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THE EQUAL VARIABLE METHOD

VASILE CÎRTOAJE

DEPARTMENT OF AUTOMATION AND COMPUTERS
UNIVERSITY OF PLOIEŞTI
BUCUREŞTI 39, ROMANIA
vcirtoaje@upg-ploiesti.ro

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ABSTRACT. The Equal Variable Method (called also $n-1$ Equal Variable Method on the Mathlinks Site - Inequalities Forum) can be used to prove some difficult symmetric inequalities involving either three power means or, more general, two power means and an expression of form $f(x_1) + f(x_2) + \cdots + f(x_n)$.

Key words and phrases: Symmetric inequalities, Power means, EV-Theorem.

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1. STATEMENT OF RESULTS

In order to state and prove the Equal Variable Theorem (EV-Theorem) we require the following lemma and proposition.

Lemma 1.1. *Let a, b, c be fixed non-negative real numbers, not all equal and at most one of them equal to zero, and let $x \le y \le z$ be non-negative real numbers such that*

$$x + y + z = a + b + c, \quad x^p + y^p + z^p = a^p + b^p + c^p,$$

where $p \in (-\infty, 0] \cup (1, \infty)$. For $p = 0$, the second equation is $xyz = abc > 0$. Then, there exist two non-negative real numbers x_1 and x_2 with $x_1 < x_2$ such that $x \in [x_1, x_2]$. Moreover,

(1) *if $x = x_1$ and $p \le 0$, then $0 < x < y = z$;*
(2) *if $x = x_1$ and $p > 1$, then either $0 = x < y \le z$ or $0 < x < y = z$;*
(3) *if $x \in (x_1, x_2)$, then $x < y < z$;*
(4) *if $x = x_2$, then $x = y < z$.*

Proposition 1.2. *Let a, b, c be fixed non-negative real numbers, not all equal and at most one of them equal to zero, and let $0 \le x \le y \le z$ such that*

$$x + y + z = a + b + c, \quad x^p + y^p + z^p = a^p + b^p + c^p,$$

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where $p \in (-\infty, 0] \cup (1, \infty)$. For $p = 0$, the second equation is $xyz = abc > 0$. Let $f(u)$ be a differentiable function on $(0, \infty)$, such that $g(x) = f'\left(x^{\frac{1}{p-1}}\right)$ is strictly convex on $(0, \infty)$, and let

$$F_3(x, y, z) = f(x) + f(y) + f(z).$$

(1) *If $p \leq 0$, then F_3 is maximal only for $0 < x = y < z$, and is minimal only for $0 < x < y = z$;*
(2) *If $p > 1$ and either $f(u)$ is continuous at $u = 0$ or $\lim_{u \to 0} f(u) = -\infty$, then F_3 is maximal only for $0 < x = y < z$, and is minimal only for either $x = 0$ or $0 < x < y = z$.*

Theorem 1.3 (Equal Variable Theorem (EV-Theorem))**.** *Let $a_1, a_2, \dots, a_n$ $(n \geq 3)$ be fixed non-negative real numbers, and let $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ such that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$x_1^p + x_2^p + \cdots + x_n^p = a_1^p + a_2^p + \cdots + a_n^p,$$

where p is a real number, $p \neq 1$. For $p = 0$, the second equation is $x_1 x_2 \cdots x_n = a_1 a_2 \cdots a_n > 0$. Let $f(u)$ be a differentiable function on $(0, \infty)$ such that

$$g(x) = f'\left(x^{\frac{1}{p-1}}\right)$$

is strictly convex on $(0, \infty)$, and let

$$F_n(x_1, x_2, \dots, x_n) = f(x_1) + f(x_2) + \cdots + f(x_n).$$

(1) *If $p \leq 0$, then F_n is maximal for $0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n$, and is minimal for $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$;*
(2) *If $p > 0$ and either $f(u)$ is continuous at $u = 0$ or $\lim_{u \to 0} f(u) = -\infty$, then F_n is maximal for $0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n$, and is minimal for either $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$.*

Remark 1.4. Let $0 < \alpha < \beta$. If the function f is differentiable on (α, β) and the function $g(x) = f'\left(x^{\frac{1}{p-1}}\right)$ is strictly convex on $(\alpha^{p-1}, \beta^{p-1})$ or $(\beta^{p-1}, \alpha^{p-1})$, then the EV-Theorem holds true for $x_1, x_2, \dots, x_n \in (\alpha, \beta)$.

By Theorem 1.3, we easily obtain some particular results, which are very useful in applications.

Corollary 1.5. *Let $a_1, a_2, \dots, a_n$ $(n \geq 3)$ be fixed non-negative numbers, and let $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ such that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$x_1^2 + x_2^2 + \cdots + x_n^2 = a_1^2 + a_2^2 + \cdots + a_n^2.$$

Let f be a differentiable function on $(0, \infty)$ such that $g(x) = f'(x)$ is strictly convex on $(0, \infty)$. Moreover, either $f(x)$ is continuous at $x = 0$ or $\lim_{x \to 0} f(x) = -\infty$. Then,

$$F_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is maximal for $0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n$, and is minimal for either $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$.

Corollary 1.6. *Let* $a_1, a_2, \dots, a_n$ $(n \ge 3)$ *be fixed positive numbers, and let* $0 < x_1 \le x_2 \le \dots \le x_n$ *such that*

$$x_1 + x_2 + \dots + x_n = a_1 + a_2 + \dots + a_n,$$
$$\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n} = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}.$$

Let f *be a differentiable function on* $(0, \infty)$ *such that* $g(x) = f'\left(\frac{1}{\sqrt{x}}\right)$ *is strictly convex on* $(0, \infty)$*. Then,*

$$F_n = f(x_1) + f(x_2) + \dots + f(x_n)$$

is maximal for $0 < x_1 = x_2 = \dots = x_{n-1} \le x_n$*, and is minimal for* $0 < x_1 \le x_2 = x_3 = \dots = x_n$*.*

Corollary 1.7. *Let* $a_1, a_2, \dots, a_n$ $(n \ge 3)$ *be fixed positive numbers, and let* $0 < x_1 \le x_2 \le \dots \le x_n$ *such that*

$$x_1 + x_2 + \dots + x_n = a_1 + a_2 + \dots + a_n, \quad x_1 x_2 \cdots x_n = a_1 a_1 \cdots a_n.$$

Let f *be a differentiable function on* $(0, \infty)$ *such that* $g(x) = f'\left(\frac{1}{x}\right)$ *is strictly convex on* $(0, \infty)$*. Then,*

$$F_n = f(x_1) + f(x_2) + \dots + f(x_n)$$

is maximal for $0 < x_1 = x_2 = \dots = x_{n-1} \le x_n$*, and is minimal for* $0 < x_1 \le x_2 = x_3 = \dots = x_n$*.*

Corollary 1.8. *Let* $a_1, a_2, \dots, a_n$ $(n \ge 3)$ *be fixed non-negative numbers, and let* $0 \le x_1 \le x_2 \le \dots \le x_n$ *such that*

$$x_1 + x_2 + \dots + x_n = a_1 + a_2 + \dots + a_n,$$
$$x_1^p + x_2^p + \dots + x_n^p = a_1^p + a_2^p + \dots + a_n^p,$$

where p *is a real number,* $p \ne 0$ *and* $p \ne 1$*.*

(a) *For* $p < 0$*,* $P = x_1 x_2 \cdots x_n$ *is minimal when* $0 < x_1 = x_2 = \dots = x_{n-1} \le x_n$*, and is maximal when* $0 < x_1 \le x_2 = x_3 = \dots = x_n$*.*
(b) *For* $p > 0$*,* $P = x_1 x_2 \cdots x_n$ *is maximal when* $0 \le x_1 = x_2 = \dots = x_{n-1} \le x_n$*, and is minimal when either* $x_1 = 0$ *or* $0 < x_1 \le x_2 = x_3 = \dots = x_n$*.*

Corollary 1.9. *Let* $a_1, a_2, \dots, a_n$ $(n \ge 3)$ *be fixed non-negative numbers, let* $0 \le x_1 \le x_2 \le \dots \le x_n$ *such that*

$$x_1 + x_2 + \dots + x_n = a_1 + a_2 + \dots + a_n,$$
$$x_1^p + x_2^p + \dots + x_n^p = a_1^p + a_2^p + \dots + a_n^p,$$

and let $E = x_1^q + x_2^q + \dots + x_n^q$*.*

Case 1. $p \le 0$ ($p = 0$ yields $x_1 x_2 \cdots x_n = a_1 a_2 \cdots a_n > 0$).

(a) For $q \in (p, 0) \cup (1, \infty)$, E is maximal when $0 < x_1 = x_2 = \dots = x_{n-1} \le x_n$, and is minimal when $0 < x_1 \le x_2 = x_3 = \dots = x_n$.
(b) For $q \in (-\infty, p) \cup (0, 1)$, E is minimal when $0 < x_1 = x_2 = \dots = x_{n-1} \le x_n$, and is maximal when $0 < x_1 \le x_2 = x_3 = \dots = x_n$.

Case 2. $0 < p < 1$.

(a) For $q \in (0, p) \cup (1, \infty)$, E is maximal when $0 \le x_1 = x_2 = \dots = x_{n-1} \le x_n$, and is minimal when either $x_1 = 0$ or $0 < x_1 \le x_2 = x_3 = \dots = x_n$.

(b) For $q \in (-\infty, 0) \cup (p, 1)$, E is minimal when $0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n$, and is maximal when either $x_1 = 0$ or $0 < x_1 \le x_2 = x_3 = \cdots = x_n$.

Case 3. $p > 1$.

(a) For $q \in (0, 1) \cup (p, \infty)$, E is maximal when $0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n$, and is minimal when either $x_1 = 0$ or $0 < x_1 \le x_2 = x_3 = \cdots = x_n$.

(b) For $q \in (-\infty, 0) \cup (1, p)$, E is minimal when $0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n$, and is maximal when either $x_1 = 0$ or $0 < x_1 \le x_2 = x_3 = \cdots = x_n$.

2. PROOFS

Proof of Lemma 1.1. Let $a \le b \le c$. Note that in the excluded cases $a = b = c$ and $a = b = 0$, there is a single triple (x, y, z) which verifies the conditions

$$x + y + z = a + b + c \quad \text{and} \quad x^p + y^p + z^p = a^p + b^p + c^p.$$

Consider now three cases: $p = 0$, $p < 0$ and $p > 1$.

A. Case $p = 0$ $(xyz = abc > 0)$. Let $S = \frac{a+b+c}{3}$ and $P = \sqrt[3]{abc}$, where $S > P > 0$ by AM-GM Inequality. We have

$$x + y + z = 3S, \quad xyz = P^3,$$

and from $0 < x \le y \le z$ and $x < z$, it follows that $0 < x < P$. Now let

$$f = y + z - 2\sqrt{yz}.$$

It is clear that $f \ge 0$, with equality if and only if $y = z$. Writing f as a function of x,

$$f(x) = 3S - x - 2P\sqrt{\frac{P}{x}},$$

we have

$$f'(x) = \frac{P}{x}\sqrt{\frac{P}{x}} - 1 > 0,$$

and hence the function $f(x)$ is strictly increasing. Since $f(P) = 3(S - P) > 0$, the equation $f(x) = 0$ has a unique positive root x_1, $0 < x_1 < P$. From $f(x) \ge 0$, it follows that $x \ge x_1$.

Sub-case $x = x_1$. Since $f(x) = f(x_1) = 0$ and $f = 0$ implies $y = z$, we have $0 < x < y = z$.

Sub-case $x > x_1$. We have $f(x) > 0$ and $y < z$. Consider now that y and z depend on x. From $x + y(x) + z(x) = 3S$ and $x \cdot y(x) \cdot z(x) = P^3$, we get $1 + y' + z' = 0$ and $\frac{1}{x} + \frac{y'}{y} + \frac{z'}{z} = 0$. Hence,

$$y'(x) = \frac{y(x - z)}{x(z - y)}, \quad z'(x) = \frac{z(y - x)}{x(z - y)}.$$

Since $y'(x) < 0$, the function $y(x)$ is strictly decreasing. Since $y(x_1) > x_1$ (see sub-case $x = x_1$), there exists $x_2 > x_1$ such that $y(x_2) = x_2$, $y(x) > x$ for $x_1 < x < x_2$ and $y(x) < x$ for $x > x_2$. Taking into account that $y \ge x$, it follows that $x_1 < x \le x_2$. On the other hand, we see that $z'(x) > 0$ for $x_1 < x < x_2$. Consequently, the function $z(x)$ is strictly increasing, and hence $z(x) > z(x_1) = y(x_1) > y(x)$. Finally, we conclude that $x < y < z$ for $x \in (x_1, x_2)$, and $x = y < z$ for $x = x_2$.

B. Case $p < 0$. Denote $S = \frac{a+b+c}{3}$ and $R = \left(\frac{a^p+b^p+c^p}{3}\right)^{\frac{1}{p}}$. Taking into account that

$$x + y + z = 3S, \quad x^p + y^p + z^p = 3R^p,$$

from $0 < x \le y \le z$ and $x < z$ we get $x < S$ and $3^{\frac{1}{p}}R < x < R$. Let

$$h = (y+z)\left(\frac{y^p+z^p}{2}\right)^{\frac{-1}{p}} - 2.$$

By the AM-GM Inequality, we have

$$h \ge 2\sqrt{yz}\frac{1}{\sqrt{yz}} - 2 = 0,$$

with equality if and only if $y = z$. Writing now h as a function of x,

$$h(x) = (3S - x)\left(\frac{3R^p - x^p}{2}\right)^{\frac{-1}{p}} - 2,$$

from

$$h'(x) = \frac{3R^p}{2}\left(\frac{3R^p - x^p}{2}\right)^{\frac{-1-p}{p}}\left[\left(\frac{S}{x}\right)\left(\frac{R}{x}\right)^{-p} - 1\right] > 0$$

it follows that $h(x)$ is strictly increasing. Since $h(x) \ge 0$ and $h\left(3^{\frac{1}{p}}R\right) = -2$, the equation $h(x) = 0$ has a unique root x_1 and $x \ge x_1 > 3^{\frac{1}{p}}R$.

Sub-case $x = x_1$. Since $f(x) = f(x_1) = 0$, and $f = 0$ implies $y = z$, we have $0 < x < y = z$.

Sub-case $x > x_1$. We have $h(x) > 0$ and $y < z$. Consider now that y and z depend on x. From $x + y(x) + z(x) = 3S$ and $x^p + y(x)^p + z(x)^p = 3R^p$, we get $1 + y' + z' = 0$ and $x^{p-1} + y^{p-1}y' + z^{p-1}z' = 0$, and hence

$$y'(x) = \frac{x^{p-1} - z^{p-1}}{z^{p-1} - y^{p-1}}, \quad z'(x) = \frac{x^{p-1} - y^{p-1}}{y^{p-1} - z^{p-1}}.$$

Since $y'(x) > 0$, the function $y(x)$ is strictly decreasing. Since $y(x_1) > x_1$ (see sub-case $x = x_1$), there exists $x_2 > x_1$ such that $y(x_2) = x_2$, $y(x) > x$ for $x_1 < x < x_2$, and $y(x) < x$ for $x > x_2$. The condition $y \ge x$ yields $x_1 < x \le x_2$. We see now that $z'(x) > 0$ for $x_1 < x < x_2$. Consequently, the function $z(x)$ is strictly increasing, and hence $z(x) > z(x_1) = y(x_1) > y(x)$. Finally, we have $x < y < z$ for $x \in (x_1, x_2)$ and $x = y < z$ for $x = x_2$.

C. Case $p > 1$. Denoting $S = \frac{a+b+c}{3}$ and $R = \left(\frac{a^p+b^p+c^p}{3}\right)^{\frac{1}{p}}$ yields

$$x + y + z = 3S, \quad x^p + y^p + z^p = 3R^p.$$

By Jensen's inequality applied to the convex function $g(u) = u^p$, we have $R > S$, and hence $x < S < R$. Let

$$h = \frac{2}{y+z}\left(\frac{y^p+z^p}{2}\right)^{\frac{1}{p}} - 1.$$

By Jensen's Inequality, we get $h \ge 0$, with equality if only if $y = z$. From

$$h(x) = \frac{2}{3S - x}\left(\frac{3R^p - x^p}{2}\right)^{\frac{1}{p}} - 1$$

and

$$h'(x) = \frac{3}{(3S-x)^2}\left(\frac{3R^p - x^p}{2}\right)^{\frac{1-p}{p}}(R^p - Sx^{p-1}) > 0,$$

it follows that the function $h(x)$ is strictly increasing, and $h(x) \ge 0$ implies $x \ge x_1$. In the case $h(0) \ge 0$ we have $x_1 = 0$, and in the case $h(0) < 0$ we have $x_1 > 0$ and $h(x_1) = 0$.

Sub-case $x = x_1$. If $h(0) \geq 0$, then $0 = x_1 < y(x_1) \leq z(x_1)$. If $h(0) < 0$, then $h(x_1) = 0$, and since $h = 0$ implies $y = z$, we have $0 < x_1 < y(x_1) = z(x_1)$.

Sub-case $x > x_1$. Since $h(x)$ is strictly increasing, for $x > x_1$ we have $h(x) > h(x_1) \geq 0$, hence $h(x) > 0$ and $y < z$. From $x + y(x) + z(x) = 3S$ and $x^p + y^p(x) + z^p(x) = 3R^p$, we get

$$y'(x) = \frac{x^{p-1} - z^{p-1}}{z^{p-1} - y^{p-1}}, \quad z'(x) = \frac{y^{p-1} - x^{p-1}}{z^{p-1} - y^{p-1}}.$$

Since $y'(x) < 0$, the function $y(x)$ is strictly decreasing. Taking account of $y(x_1) > x_1$ (see sub-case $x = x_1$), there exists $x_2 > x_1$ such that $y(x_2) = x_2$, $y(x) > x$ for $x_1 < x < x_2$, and $y(x) < x$ for $x > x_2$. The condition $y \geq x$ implies $x_1 < x \leq x_2$. We see now that $z'(x) > 0$ for $x_1 < x < x_2$. Consequently, the function $z(x)$ is strictly increasing, and hence $z(x) > z(x_1) \geq y(x_1) > y(x)$. Finally, we conclude that $x < y < z$ for $x \in (x_1, x_2)$, and $x = y < z$ for $x = x_2$. □

Proof of Proposition 1.2. Consider the function

$$F(x) = f(x) + f(y(x)) + f(z(x))$$

defined on $x \in [x_1, x_2]$. We claim that $F(x)$ is minimal for $x = x_1$ and is maximal for $x = x_2$. If this assertion is true, then by Lemma 1.1 it follows that:

(a) $F(x)$ is minimal for $0 < x = y < z$ in the case $p \leq 0$, or for either $x = 0$ or $0 < x < y = z$ in the case $p > 1$;

(b) $F(x)$ is maximal for $0 < x = y < z$.

In order to prove the claim, assume that $x \in (x_1, x_2)$. By Lemma 1.1, we have $0 < x < y < z$. From

$$x + y(x) + z(x) = a + b + c \quad \text{and}$$
$$x^p + y^p(x) + z^p(x) = a^p + b^p + c^p,$$

we get

$$y' + z' = -1, \quad y^{p-1}y' + z^{p-1}z' = -x^{p-1},$$

whence

$$y' = \frac{x^{p-1} - z^{p-1}}{z^{p-1} - y^{p-1}}, \quad z' = \frac{x^{p-1} - y^{p-1}}{y^{p-1} - z^{p-1}}.$$

It is easy to check that this result is also valid for $p = 0$. We have

$$F'(x) = f'(x) + y'f'(y) + z'f'(z)$$

and

$$\frac{F'(x)}{(x^{p-1} - y^{p-1})(x^{p-1} - z^{p-1})} = \frac{g(x^{p-1})}{(x^{p-1} - y^{p-1})(x^{p-1} - z^{p-1})} + \frac{g(y^{p-1})}{(y^{p-1} - z^{p-1})(y^{p-1} - x^{p-1})} + \frac{g(z^{p-1})}{(z^{p-1} - x^{p-1})(z^{p-1} - y^{p-1})}.$$

Since g is strictly convex, the right hand side is positive. On the other hand,

$$(x^{p-1} - y^{p-1})(x^{p-1} - z^{p-1}) > 0.$$

These results imply $F'(x) > 0$. Consequently, the function $F(x)$ is strictly increasing for $x \in (x_1, x_2)$. Excepting the trivial case when $p > 1$, $x_1 = 0$ and $\lim\limits_{u \to 0} f(u) = -\infty$, the function

$F(x)$ is continuous on $[x_1, x_2]$, and hence is minimal only for $x = x_1$, and is maximal only for $x = x_2$. □

Proof of Theorem 1.3. We will consider two cases.

Case $p \in (-\infty, 0] \cup (1, \infty)$. Excepting the trivial case when $p > 1$, $x_1 = 0$ and $\lim_{u \to 0} f(u) = -\infty$, the function $F_n(x_1, x_2, \ldots, x_n)$ attains its minimum and maximum values, and the conclusion follows from Proposition 1.2 above, via contradiction. For example, let us consider the case $p \leq 0$. In order to prove that F_n is maximal for $0 < x_1 = x_2 = \cdots = x_{n-1} \leq x_n$, we assume, for the sake of contradiction, that F_n attains its maximum at $(b_1, b_2, \ldots, b_n)$ with $b_1 \leq b_2 \leq \cdots \leq b_n$ and $b_1 < b_{n-1}$. Let x_1, x_{n-1}, x_n be positive numbers such that $x_1 + x_{n-1} + x_n = b_1 + b_{n-1} + b_n$ and $x_1^p + x_{n-1}^p + x_n^p = b_1^p + b_{n-1}^p + b_n^p$. According to Proposition 1.2, the expression

$$F_3(x_1, x_{n-1}, x_n) = f(x_1) + f(x_{n-1}) + f(x_n)$$

is maximal only for $x_1 = x_{n-1} < x_n$, which contradicts the assumption that F_n attains its maximum at $(b_1, b_2, \ldots, b_n)$ with $b_1 < b_{n-1}$.

Case $p \in (0, 1)$. This case reduces to the case $p > 1$, replacing each of the a_i by $a_i^{\frac{1}{p}}$, each of the x_i by $x_i^{\frac{1}{p}}$, and then p by $\frac{1}{p}$. Thus, we obtain the sufficient condition that $h(x) = xf'\left(x^{\frac{1}{1-p}}\right)$ to be strictly convex on $(0, \infty)$. We claim that this condition is equivalent to the condition that $g(x) = f'\left(x^{\frac{1}{p-1}}\right)$ to be strictly convex on $(0, \infty)$. Actually, for our proof, it suffices to show that if $g(x)$ is strictly convex on $(0, \infty)$, then $h(x)$ is strictly convex on $(0, \infty)$. To show this, we see that $g\left(\frac{1}{x}\right) = \frac{1}{x}h(x)$. Since $g(x)$ is strictly convex on $(0, \infty)$, by Jensen's inequality we have

$$ug\left(\frac{1}{x}\right) + vg\left(\frac{1}{y}\right) > (u+v)g\left(\frac{\frac{u}{x} + \frac{v}{y}}{u+v}\right)$$

for any $x, y, u, v > 0$ with $x \neq y$. This inequality is equivalent to

$$\frac{u}{x}h(x) + \frac{v}{y}h(y) > \left(\frac{u}{x} + \frac{v}{y}\right) h\left(\frac{u+v}{\frac{u}{x} + \frac{v}{y}}\right).$$

Substituting $u = tx$ and $v = (1-t)y$, where $t \in (0, 1)$, reduces the inequality to

$$th(x) + (1-t)h(y) > h(tx + (1-t)y),$$

which shows us that $h(x)$ is strictly convex on $(0, \infty)$. □

Proof of Corollary 1.8. We will apply Theorem 1.3 to the function $f(u) = p \ln u$. We see that $\lim_{u \to 0} f(u) = -\infty$ for $p > 0$, and

$$f'(u) = \frac{p}{u}, \quad g(x) = f'\left(x^{\frac{1}{p-1}}\right) = px^{\frac{1}{1-p}}, \quad g''(x) = \frac{p^2}{(1-p)^2}x^{\frac{2p-1}{1-p}}.$$

Since $g''(x) > 0$ for $x > 0$, the function $g(x)$ is strictly convex on $(0, \infty)$, and the conclusion follows by Theorem 1.3. □

Proof of Corollary 1.9. We will apply Theorem 1.3 to the function

$$f(u) = q(q-1)(q-p)u^q.$$

For $p > 0$, it is easy to check that either $f(u)$ is continuous at $u = 0$ (in the case $q > 0$) or $\lim_{u \to 0} f(u) = -\infty$ (in the case $q < 0$). We have

$$f'(u) = q^2(q-1)(q-p)u^{q-1}$$

and

$$g(x) = f'\left(x^{\frac{1}{p-1}}\right) = q^2(q-1)(q-p)x^{\frac{q-1}{p-1}},$$

$$g''(x) = \frac{q^2(q-1)^2(q-p)^2}{(p-1)^2} x^{\frac{2p-1}{1-p}}.$$

Since $g''(x) > 0$ for $x > 0$, the function $g(x)$ is strictly convex on $(0, \infty)$, and the conclusion follows by Theorem 1.3. □

3. APPLICATIONS

Proposition 3.1. *Let x, y, z be non-negative real numbers such that $x+y+z=2$. If $r_0 \le r \le 3$, where $r_0 = \frac{\ln 2}{\ln 3 - \ln 2} \approx 1.71$, then*

$$x^r(y+z) + y^r(z+x) + z^r(x+y) \le 2.$$

Proof. Rewrite the inequality in the homogeneous form

$$x^{r+1} + y^{r+1} + z^{r+1} + 2\left(\frac{x+y+z}{2}\right)^{r+1} \ge (x+y+z)(x^r + y^r + z^r),$$

and apply Corollary 1.9 (case $p = r$ and $q = r+1$):

If $0 \le x \le y \le z$ such that

$$x + y + z = \text{constant} \quad \text{and}$$
$$x^r + y^r + z^r = \text{constant},$$

then the sum $x^{r+1} + y^{r+1} + z^{r+1}$ is minimal when either $x = 0$ or $0 < x \le y = z$.

Case $x = 0$. The initial inequality becomes

$$yz(y^{r-1} + z^{r-1}) \le 2,$$

where $y + z = 2$. Since $0 < r - 1 \le 2$, by the Power Mean inequality we have

$$\frac{y^{r-1} + z^{r-1}}{2} \le \left(\frac{y^2+z^2}{2}\right)^{\frac{r-1}{2}}.$$

Thus, it suffices to show that

$$yz\left(\frac{y^2+z^2}{2}\right)^{\frac{r-1}{2}} \le 1.$$

Taking account of

$$\frac{y^2+z^2}{2} = \frac{2(y^2+z^2)}{(y+z)^2} \ge 1 \quad \text{and} \quad \frac{r-1}{2} \le 1,$$

we have

$$\begin{aligned} 1 - yz\left(\frac{y^2+z^2}{2}\right)^{\frac{r-1}{2}} &\ge 1 - yz\left(\frac{y^2+z^2}{2}\right) \\ &= \frac{(y+z)^4}{16} - \frac{yz(y^2+z^2)}{2} \\ &= \frac{(y-z)^4}{16} \ge 0. \end{aligned}$$

Case $0 < x \le y = z$. In the homogeneous inequality we may leave aside the constraint $x + y + z = 2$, and consider $y = z = 1$, $0 < x \le 1$. The inequality reduces to

$$\left(1 + \frac{x}{2}\right)^{r+1} - x^r - x - 1 \ge 0.$$

Since $\left(1+\frac{x}{2}\right)^{r+1}$ is increasing and x^r is decreasing in respect to r, it suffices to consider $r = r_0$. Let

$$f(x) = \left(1+\frac{x}{2}\right)^{r_0+1} - x^{r_0} - x - 1.$$

We have

$$f'(x) = \frac{r_0+1}{2}\left(1+\frac{x}{2}\right)^{r_0} - r_0 x^{r_0-1} - 1,$$
$$\frac{1}{r_0}f''(x) = \frac{r_0+1}{4}\left(1+\frac{x}{2}\right)^{r_0} - \frac{r_0-1}{x^{2-r_0}}.$$

Since $f''(x)$ is strictly increasing on $(0,1]$, $f''(0_+) = -\infty$ and

$$\frac{1}{r_0}f''(1) = \frac{r_0+1}{4}\left(\frac{3}{2}\right)^{r_0} - r_0 + 1$$
$$= \frac{r_0+1}{2} - r_0 + 1 = \frac{3-r_0}{2} > 0,$$

there exists $x_1 \in (0,1)$ such that $f''(x_1) = 0$, $f''(x) < 0$ for $x \in (0, x_1)$, and $f''(x) > 0$ for $x \in (x_1, 1]$. Therefore, the function $f'(x)$ is strictly decreasing for $x \in [0, x_1]$, and strictly increasing for $x \in [x_1, 1]$. Since

$$f'(0) = \frac{r_0-1}{2} > 0 \quad \text{and} \quad f'(1) = \frac{r_0+1}{2}\left[\left(\frac{3}{2}\right)^{r_0} - 2\right] = 0,$$

there exists $x_2 \in (0, x_1)$ such that $f'(x_2) = 0$, $f'(x) > 0$ for $x \in [0, x_2)$, and $f'(x) < 0$ for $x \in (x_2, 1)$. Thus, the function $f(x)$ is strictly increasing for $x \in [0, x_2]$, and strictly decreasing for $x \in [x_2, 1]$. Since $f(0) = f(1) = 0$, it follows that $f(x) \geq 0$ for $0 < x \leq 1$, establishing the desired result.

For $x \leq y \leq z$, equality occurs when $x = 0$ and $y = z = 1$. Moreover, for $r = r_0$, equality holds again when $x = y = z = 1$. □

Proposition 3.2 ([12]). *Let x, y, z be non-negative real numbers such that $xy + yz + zx = 3$. If $1 < r \leq 2$, then*

$$x^r(y+z) + y^r(z+x) + z^r(x+y) \geq 6.$$

Proof. Rewrite the inequality in the homogeneous form

$$x^r(y+z) + y^r(z+x) + z^r(x+y) \geq 6\left(\frac{xy+yz+zx}{3}\right)^{\frac{r+1}{2}}.$$

For convenience, we may leave aside the constraint $xy+yz+zx = 3$. Using now the constraint $x+y+z = 1$, the inequality becomes

$$x^r(1-x) + y^r(1-y) + z^r(1-z) \geq 6\left(\frac{1-x^2-y^2-z^2}{6}\right)^{\frac{r+1}{2}}.$$

To prove it, we will apply Corollary 1.5 to the function $f(u) = -u^r(1-u)$ for $0 \leq u \leq 1$. We have $f'(u) = -ru^{r-1} + (r+1)u^r$ and

$$g(x) = f'(x) = -rx^{r-1} + (r+1)x^r, \quad g''(x) = r(r-1)x^{r-3}[(r+1)x + 2 - r].$$

Since $g''(x) > 0$ for $x > 0$, $g(x)$ is strictly convex on $[0, \infty)$. According to Corollary 1.5, if $0 \leq x \leq y \leq z$ such that $x+y+z = 1$ and $x^2+y^2+z^2 =$ constant, then the sum $f(x) + f(y) + f(z)$ is maximal for $0 \leq x = y \leq z$.

Thus, we have only to prove the original inequality in the case $x = y \le z$. This means, to prove that $0 < x \le 1 \le y$ and $x^2 + 2xz = 3$ implies

$$x^r(x+z) + xz^r \ge 3.$$

Let $f(x) = x^r(x+z) + xz^r - 3$, with $z = \frac{3-x^2}{2x}$.

Differentiating the equation $x^2 + 2xz = 3$ yields $z' = \frac{-(x+z)}{x}$. Then,

$$\begin{aligned} f'(x) &= (r+1)x^r + rx^{r-1}z + z^r + (x^r + rxz^{r-1})z' \\ &= (x^{r-1} - z^{r-1})[rx + (r-1)z] \le 0. \end{aligned}$$

The function $f(x)$ is strictly decreasing on $[0,1]$, and hence $f(x) \ge f(1) = 0$ for $0 < x \le 1$. Equality occurs if and only if $x = y = z = 1$. □

Proposition 3.3 ([5]). *If $x_1, x_2, \ldots, x_n$ are positive real numbers such that*

$$x_1 + x_2 + \cdots + x_n = \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n},$$

then

$$\frac{1}{1+(n-1)x_1} + \frac{1}{1+(n-1)x_2} + \cdots + \frac{1}{1+(n-1)x_n} \ge 1.$$

Proof. We have to consider two cases.

Case $n = 2$. The inequality is verified as equality.

Case $n \ge 3$. Assume that $0 < x_1 \le x_2 \le \cdots \le x_n$, and then apply Corollary 1.6 to the function $f(u) = \frac{1}{1+(n-1)u}$ for $u > 0$. We have $f'(u) = \frac{-(n-1)}{[1+(n-1)u]^2}$ and

$$\begin{aligned} g(x) &= f'\left(\frac{1}{\sqrt{x}}\right) = \frac{-(n-1)x}{(\sqrt{x}+n-1)^2}, \\ g''(x) &= \frac{3(n-1)^2}{2\sqrt{x}\,(\sqrt{x}+n-1)^4}. \end{aligned}$$

Since $g''(x) > 0$, $g(x)$ is strictly convex on $(0, \infty)$. According to Corollary 1.6, if $0 < x_1 \le x_2 \le \cdots \le x_n$ such that

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= \text{constant} \quad \text{and} \\ \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} &= \text{constant}, \end{aligned}$$

then the sum $f(x_1) + f(x_2) + \cdots + f(x_n)$ is minimal when $0 < x_1 \le x_2 = x_3 = \cdots = x_n$.

Thus, we have to prove the inequality

$$\frac{1}{1+(n-1)x} + \frac{n-1}{1+(n-1)y} \ge 1,$$

under the constraints $0 < x \le 1 \le y$ and

$$x + (n-1)y = \frac{1}{x} + \frac{n-1}{y}.$$

The last constraint is equivalent to

$$(n-1)(y-1) = \frac{y(1-x^2)}{x(1+y)}.$$

Since

$$\begin{aligned}\frac{1}{1+(n-1)x}&+\frac{n-1}{1+(n-1)y}-1\\&=\frac{1}{1+(n-1)x}-\frac{1}{n}+\frac{n-1}{1+(n-1)y}-\frac{n-1}{n}\\&=\frac{(n-1)(1-x)}{n[1+(n-1)x]}-\frac{(n-1)^2(y-1)}{n[1+(n-1)y]}\\&=\frac{(n-1)(1-x)}{n[1+(n-1)x]}-\frac{(n-1)y(1-x^2)}{nx(1+y)[1+(n-1)y]},\end{aligned}$$

we must show that

$$x(1+y)[1+(n-1)y]\ge y(1+x)[1+(n-1)x],$$

which reduces to

$$(y-x)[(n-1)xy-1]\ge 0.$$

Since $y-x\ge 0$, we have still to prove that

$$(n-1)xy\ge 1.$$

Indeed, from $x+(n-1)y=\frac{1}{x}+\frac{n-1}{y}$ we get $xy=\frac{y+(n-1)x}{x+(n-1)y}$, and hence

$$(n-1)xy-1=\frac{n(n-2)x}{x+(n-1)y}>0.$$

For $n\ge 3$, one has equality if and only if $x_1=x_2=\cdots=x_n=1$. □

Proposition 3.4 ([10]). *Let $a_1,a_2,\dots,a_n$ be positive real numbers such that $a_1a_2\cdots a_n=1$. If m is a positive integer satisfying $m\ge n-1$, then*

$$a_1^m+a_2^m+\cdots+a_n^m+(m-1)n\ge m\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right).$$

Proof. For $n=2$ (hence $m\ge 1$), the inequality reduces to

$$a_1^m+a_2^m+2m-2\ge m(a_1+a_2).$$

We can prove it by summing the inequalities $a_1^m\ge 1+m(a_1-1)$ and $a_2^m\ge 1+m(a_2-1)$, which are straightforward consequences of Bernoulli's inequality. For $n\ge 3$, replacing $a_1,a_2,\dots,a_n$ by $\frac{1}{x_1},\frac{1}{x_2},\dots,\frac{1}{x_n}$, respectively, we have to show that

$$\frac{1}{x_1^m}+\frac{1}{x_2^m}+\cdots+\frac{1}{x_n^m}+(m-1)n\ge m(x_1+x_2+\cdots+x_n)$$

for $x_1x_2\cdots x_n=1$. Assume $0<x_1\le x_2\le\cdots\le x_n$ and apply Corollary 1.9 (case $p=0$ and $q=-m$):

If $0<x_1\le x_2\le\cdots\le x_n$ such that

$$x_1+x_2+\cdots+x_n=\text{constant}\quad\text{and}$$
$$x_1x_2\cdots x_n=1,$$

then the sum $\frac{1}{x_1^m}+\frac{1}{x_2^m}+\cdots+\frac{1}{x_n^m}$ is minimal when $0<x_1=x_2=\cdots=x_{n-1}\le x_n$.

Thus, it suffices to prove the inequality for $x_1=x_2=\cdots=x_{n-1}=x\le 1$, $x_n=y$ and $x^{n-1}y=1$, when it reduces to:

$$\frac{n-1}{x^m}+\frac{1}{y^m}+(m-1)n\ge m(n-1)x+my.$$

By the AM-GM inequality, we have

$$\frac{n-1}{x^m} + (m-n+1) \geq \frac{m}{x^{n-1}} = my.$$

Then, we have still to show that

$$\frac{1}{y^m} - 1 \geq m(n-1)(x-1).$$

This inequality is equivalent to

$$x^{mn-m} - 1 - m(n-1)(x-1) \geq 0$$

and

$$(x-1)[(x^{mn-m-1}-1) + (x^{mn-m-2}-1) + \cdots + (x-1)] \geq 0.$$

The last inequality is clearly true. For $n = 2$ and $m = 1$, the inequality becomes equality. Otherwise, equality occurs if and only if $a_1 = a_2 = \cdots = a_n = 1$. □

Proposition 3.5 ([6]). *Let $x_1, x_2, \ldots, x_n$ be non-negative real numbers such that $x_1 + x_2 + \cdots + x_n = n$. If k is a positive integer satisfying $2 \leq k \leq n+2$, and $r = \left(\frac{n}{n-1}\right)^{k-1} - 1$, then*

$$x_1^k + x_2^k + \cdots + x_n^k - n \geq nr(1 - x_1 x_2 \cdots x_n).$$

Proof. If $n = 2$, then the inequality reduces to $x_1^k + x_2^k - 2 \geq (2^k - 2)x_1 x_2$. For $k = 2$ and $k = 3$, this inequality becomes equality, while for $k = 4$ it reduces to $6x_1x_2(1 - x_1x_2) \geq 0$, which is clearly true.

Consider now $n \geq 3$ and $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$. Towards proving the inequality, we will apply Corollary 1.8 (case $p = k > 0$): If $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ such that $x_1 + x_2 + \cdots + x_n = n$ and $x_1^k + x_2^k + \cdots + x_n^k =$ constant, then the product $x_1 x_2 \cdots x_n$ is minimal when either $x_1 = 0$ or $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$.

Case $x_1 = 0$. The inequality reduces to

$$x_2^k + \cdots + x_n^k \geq \frac{n^k}{(n-1)^{k-1}},$$

with $x_2 + \cdots + x_n = n$, This inequality follows by applying Jensen's inequality to the convex function $f(u) = u^k$:

$$x_2^k + \cdots + x_n^k \geq (n-1)\left(\frac{x_2 + \cdots + x_n}{n-1}\right)^k.$$

Case $0 < x_1 \leq x_2 = x_3 = \cdots = x_n$. Denoting $x_1 = x$ and $x_2 = x_3 = \cdots = x_n = y$, we have to prove that for $0 < x \leq 1 \leq y$ and $x + (n-1)y = n$, the inequality holds:

$$x^k + (n-1)y^k + nrxy^{n-1} - n(r+1) \geq 0.$$

Write the inequality as $f(x) \geq 0$, where

$$f(x) = x^k + (n-1)y^k + nrxy^{n-1} - n(r+1), \quad \text{with} \quad y = \frac{n-x}{n-1}.$$

We see that $f(0) = f(1) = 0$. Since $y' = \frac{-1}{n-1}$, we have

$$\begin{aligned} f'(x) &= k(x^{k-1} - y^{k-1}) + nry^{n-2}(y-x) \\ &= (y-x)[nry^{n-2} - k(y^{k-2} + y^{k-3}x + \cdots + x^{k-2})] \\ &= (y-x)y^{n-2}[nr - kg(x)], \end{aligned}$$

where

$$g(x) = \frac{1}{y^{n-k}} + \frac{x}{y^{n-k+1}} + \cdots + \frac{x^{k-2}}{y^{n-2}}.$$

Since the function $y(x) = \frac{n-x}{n-1}$ is strictly decreasing, the function $g(x)$ is strictly increasing for $2 \le k \le n$. For $k = n+1$, we have

$$\begin{aligned} g(x) &= y + x + \frac{x^2}{y} + \cdots + \frac{x^{n-1}}{y^{n-2}} \\ &= \frac{(n-2)x + n}{n-1} + \frac{x^2}{y} + \cdots + \frac{x^{n-1}}{y^{n-2}}, \end{aligned}$$

and for $k = n+2$, we have

$$\begin{aligned} g(x) &= y^2 + yx + x^2 + \frac{x^3}{y} + \cdots + \frac{x^n}{y^{n-2}} \\ &= \frac{(n^2-3n+3)x^2 + n(n-3)x + n^2}{(n-1)^2} + \frac{x^3}{y} + \cdots + \frac{x^n}{y^{n-2}}. \end{aligned}$$

Therefore, the function $g(x)$ is strictly increasing for $2 \le k \le n+2$, and the function

$$h(x) = nr - kg(x)$$

is strictly decreasing. Note that

$$f'(x) = (y-x)y^{n-2}h(x).$$

We assert that $h(0) > 0$ and $h(1) < 0$. If our claim is true, then there exists $x_1 \in (0,1)$ such that $h(x_1) = 0$, $h(x) > 0$ for $x \in [0, x_1)$, and $h(x) < 0$ for $x \in (x_1, 1]$. Consequently, $f(x)$ is strictly increasing for $x \in [0, x_1]$, and strictly decreasing for $x \in [x_1, 1]$. Since $f(0) = f(1) = 0$, it follows that $f(x) \ge 0$ for $0 < x \le 1$, and the proof is completed.

In order to prove that $h(0) > 0$, we assume that $h(0) \le 0$. Then, $h(x) < 0$ for $x \in (0,1)$, $f'(x) < 0$ for $x \in (0,1)$, and $f(x)$ is strictly decreasing for $x \in [0,1]$, which contradicts $f(0) = f(1)$. Also, if $h(1) \ge 0$, then $h(x) > 0$ for $x \in (0,1)$, $f'(x) > 0$ for $x \in (0,1)$, and $f(x)$ is strictly increasing for $x \in [0,1]$, which also contradicts $f(0) = f(1)$.

For $n \ge 3$ and $x_1 \le x_2 \le \cdots \le x_n$, equality occurs when $x_1 = x_2 = \cdots = x_n = 1$, and also when $x_1 = 0$ and $x_2 = \cdots = x_n = \frac{n}{n-1}$. □

Remark 3.6. For $k = 2$, $k = 3$ and $k = 4$, we get the following nice inequalities:

$$(n-1)(x_1^2 + x_2^2 + \cdots + x_n^2) + nx_1x_2\cdots x_n \ge n^2,$$

$$(n-1)^2(x_1^3 + x_2^3 + \cdots + x_n^3) + n(2n-1)x_1x_2\cdots x_n \ge n^3,$$

$$(n-1)^3(x_1^4 + x_2^4 + \cdots + x_n^4) + n(3n^2-3n+1)x_1x_2\cdots x_n \ge n^4.$$

Remark 3.7. The inequality for $k = n$ was posted in 2004 on the Mathlinks Site - Inequalities Forum by Gabriel Dospinescu and Călin Popa.

Proposition 3.8 ([11]). *Let $x_1, x_2, \ldots, x_n$ be positive real numbers such that $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = n$. Then*

$$x_1 + x_2 + \cdots + x_n - n \le e_{n-1}(x_1x_2\cdots x_n - 1),$$

where $e_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1} < e$.

Proof. Replacing each of the x_i by $\frac{1}{a_i}$, the statement becomes as follows:

If $a_1, a_2, \ldots, a_n$ are positive numbers such that $a_1 + a_2 + \cdots + a_n = n$, then

$$a_1 a_2 \cdots a_n \left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n + e_{n-1}\right) \le e_{n-1}.$$

It is easy to check that the inequality holds for $n = 2$. Consider now $n \ge 3$, assume that $0 < a_1 \le a_2 \le \cdots \le a_n$ and apply Corollary 1.8 (case $p = -1$): If $0 < a_1 \le a_2 \le \cdots \le a_n$ such that $a_1 + a_2 + \cdots + a_n = n$ and $\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} =$ constant, then the product $a_1 a_2 \cdots a_n$ is maximal when $0 < a_1 \le a_2 = a_3 = \cdots = a_n$.

Denoting $a_1 = x$ and $a_2 = a_3 = \cdots = a_n = y$, we have to prove that for $0 < x \le 1 \le y < \frac{n}{n-1}$ and $x + (n-1)y = n$, the inequality holds:

$$y^{n-1} + (n-1)xy^{n-2} - (n - e_{n-1})xy^{n-1} \le e_{n-1}.$$

Letting

$$f(x) = y^{n-1} + (n-1)xy^{n-2} - (n - e_{n-1})xy^{n-1} - e_{n-1}, \quad \text{with}$$

$$y = \frac{n-x}{n-1},$$

we must show that $f(x) \le 0$ for $0 < x \le 1$. We see that $f(0) = f(1) = 0$. Since $y' = \frac{-1}{n-1}$, we have

$$\frac{f'(x)}{y^{n-3}} = (y - x)[n - 2 - (n - e_{n-1})y] = (y - x)h(x),$$

where

$$h(x) = n - 2 - (n - e_{n-1})\frac{n-x}{n-1}$$

is a linear increasing function.

Let us show that $h(0) < 0$ and $h(1) > 0$. If $h(0) \ge 0$, then $h(x) > 0$ for $x \in (0, 1)$, hence $f'(x) > 0$ for $x \in (0, 1)$, and $f(x)$ is strictly increasing for $x \in [0, 1]$, which contradicts $f(0) = f(1)$. Also, $h(1) = e_{n-1} - 2 > 0$.

From $h(0) < 0$ and $h(1) > 0$, it follows that there exists $x_1 \in (0, 1)$ such that $h(x_1) = 0$, $h(x) < 0$ for $x \in [0, x_1)$, and $h(x) > 0$ for $x \in (x_1, 1]$. Consequently, $f(x)$ is strictly decreasing for $x \in [0, x_1]$, and strictly increasing for $x \in [x_1, 1]$. Since $f(0) = f(1) = 0$, it follows that $f(x) \le 0$ for $0 \le x \le 1$.

For $n \ge 3$, equality occurs when $x_1 = x_2 = \cdots = x_n = 1$. □

Proposition 3.9 ([9]). *If $x_1, x_2, \ldots, x_n$ are positive real numbers, then*

$$x_1^n + x_2^n + \cdots + x_n^n + n(n-1)x_1x_2\cdots x_n \ge x_1x_2\cdots x_n(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right).$$

Proof. For $n = 2$, one has equality. Assume now that $n \ge 3$, $0 < x_1 \le x_2 \le \cdots \le x_n$ and apply Corollary 1.9 (case $p = 0$): If $0 < x_1 \le x_2 \le \cdots \le x_n$ such that

$$x_1 + x_2 + \cdots + x_n = \text{constant} \quad \text{and}$$

$$x_1x_2\cdots x_n = \text{constant},$$

then the sum $x_1^n + x_2^n + \cdots + x_n^n$ is minimal and the sum $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}$ is maximal when $0 < x_1 \le x_2 = x_3 = \cdots = x_n$.

Thus, it suffices to prove the inequality for $0 < x_1 \le 1$ and $x_2 = x_3 = \cdots = x_n = 1$. The inequality becomes

$$x_1^n + (n-2)x_1 \ge (n-1)x_1^2,$$

and is equivalent to

$$x_1(x_1-1)[(x_1^{n-2}-1)+(x_1^{n-3}-1)+\cdots+(x_1-1)]\geq 0,$$

which is clearly true. For $n\geq 3$, equality occurs if and only if $x_1=x_2=\cdots=x_n$. □

Proposition 3.10 ([14]). *If $x_1,x_2,\ldots,x_n$ are non-negative real numbers, then*

$$(n-1)(x_1^n+x_2^n+\cdots+x_n^n)+nx_1x_2\cdots x_n \geq (x_1+x_2+\cdots+x_n)(x_1^{n-1}+x_2^{n-1}+\cdots+x_n^{n-1}).$$

Proof. For $n=2$, one has equality. For $n\geq 3$, assume that $0\leq x_1\leq x_2\leq\cdots\leq x_n$ and apply Corollary 1.9 (case $p=n$ and $q=n-1$) and Corollary 1.8 (case $p=n$): If $0\leq x_1\leq x_2\leq\cdots\leq x_n$ such that

$$x_1+x_2+\cdots+x_n=\text{constant} \quad \text{and}$$
$$x_1^n+x_2^n+\cdots+x_n^n=\text{constant},$$

then the sum $x_1^{n-1}+x_2^{n-1}+\cdots+x_n^{n-1}$ is maximal and the product $x_1x_2\cdots x_n$ is minimal when either $x_1=0$ or $0<x_1\leq x_2=x_3=\cdots=x_n$.

So, it suffices to consider the cases $x_1=0$ and $0<x_1\leq x_2=x_3=\cdots=x_n$.

Case $x_1=0$. The inequality reduces to

$$(n-1)(x_2^n+\cdots+x_n^n)\geq (x_2+\cdots+x_n)(x_2^{n-1}+\cdots+x_n^{n-1}),$$

which immediately follows by Chebyshev's inequality.

Case $0<x_1\leq x_2=x_3=\cdots=x_n$. Setting $x_2=x_3=\cdots=x_n=1$, the inequality reduces to:

$$(n-2)x_1^n+x_1\geq (n-1)x_1^{n-1}.$$

Rewriting this inequality as

$$x_1(x_1-1)[x_1^{n-3}(x_1-1)+x_1^{n-4}(x_1^2-1)+\cdots+(x_1^{n-2}-1)]\geq 0,$$

we see that it is clearly true. For $n\geq 3$ and $x_1\leq x_2\leq\cdots\leq x_n$ equality occurs when $x_1=x_2=\cdots=x_n$, and for $x_1=0$ and $x_2=\cdots=x_n$. □

Proposition 3.11 ([8]). *If $x_1,x_2,\ldots,x_n$ are positive real numbers, then*

$$(x_1+x_2+\cdots+x_n-n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}-n\right)+x_1x_2\cdots x_n+\frac{1}{x_1x_2\cdots x_n}\geq 2.$$

Proof. For $n=2$, the inequality reduces to

$$\frac{(1-x_1)^2(1-x_2)^2}{x_1x_2}\geq 0.$$

For $n\geq 3$, assume that $0<x_1\leq x_2\leq\cdots\leq x_n$. Since the inequality preserves its form by replacing each number x_i with $\frac{1}{x_i}$, we may consider $x_1x_2\cdots x_n\geq 1$. So, by the AM-GM inequality we get

$$x_1+x_2+\cdots+x_n-n\geq n\sqrt[n]{x_1x_2\cdots x_n}-n\geq 0,$$

and we may apply Corollary 1.9 (case $p=0$ and $q=-1$): If $0\leq x_1\leq x_2\leq\cdots\leq x_n$ such that

$$x_1+x_2+\cdots+x_n=\text{constant} \quad \text{and}$$
$$x_1x_2\cdots x_n=\text{constant},$$

then the sum $\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}$ is minimal when $0<x_1=x_2=\cdots=x_{n-1}\leq x_n$.

According to this statement, it suffices to consider $x_1 = x_2 = \cdots = x_{n-1} = x$ and $x_n = y$, when the inequality reduces to

$$((n-1)x + y - n)\left(\frac{n-1}{x} + \frac{1}{y} - n\right) + x^{n-1}y + \frac{1}{x^{n-1}y} \geq 2,$$

or

$$\left(x^{n-1} + \frac{n-1}{x} - n\right)y + \left[\frac{1}{x^{n-1}} + (n-1)x - n\right]\frac{1}{y} \geq \frac{n(n-1)(x-1)^2}{x}.$$

Since

$$\begin{aligned} x^{n-1} + \frac{n-1}{x} - n &= \frac{x-1}{x}[(x^{n-1}-1) + (x^{n-2}-1) + \cdots + (x-1)] \\ &= \frac{(x-1)^2}{x}[x^{n-2} + 2x^{n-3} + \cdots + (n-1)] \end{aligned}$$

and

$$\frac{1}{x^{n-1}} + (n-1)x - n = \frac{(x-1)^2}{x}\left[\frac{1}{x^{n-2}} + \frac{2}{x^{n-3}} + \cdots + (n-1)\right],$$

it is enough to show that

$$[x^{n-2} + 2x^{n-3} + \cdots + (n-1)]y + \left[\frac{1}{x^{n-2}} + \frac{2}{x^{n-3}} + \cdots + (n-1)\right]\frac{1}{y} \geq n(n-1).$$

This inequality is equivalent to

$$\left(x^{n-2}y + \frac{1}{x^{n-2}y} - 2\right) + 2\left(x^{n-3}y + \frac{1}{x^{n-3}y} - 2\right) + \cdots + (n-1)\left(y + \frac{1}{y} - 2\right) \geq 0,$$

or

$$\frac{(x^{n-2}y-1)^2}{x^{n-2}y} + \frac{2(x^{n-3}y-1)^2}{x^{n-3}y} + \cdots + \frac{(n-1)(y-1)^2}{y} \geq 0,$$

which is clearly true. Equality occurs if and only if $n-1$ of the numbers x_i are equal to 1. □

Proposition 3.12 ([15]). *If $x_1, x_2, \ldots, x_n$ are non-negative real numbers such that $x_1 + x_2 + \cdots + x_n = n$, then*

$$(x_1x_2\cdots x_n)^{\frac{1}{\sqrt{n-1}}}(x_1^2 + x_2^2 + \cdots + x_n^2) \leq n.$$

Proof. For $n = 2$, the inequality reduces to $2(x_1x_2 - 1)^2 \geq 0$. For $n \geq 3$, assume that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ and apply Corollary 1.8 (case $p = 2$): If $0 \leq x_1 \leq x_2 \leq \cdots \leq x_n$ such that $x_1 + x_2 + \cdots + x_n = n$ and $x_1^2 + x_2^2 + \cdots + x_n^2 =$ constant, then the product $x_1x_2\cdots x_n$ is maximal when $0 \leq x_1 = x_2 = \cdots = x_{n-1} \leq x_n$.

Consequently, it suffices to show that the inequality holds for $x_1 = x_2 = \cdots = x_{n-1} = x$ and $x_n = y$, where $0 \leq x \leq 1 \leq y$ and $(n-1)x + y = n$. Under the circumstances, the inequality reduces to

$$x^{\sqrt{n-1}}y^{\frac{1}{\sqrt{n-1}}}[(n-1)x^2 + y^2] \leq n.$$

For $x = 0$, the inequality is trivial. For $x > 0$, it is equivalent to $f(x) \leq 0$, where

$$f(x) = \sqrt{n-1}\ln x + \frac{1}{\sqrt{n-1}}\ln y + \ln[(n-1)x^2 + y^2] - \ln n, \quad \text{with} \quad y = n - (n-1)x.$$

We have $y' = -(n-1)$ and

$$\frac{f'(x)}{\sqrt{n-1}} = \frac{1}{x} - \frac{1}{y} + \frac{2\sqrt{n-1}(x-y)}{(n-1)x^2+y^2} = \frac{(y-x)(\sqrt{n-1}x-y)^2}{xy[(n-1)x^2+y^2]} \geq 0.$$

Therefore, the function $f(x)$ is strictly increasing on $(0,1]$ and hence $f(x) \leq f(1) = 0$. Equality occurs if and only if $x_1 = x_2 = \cdots = x_n = 1$. □

Remark 3.13. For $n = 5$, we get the following nice statement:

If a, b, c, d, e are positive real numbers such that $a^2 + b^2 + c^2 + d^2 + e^2 = 5$, then

$$abcde(a^4 + b^4 + c^4 + d^4 + e^4) \leq 5.$$

Proposition 3.14 ([4]). *Let x, y, z be non-negative real numbers such that $xy + yz + zx = 3$, and let*

$$p \geq \frac{\ln 9 - \ln 4}{\ln 3} \approx 0.738.$$

Then,

$$x^p + y^p + z^p \geq 3.$$

Proof. Let $r = \frac{\ln 9 - \ln 4}{\ln 3}$. By the Power-Mean inequality, we have

$$\frac{x^p + y^p + z^p}{3} \geq \left(\frac{x^r + y^r + z^r}{3}\right)^{\frac{p}{r}}.$$

Thus, it suffices to show that

$$x^r + y^r + z^r \geq 3.$$

Let $x \leq y \leq z$. We consider two cases.

Case $x = 0$. We have to show that $y^r + z^r \geq 3$ for $yz = 3$. Indeed, by the AM-GM inequality, we get

$$y^r + z^r \geq 2(yz)^{r/2} = 2 \cdot 3^{r/2} = 3.$$

Case $x > 0$. The inequality $x^r + y^r + z^r \geq 3$ is equivalent to the homogeneous inequality

$$x^r + y^r + z^r \geq 3\left(\frac{xyz}{3}\right)^{\frac{r}{2}} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)^{\frac{r}{2}}.$$

Setting $x = a^{\frac{1}{r}}$, $y = b^{\frac{1}{r}}$, $z = c^{\frac{1}{r}}$ $(0 < a \leq b \leq c)$, the inequality becomes

$$a + b + c \geq 3\left(\frac{abc}{3}\right)^{\frac{1}{2}} \left(a^{\frac{-1}{r}} + b^{\frac{-1}{r}} + c^{\frac{-1}{r}}\right)^{\frac{r}{2}}.$$

Towards proving this inequality, we apply Corollary 1.9 (case $p = 0$, $q = \frac{-1}{r}$): If $0 < a \leq b \leq c$ such that $a + b + c =$ constant and $abc =$ constant, then the sum $a^{\frac{-1}{r}} + b^{\frac{-1}{r}} + c^{\frac{-1}{r}}$ is maximal when $0 < a \leq b = c$.

So, it suffices to prove the inequality for $0 < a \leq b = c$; that is, to prove the homogeneous inequality in x, y, z for $0 < x \leq y = z$. Without loss of generality, we may leave aside the constraint $xy + yz + zx = 3$, and consider $y = z = 1$ and $0 < x \leq 1$. The inequality reduces to

$$x^r + 2 \geq 3\left(\frac{2x+1}{3}\right)^{\frac{r}{2}}.$$

Denoting

$$f(x) = \ln\frac{x^r + 2}{3} - \frac{r}{2}\ln\frac{2x+1}{3},$$

we have to show that $f(x) \geq 0$ for $0 < x \leq 1$. The derivative

$$f'(x) = \frac{rx^{r-1}}{x^r + 2} - \frac{r}{2x+1} = \frac{r(x - 2x^{1-r} + 1)}{x^{1-r}(x^r + 2)(2x+1)}$$

has the same sign as $g(x) = x - 2x^{1-r} + 1$. Since $g'(x) = 1 - \frac{2(1-r)}{x^r}$, we see that $g'(x) < 0$ for $x \in (0, x_1)$, and $g'(x) > 0$ for $x \in (x_1, 1]$, where $x_1 = (2 - 2r)^{1/r} \approx 0.416$. The function $g(x)$ is strictly decreasing on $[0, x_1]$, and strictly increasing on $[x_1, 1]$. Since $g(0) = 1$ and $g(1) = 0$, there exists $x_2 \in (0, 1)$ such that $g(x_2) = 0$, $g(x) > 0$ for $x \in [0, x_2)$ and $g(x) < 0$ for $x \in (x_2, 1)$. Consequently, the function $f(x)$ is strictly increasing on $[0, x_2]$ and strictly decreasing on $[x_2, 1]$. Since $f(0) = f(1) = 0$, we have $f(x) \geq 0$ for $0 < x \leq 1$, establishing the desired result.

Equality occurs for $x = y = z = 1$. Additionally, for $p = \frac{\ln 9 - \ln 4}{\ln 3}$ and $x \leq y \leq z$, equality holds again for $x = 0$ and $y = z = \sqrt{3}$. □

Proposition 3.15 ([7]). *Let x, y, z be non-negative real numbers such that $x + y + z = 3$, and let $p \geq \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2} \approx 0.29$. Then,*

$$x^p + y^p + z^p \geq xy + yz + zx.$$

Proof. For $p \geq 1$, by Jensen's inequality we have

$$\begin{aligned} x^p + y^p + z^p &\geq 3\left(\frac{x+y+z}{3}\right)^p \\ &= 3 = \frac{1}{3}(x+y+z)^2 \geq xy + yz + zx. \end{aligned}$$

Assume now $p < 1$. Let $r = \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2}$ and $x \leq y \leq z$. The inequality is equivalent to the homogeneous inequality

$$2(x^p + y^p + z^p)\left(\frac{x+y+z}{3}\right)^{2-p} + x^2 + y^2 + z^2 \geq (x+y+z)^2.$$

By Corollary 1.9 (case $0 < p < 1$ and $q = 2$), if $x \leq y \leq z$ such that $x + y + z =$ constant and $x^p + y^p + z^p =$ constant, then the sum $x^2 + y^2 + z^2$ is minimal when either $x = 0$ or $0 < x \leq y = z$.

Case $x = 0$. Returning to our original inequality, we have to show that $y^p + z^p \geq yz$ for $y + z = 3$. Indeed, by the AM-GM inequality, we get

$$\begin{aligned} y^p + z^p - yz &\geq 2(yz)^{\frac{p}{2}} - yz \\ &= (yz)^{\frac{p}{2}}[2 - (yz)^{\frac{2-p}{2}}] \\ &\geq (yz)^{\frac{p}{2}}\left[2 - \left(\frac{y+z}{2}\right)^{2-p}\right] \\ &= (yz)^{\frac{p}{2}}\left[2 - \left(\frac{3}{2}\right)^{2-p}\right] \\ &\geq (yz)^{\frac{p}{2}}\left[2 - \left(\frac{3}{2}\right)^{2-r}\right] = 0. \end{aligned}$$

Case $0 < x \le y = z$. In the homogeneous inequality, we may leave aside the constraint $x + y + z = 3$, and consider $y = z = 1$ and $0 < x \le 1$. Thus, the inequality reduces to

$$(x^p + 2)\left(\frac{x+2}{3}\right)^{2-p} \ge 2x + 1.$$

To prove this inequality, we consider the function

$$f(x) = \ln(x^p + 2) + (2 - p)\ln\frac{x+2}{3} - \ln(2x + 1).$$

We have to show that $f(x) \ge 0$ for $0 < x \le 1$ and $r \le p < 1$. We have

$$f'(x) = \frac{px^{p-1}}{x^p + 2} + \frac{2-p}{x+2} - \frac{2}{2x+1} = \frac{2g(x)}{x^{1-p}(x^p+2)(2x+1)},$$

where

$$g(x) = x^2 + (2p - 1)x + p + 2(1 - p)x^{2-p} - (p + 2)x^{1-p},$$

and

$$g'(x) = 2x + 2p - 1 + 2(1 - p)(2 - p)x^{1-p} - (p + 2)(1 - p)x^{-p},$$
$$g''(x) = 2 + 2(1 - p)^2(2 - p)x^{-p} + p(p + 2)(1 - p)x^{-p-1}.$$

Since $g''(x) > 0$, the first derivative $g'(x)$ is strictly increasing on $(0, 1]$. Taking into account that $g'(0+) = -\infty$ and $g'(1) = 3(1 - p) + 3p^2 > 0$, there is $x_1 \in (0, 1)$ such that $g'(x_1) = 0$, $g'(x) < 0$ for $x \in (0, x_1)$and $g'(x) > 0$ for $x \in (x_1, 1]$. Therefore, the function $g(x)$ is strictly decreasing on $[0, x_1]$ and strictly increasing on $[x_1, 1]$. Since $g(0) = p > 0$ and $g(1) = 0$, there is $x_2 \in (0, x_1)$ such that $g(x_2) = 0$, $g(x) > 0$ for $x \in [0, x_2)$ and $g(x) < 0$ for $x \in (x_2, 1]$. We have also $f'(x_2) = 0$, $f'(x) > 0$ for $x \in (0, x_2)$ and $f'(x) < 0$ for $x \in (x_2, 1]$. According to this result, the function $f(x)$ is strictly increasing on $[0, x_2]$ and strictly decreasing on $[x_2, 1]$. Since

$$f(0) = \ln 2 + (2 - p)\ln\frac{2}{3} \ge \ln 2 + (2 - r)\ln\frac{2}{3} = 0$$

and $f(1) = 0$, we get $f(x) \ge \min\{f(0), f(1)\} = 0$.

Equality occurs for $x = y = z = 1$. Additionally, for $p = \frac{\ln 9 - \ln 8}{\ln 3 - \ln 2}$ and $x \le y \le z$, equality holds again when $x = 0$ and $y = z = \frac{3}{2}$. □

Proposition 3.16 ([8]). *If $x_1, x_2, \dots, x_n$ $(n \ge 4)$ are non-negative numbers such that $x_1 + x_2 + \cdots + x_n = n$, then*

$$\frac{1}{n + 1 - x_2x_3\cdots x_n} + \frac{1}{n + 1 - x_3x_4\cdots x_1} + \cdots + \frac{1}{n + 1 - x_1x_2\cdots x_{n-1}} \le 1.$$

Proof. Let $x_1 \le x_2 \le \cdots \le x_n$ and $e_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1}$. By the AM-GM inequality, we have

$$x_2\cdots x_n \le \left(\frac{x_2 + \cdots + x_n}{n-1}\right)^{n-1} \le \left(\frac{x_1 + x_2 + \cdots + x_n}{n-1}\right)^{n-1} = e_{n-1}.$$

Hence

$$n + 1 - x_2x_3\cdots x_n \ge n + 1 - e_{n-1} > 0,$$

and all denominators of the inequality are positive.

Case $x_1 = 0$. It is easy to show that the inequality holds.

Case $x_1 > 0$. Suppose that $x_1x_2\cdots x_n = (n + 1)r =$ constant, $r > 0$. The inequality becomes

$$\frac{x_1}{x_1 - r} + \frac{x_2}{x_2 - r} + \cdots + \frac{x_n}{x_n - r} \le n + 1,$$

or

$$\frac{1}{x_1-r}+\frac{1}{x_2-r}+\cdots+\frac{1}{x_n-r}\le\frac{1}{r}.$$

By the AM-GM inequality, we have

$$(n+1)r=x_1x_2\cdots x_n\le\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^n=1,$$

hence $r\le\frac{1}{n+1}$. From $x_n<x_1+x_2+\cdots+x_n=n<n+1\le\frac{1}{r}$, we get $x_n<\frac{1}{r}$. Therefore, we have $r<x_i<\frac{1}{r}$ for all numbers x_i.

We will apply now Corollary 1.7 to the function $f(u)=\frac{-1}{u-r}$, $u>r$. We have $f'(u)=\frac{1}{(u-r)^2}$ and

$$g(x)=f'\left(\frac{1}{x}\right)=\frac{x^2}{(1-rx)^2},\quad g''(x)=\frac{4rx+2}{(1-rx)^4}.$$

Since $g''(x)>0$, $g(x)$ is strictly convex on $\left(r,\frac{1}{r}\right)$. According to Corollary 1.7, if $0\le x_1\le x_2\le\cdots\le x_n$ such that for $x_1+x_2+\cdots+x_n=$ constant and $x_1x_2\cdots x_n=$ constant, then the sum $f(x_1)+f(x_2)+\cdots+f(x_n)$ is minimal when $x_1\le x_2=x_3=\cdots=x_n$. Thus, to prove the original inequality, it suffices to consider the case $x_1=x$ and $x_2=x_3=\cdots=x_n=y$, where $0<x\le 1\le y$ and $x+(n-1)y=n$. We leave ending the proof to the reader. □

Remark 3.17. The inequality is a particular case of the following more general statement:

Let $n\ge 3$, $e_{n-1}=\left(1+\frac{1}{n-1}\right)^{n-1}$, $k_n=\frac{(n-1)e_{n-1}}{n-e_{n-1}}$ and let $a_1,a_2,\ldots,a_n$ be non-negative numbers such that $a_1+a_2+\cdots+a_n=n$.

(a) If $k\ge k_n$, then

$$\frac{1}{k-a_2a_2\cdots a_n}+\frac{1}{k-a_3a_4\cdots a_1}+\cdots+\frac{1}{k-a_1a_2\cdots a_{n-1}}\le\frac{n}{k-1};$$

(b) If $e_{n-1}<k<k_n$, then

$$\frac{1}{k-a_2a_3\cdots a_n}+\frac{1}{k-a_3a_4\cdots a_1}+\cdots+\frac{1}{k-a_1a_2\cdots a_{n-1}}\le\frac{n-1}{k}+\frac{1}{k-e_{n-1}}.$$

Finally, we mention that many other applications of the EV-Method are given in the book [2].

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