Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 6, Issue 2, Article 51, 2005

GENERALIZED INTEGRAL OPERATOR AND MULTIVALENT FUNCTIONS

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Received 20 February, 2005; accepted 02 March, 2005
Communicated by Th.M. Rassias

ABSTRACT. Let $\mathcal{A}(p)$ be the class of functions $f : f(z) = z^p + \sum_{j=1}^{\infty} a_j z^{p+j}$ analytic in the open unit disc E. Let, for any integer $n > -p$, $f_{n+p-1}(z) = \frac{z^p}{(1-z)^{n+p}}$. We define $f_{n+p-1}^{(-1)}(z)$ by using convolution $\star$ as $f_{n+p-1}(z) \star f_{n+p-1}^{(-1)}(z) = \frac{z^P}{(1-z)^{n+p}}$. A function p, analytic in E with $p(0) = 1$, is in the class $P_k(\rho)$ if $\int_0^{2\pi} \left| \frac{Rep(z)-\rho}{p-\rho} \right| d\theta \leq k\pi$, where $z = re^{i\theta}, k \geq 2$ and $0 \leq \rho < p$. We use the class $P_k(\rho)$ to introduce a new class of multivalent analytic functions and define an integral operator $I_{n+p-1}(f) = f_{n+p-1}^{(-1)} \star f(z)$ for $f(z)$ belonging to this class. We derive some interesting properties of this generalized integral operator which include inclusion results and radius problems.

Key words and phrases: Convolution (Hadamard product), Integral operator, Functions with positive real part, Convex functions.

2000 *Mathematics Subject Classification.* Primary 30C45, 30C50.

1. INTRODUCTION

Let $\mathcal{A}(p)$ denote the class of functions f given by

$$f(z) = z^p + \sum_{j=1}^{\infty} a_j z^{p+j}, \quad p \in N = \{1, 2, \dots\}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. The Hadamard product or convolution $(f \star g)$ of two functions with

$$f(z) = z^p + \sum_{j=1}^{\infty} a_{j,1} z^{p+j} \quad \text{and} \quad g(z) = z^p + \sum_{j=1}^{\infty} z^{p+j}$$

ISSN (electronic): 1443-5756

061-05

is given by

$$(f \star g)(z) = z^p + \sum_{j=1}^{\infty} a_{j,1} a_{j,2} z^{p+j}.$$

The integral operator $I_{n+p-1} : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ is defined as follows, see [2].

For any integer n greater than $-p$, let $f_{n+p-1}(z) = \frac{z^p}{(1-z)^{n+p}}$ and let $f_{n+p-1}^{(-1)}(z)$ be defined such that

$$f_{n+p-1}(z) \star f_{n+p-1}^{(-1)}(z) = \frac{z^p}{(1-z)^{p+1}}. \tag{1.1}$$

Then

$$I_{n+p-1} f(z) = f_{n+p-1}^{(-1)}(z) \star f(z) = \left[\frac{z^p}{(1-z)^{n+p}}\right]^{(-1)} \star f(z). \tag{1.2}$$

From (1.1) and (1.2) and a well known identity for the Ruscheweyh derivative [1, 8], it follows that

$$z\left(I_{n+p} f(z)\right)' = (n+p) I_{n+p-1} f(z) - n I_{n+p} f(z). \tag{1.3}$$

For $p = 1$, the identity (1.3) is given by Noor and Noor [3].

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in E satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \rho}{p - \rho} \right| d\theta \le k\pi, \tag{1.4}$$

where $z = re^{i\theta}, k \ge 2$ and $0 \le \rho < p$. For $p = 1$, this class was introduced in [5] and for $\rho = 0$, see [6]. For $\rho = 0, \quad k = 2$, we have the well known class P of functions with positive real part and the class $k = 2$ gives us the class $P(\rho)$ of functions with positive real part greater than ρ. Also from (1.4), we note that $p \in P_k(\rho)$ if and only if there exist $p_1, p_2 \in P_k(\rho)$ such that

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z). \tag{1.5}$$

It is known [4] that the class $P_k(\rho)$ is a convex set.

Definition 1.1. Let $f \in \mathcal{A}(p)$. Then $f \in T_k(\alpha, p, n, \rho)$ if and only if

$$\left[(1-\alpha)\frac{I_{n+p-1} f(z)}{z^p} + \alpha \frac{I_{n+p} f(z)}{z^p}\right] \in P_k(\rho),$$

for $\alpha \ge 0, n > -p, 0 \le \rho < p, k \ge 2$ and $z \in E$.

2. PRELIMINARY RESULTS

Lemma 2.1. *Let $p(z) = 1 + b_1 z + b_2 z^2 + \cdots \in P(\rho)$. Then*

$$\operatorname{Re} p(z) \ge 2\rho - 1 + \frac{2(1-\rho)}{1+|z|}.$$

This result is well known.

Lemma 2.2 ([7]). *If $p(z)$ is analytic in E with $p(0) = 1$ and if λ_1 is a complex number satisfying $\operatorname{Re} \lambda_1 \ge 0, \quad (\lambda_1 \ne 0)$, then $\operatorname{Re}\{p(z) + \lambda_1 z p'(z)\} > \beta \quad (0 \le \beta < p)$ implies*

$$\operatorname{Re} p(z) > \beta + (1-\beta)(2\gamma_1 - 1),$$

where γ_1 is given by

$$\gamma_1 = \int_0^1 \left(1 + t^{\operatorname{Re} \lambda_1}\right)^{-1} dt.$$

Lemma 2.3 ([9]). *If $p(z)$ is analytic in E, $\quad p(0)=1$ and $\operatorname{Re} p(z) > \frac{1}{2}$, $\quad z \in E$, then for any function F analytic in E, the function $p \star F$ takes values in the convex hull of the image E under F.*

3. MAIN RESULTS

Theorem 3.1. *Let $f \in T_k(\alpha, p, n, \rho_1)$ and $g \in T_k(\alpha, p, n, \rho_2)$, and let $F = f \star g$. Then $F \in T_k(\alpha, p, n, \rho_3)$ where*

$$\rho_3 = 1 - 4(1-\rho_1)(1-\rho_2)\left[1 - \frac{n+p}{1-\alpha}\int_0^1 \frac{u^{\left(\frac{n+p}{1-\alpha}\right)-1}}{1+u}du\right]. \tag{3.1}$$

This results is sharp.

Proof. Since $f \in T_k(\alpha, p, n, \rho_1)$, it follows that

$$H(z) = \left[(1-\alpha)\frac{I_{n+p-1}f(z)}{z^p} + \alpha\frac{I_{n+p}f(z)}{z^p}\right] \in P_k(\rho_1),$$

and so using (1.3), we have

$$I_{n+p}f(z) = \frac{n+p}{1-\alpha} z^{-\left(\frac{n+p}{1-\alpha}\right)} \int_0^z t^{\frac{n+p}{1-\alpha}-1} H(t)dt. \tag{3.2}$$

Similarly

$$I_{n+p}g(z) = \frac{n+p}{1-\alpha} z^{-\left(\frac{n+p}{1-\alpha}\right)} \int_0^z t^{\frac{n+p}{1-\alpha}-1} H^\star(t)dt, \tag{3.3}$$

where $H^\star \in P_k(\rho_2)$.

Using (3.1) and (3.2), we have

$$I_{n+p}F(z) = \frac{n+p}{1-\alpha} z^{-\left(\frac{n+p}{1-\alpha}\right)} \int_0^z t^{\frac{n+p}{1-\alpha}-1} Q(t)dt, \tag{3.4}$$

where

$$\begin{aligned} Q(z) &= \left(\frac{k}{4}+\frac{1}{2}\right) q_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) q_2(z) \\ &= \frac{n+p}{1-\alpha} z^{-\left(\frac{n+p}{1-\alpha}\right)} \int_0^z t^{\frac{n+p}{1-\alpha}-1} (H \star H^\star)(t)dt. \end{aligned} \tag{3.5}$$

Now

$$\begin{aligned} H(z) &= \left(\frac{k}{4}+\frac{1}{2}\right) h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right) h_2(z) \\ H(z)^\star &= \left(\frac{k}{4}+\frac{1}{2}\right) h_1^\star(z) - \left(\frac{k}{4}-\frac{1}{2}\right) h_2^\star(z), \end{aligned} \tag{3.6}$$

where $h_i \in P(\rho_1)$ and $h_i^\star \in P_k(\rho_2)$, $\quad i = 1, 2$.

Since

$$p_i^\star(z) = \frac{h_i^\star(z) - \rho_2}{2(1-\rho_2)} + \frac{1}{2} \in P\left(\frac{1}{2}\right), \quad i = 1, 2,$$

we obtain that $(h_i \star p_i^\star)(z) \in P(\rho_1)$, by using the Herglotz formula.

Thus

$$(h_i \star h_i^\star)(z) \in P(\rho_3)$$

with

$$\rho_3 = 1 - 2(1-\rho_1)(1-\rho_2). \tag{3.7}$$

Using (3.4), (3.5), (3.6), (3.7) and Lemma 2.1, we have

$$\begin{aligned}
\operatorname{Re} q_i(z) &= \frac{n+p}{1-\alpha}\int_0^1 u^{\frac{n+p}{1-\alpha}-1}\operatorname{Re}\{(h_i \star h_i^\star)(uz)\}du \\
&\geq \frac{n+p}{1-\alpha}\int_0^1 u^{\frac{n+p}{1-\alpha}-1}\left(2\rho_3 - 1 + \frac{2(1-\rho_3)}{1+u|z|}\right)du \\
&> \frac{n+p}{1-\alpha}\int_0^1 u^{\frac{n+p}{1-\alpha}-1}\left(2\rho_3 - 1 + \frac{2(1-\rho_3)}{1+u}\right)du \\
&= 1 - 4(1-\rho_1)(1-\rho_2)\left[1 - \frac{n+p}{1-\alpha}\int_0^1 \frac{u^{\frac{n+p}{1-\alpha}-1}}{1+u}du\right].
\end{aligned}$$

From this we conclude that $F \in T_k(\alpha, p, n, \rho_3)$, where ρ_3 is given by (3.1).

We discuss the sharpness as follows:

We take

$$\begin{aligned}
H(z) &= \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+(1-2\rho_1)z}{1-z} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-(1-2\rho_1)z}{1+z}, \\
H^\star(z) &= \left(\frac{k}{4}+\frac{1}{2}\right)\frac{1+(1-2\rho_2)z}{1-z} - \left(\frac{k}{4}-\frac{1}{2}\right)\frac{1-(1-2\rho_2)z}{1+z}.
\end{aligned}$$

Since

$$\left(\frac{1+(1-2\rho_1)z}{1-z}\right) \star \left(\frac{1+(1-2\rho_2)z}{1-z}\right) = 1 - 4(1-\rho_1)(1-\rho_2) + \frac{4(1-\rho_1)(1-\rho_2)}{1-z},$$

it follows from (3.5) that

$$\begin{aligned}
q_i(z) &= \frac{n+p}{1-\alpha}\int_0^1 u^{\frac{n+p}{1-\alpha}-1}\left\{1 - 4(1-\rho_1)(1-\rho_2) + \frac{4(1-\rho_1)(1-\rho_2)}{1-uz}\right\}du \\
&\longrightarrow 1 - 4(1-\rho_1)(1-\rho_2)\left\{1 - \frac{n+p}{1-\alpha}\int_0^1 \frac{u^{\frac{n+p}{1-\alpha}-1}}{1+u}du\right\} \quad \text{as} \quad z \longrightarrow 1.
\end{aligned}$$

This completes the proof. □

We define $J_c : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ as follows:

$$J_c(f) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)dt, \tag{3.8}$$

where c is real and $c > -p$.

Theorem 3.2. *Let $f \in T_k(\alpha, p, n, \rho)$ and $J_c(f)$ be given by (3.8). If*

$$\left[(1-\alpha)\frac{I_{n+p}f(z)}{z^p} + \alpha\frac{I_{n+p}J_c(f)}{z^p}\right] \in P_k(\rho), \tag{3.9}$$

then

$$\left\{\frac{I_{n+p}J_c(f)}{z^p}\right\} \in P_k(\gamma), \quad z \in E$$

and

$$\gamma = \rho(1-\rho)(2\sigma - 1)$$

(3.10)
$$\sigma = \int_0^1 \left[1 + t^{\operatorname{Re}\frac{1-\alpha}{\lambda+p}}\right]^{-1} dt.$$

Proof. From (3.8), we have

$$(c+p)I_{n+p}f(z) = cI_{n+p}J_c(f) + z(I_{n+p}J_c(f))'.$$

Let

(3.11)
$$H_c(z) = \left(\frac{k}{4}+\frac{1}{2}\right)s_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)s_2(z) = \frac{I_{n+p}J_c(f)}{z^p}.$$

From (3.9), (3.10) and (3.11), we have

$$\left[(1-\alpha)\frac{I_{n+p}f(z)}{z^p} + \alpha\frac{I_{n+p}J_c(f)}{z^p}\right] = \left[H_c(z) + \frac{1-\alpha}{\lambda+p}zH_c'(z)\right]$$

and consequently

$$\left[s_i(z) + \frac{1-\alpha}{\lambda+p}zs_i'(z)\right] \in P(\rho), \quad i = 1,2.$$

Using Lemma 2.2, we have $\operatorname{Re}\{s_i(z)\} > \gamma$ where γ is given by (3.10). Thus

$$H_c(z) = \frac{I_{n+p}J_c(f)}{z^p} \in P_k(\gamma)$$

and this completes the proof. □

Let

(3.12)
$$J_n(f(z)) := J_n(f) = \frac{n+p}{z^p}\int_0^z t^{n-1}f(t)dt.$$

Then

$$I_{n+p-1}J_n(f) = I_{n+p}(f),$$

and we have the following.

Theorem 3.3. *Let $f \in T_k(\alpha,p,n+1,\rho)$. Then $J_n(f) \in T_k(\alpha,p,n,\rho)$ for $z \in E$.*

Theorem 3.4. *Let $\phi \in C_p$, where C_p is the class of p-valent convex functions, and let $f \in T_k(\alpha,p,n,\rho)$. Then $\phi \star f \in T_k(\alpha,p,n,\rho)$ for $z \in E$.*

Proof. Let $G = \phi \star f$. Then

$$\begin{aligned}
(1-\alpha)\frac{I_{n+p-1}G(z)}{z^p} + \alpha\frac{I_{n+p}G(z)}{z^p} &= (1-\alpha)\frac{I_{n+p-1}(\phi\star f)(z)}{z^p} + \alpha\frac{I_{n+p}(\phi\star f)(z)}{z^p} \\
&= \frac{\phi(z)}{z^p}\star\left[(1-\alpha)\frac{I_{n+p-1}f(z)}{z^p} + \alpha\frac{I_{n+p}f(z)}{z^p}\right] \\
&= \frac{\phi(z)}{z^p}\star H(z), \quad H \in P_k(\rho) \\
&= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{(p-\rho)\left(\frac{\phi(z)}{z^p}\star h_1(z)\right)+\rho\right\} \\
&\qquad - \left(\frac{k}{4}-\frac{1}{2}\right)\left\{(p-\rho)\left(\frac{\phi(z)}{z^p}\star h_2(z)\right)+\rho\right\}, \quad h_1, h_2 \in P.
\end{aligned}$$

Since $\phi \in C_p, \quad \operatorname{Re}\left\{\frac{\phi(z)}{z^P}\right\} > \frac{1}{2}, z \in E$ and so using Lemma 2.3, we conclude that $G \in T_k(\alpha,p,n,\rho)$. □

3.1. **Applications.**

(1) We can write $J_c(f)$ defined by (3.8) as

$$J_c(f) = \phi_c \star f,$$

where ϕ_c is given by

$$\phi_c(z) = \sum_{m=p}^{\infty} \frac{p+c}{m+c} z^m, \quad (c > -p)$$

and $\phi_c \in C_p$. Therefore, from Theorem 3.4, it follows that $J_c(f) \in T_k(\alpha, p, n, \rho)$.

(2) Let $J_n(f)$, defined by (3.12), belong to $T_k(\alpha, p, n, \rho)$. Then $f \in T_k(\alpha, p, n, \rho)$ for $|z| < r_n = \frac{(1+n)}{2+\sqrt{3+n^2}}$. In fact, $J_n(f) = \Psi_n \star f$, where

$$\begin{aligned}\Psi_n(z) &= z^p + \sum_{j=2}^{\infty} \frac{n+j-1}{n+1} z^{j+p-1} \\ &= \frac{n}{n+1} \cdot \frac{z^p}{1-z} + \frac{1}{n+1} \cdot \frac{z^p}{(1-z)^2}\end{aligned}$$

and $\Psi_n \in C_p$ for

$$|z| < r_n = \frac{1+n}{2+\sqrt{3+n^2}}.$$

Now $I_{n+p-1}J_n(f) = \Psi_n \star I_{n+p-1}f$, and using Theorem 3.4, we obtain the result.

Theorem 3.5. *For* $0 \le \alpha_2 < \alpha_1, \quad T_k(\alpha_1, p, n, \rho) \subset T_k(\alpha_2, p, n, \rho), \quad z \in E.$

Proof. For $\alpha_2 = 0$, the proof is immediate. Let $\alpha_2 > 0$ and let $f \in T_k(\alpha_1, p, n, \rho)$. Then

$$\begin{aligned}&(1-\alpha_2)\frac{I_{n+p-1}f(z)}{z^p} + \alpha_2 \frac{I_{n+p}f(z)}{z^p} \\ &\qquad + \frac{\alpha_2}{\alpha_1}\left[\left(\frac{\alpha_1}{\alpha_2} - 1\right)\frac{I_{n+p-1}f(z)}{z^p} + (1-\alpha_1)\frac{I_{n+p-1}f(z)}{z^p} + \alpha_1 \frac{I_{n+p-1}f(z)}{z^p}\right] \\ &= \left(1 - \frac{\alpha_2}{\alpha_1}\right) H_1(z) + \frac{\alpha_2}{\alpha_1} H_2(z), \quad H_1, H_2 \in P_k(\rho).\end{aligned}$$

Since $P_k(\rho)$ is a convex set, we conclude that $f \in T_k(\alpha_2, p, n, \rho)$ for $z \in E$. □

Theorem 3.6. *Let* $f \in T_k(0, p, n, \rho)$. *Then* $f \in T_k(\alpha, p, n, \rho)$ *for*

$$|z| < r_\alpha = \frac{1}{2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1}}, \quad \alpha \ne \frac{1}{2}, \quad 0 < \alpha < 1.$$

Proof. Let

$$\begin{aligned}\Psi_\alpha(z) &= (1-\alpha)\frac{z^p}{1-z} + \alpha \frac{z^p}{(1-z)^2} \\ &= z^p + \sum_{m=2}^{\infty} (1 + (m-1)\alpha) z^{m+p-1}.\end{aligned}$$

$\Psi_\alpha \in C_p$ for

$$|z| < r_\alpha = \frac{1}{2\alpha + \sqrt{4\alpha^2 - 2\alpha + 1}} \quad \left(\alpha \ne \frac{1}{2}, \quad 0 < \alpha < 1\right)$$

We can write

$$\left[(1-\alpha)\frac{I_{n+p-1}f(z)}{z^p}+\alpha\frac{I_{n+p}f(z)}{z^p}\right]=\frac{\Psi_\alpha(z)}{z^p}\star\frac{I_{n+p-1}f(z)}{z^p}.$$

Applying Theorem 3.4, we see that $f\in T_k(\alpha,p,n,\rho)$ for $|z|<r_\alpha$. □

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