \cdot (n-a)^{-(a+1)} \\ &\textit{(ii)} \quad |C(a,n)| < C^*(a,m) \cdot J(a,m) \cdot (n-a)^{-(a+1)}, \end{aligned}$$

where

$$C^*(a,m) \;=\; |C(a,m)| \cdot \frac{e^{-a}\, m^{m+1/2}}{(m-a)^{m-a-1/2}}$$

$$\text{(4.48)} \qquad I(a,m) = \begin{cases} \exp\left(-\frac{a^2}{2m} - \frac{|a|}{36\sqrt{3}\, m(m+|a|)^2}\right), & \textit{if} \quad a < 0 \\ \left(1 - \frac{a}{m}\right)\exp\left(-\frac{a^2}{m} - \frac{a}{12\, m(m-a)}\right), & \textit{if} \quad a > 0 \end{cases}$$

and

$$\text{(4.49)} \qquad J(a,m) = \begin{cases} \left(1 + \frac{|a|}{2m}\right)\exp\left(\frac{|a|}{12\, m(m+|a|)}\right), & \textit{if} \quad a < 0 \\ \exp\left(\frac{a}{36\sqrt{3}\, m(m-a)^2}\right), & \textit{if} \quad a > 0\,. \end{cases}$$

We emphasize that $|C(a,n)| \approx C^*(a,m) \cdot (n-a)^{-(a+1)})$ represents a good approximation for large n. From relations (4.47)–(4.49) we conclude that the following proposition also holds.

Proposition 4.6. *For every*[1] *real number a we have*

$$\text{(4.50)} \qquad \lim_{\substack{n\in\mathbb{N} \\ n\to\infty}} |C(a,n)| = \begin{cases} \infty, & \textit{if} \quad a < -1 \\ 1, & \textit{if} \quad a = -1 \\ 0, & \textit{if} \quad a > -1\,. \end{cases}$$

Computing $C^*(a,m)$, $|I(a,m)|$ and $|J(a,m)|$ directly, using e.g. [19], we obtain from relations (4.47)–(4.49) good estimates of binomial coefficients. For example, because

$$C^*(-\pi, 2099) \cdot I(-\pi, 2099) = 0.436029\ldots > 0.436$$

and

$$C^*(-\pi, 2099) \cdot J(-\pi, 2099) = 0.437382\ldots < 0.438,$$

we have

$$0.436 \cdot (n+\pi)^{\pi-1} < |C(-\pi, n)| < 0.438 \cdot (n+\pi)^{\pi-1},$$

true for $n \geq 2100$.

Similarly, since

$$C^*(\pi, 8999) \cdot I(\pi, 8999) = 0.983122\ldots > 0.983$$

and

$$C^*(\pi, 8999) \cdot J(\pi, 8999) = 0.984545\ldots < 0.985,$$

we have

$$0.983 \cdot (n-\pi)^{-(\pi+1)} < |C(\pi, n)| < 0.985 \cdot (n-\pi)^{-(\pi+1)},$$

valid for $n \geq 9000$.

[1] Obviously $\binom{-1}{n} \equiv (-1)^n$.

According to (4.47), the quotient

$$Q(a,n) := \frac{|C(a,n)|}{C^*(a,m)\cdot(n-a)^{-(a+1)}}$$

lies close to 1 and is bounded as

$$I(a,m) < Q(a,n) < J(a,m), \tag{4.51}$$

for a, m and n, which obey (4.31). Figure 4.4 illustrates the estimate (4.51) for $a \in \{-\pi,\, \pi\}$ and $n = m+1 \in \{5,\, 6,\, \ldots,\, 101\}$.

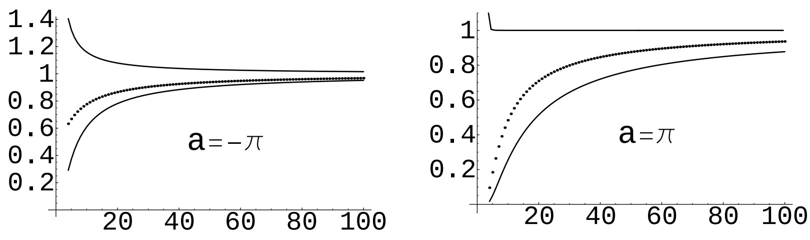

Figure 4.4: Estimating the sequence of binomial coefficients.

Remark 4.7. Using the Gamma function, the definition of binomial coefficient $C(a,b)$ could be extended as

$$C(a,b) := \frac{\Gamma(a+1)}{\Gamma(b+1)\cdot\Gamma(a-b+1)}$$

for a and b being arbitrary complex numbers, different from any negative integer. From this expression the symmetric property, $C(a,b) = C(a,a-b)$, is evident.

Using Stirling's approximation to the Gamma function, see e.g. [1],

$$\Gamma(x+1) = x\,\Gamma(x) = \sqrt{2\pi\, x}\cdot\left(\frac{x}{e}\right)^x \cdot e^{\frac{\vartheta}{12x}}$$

for $x \in \mathbb{R}^+$ and some $\vartheta \in (0,1)$, which depends on x, it is possible to obtain approximations, which are close to ours. For example, for positive integer n and real $a > n$ we obtain the formula

$$C(a,n) = \sqrt{\frac{a}{2\pi\, n(a-n)}}\cdot\frac{a^a}{n^n\,(a-n)^{a-n}}\cdot\exp\left(\frac{1}{12}\left(\frac{\vartheta}{a}-\frac{\vartheta'}{n}-\frac{\vartheta''}{a-n}\right)\right),$$

true for some $\vartheta, \vartheta', \vartheta'' \in (0,1)$ which depend on a and n.

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