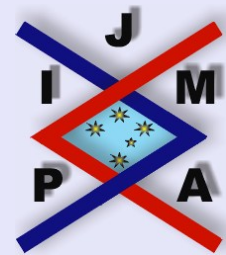


## AN INEQUALITY FOR THE ASYMMETRY OF DISTRIBUTIONS AND A BERRY-ESSEEN THEOREM FOR RANDOM SUMMATION

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[Abstract](#)

[Contents](#)



[Home Page](#)

[Go Back](#)

[Close](#)

[Quit](#)

## Abstract

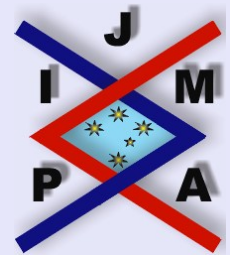
We consider random numbers  $N_n$  of independent, identically distributed (i.i.d.) random variables  $X_i$  and their sums  $\sum_{i=1}^{N_n} X_i$ . Whereas Blum, Hanson and Rosenblatt [3] proved a central limit theorem for such sums and Landers and Rogge [8] derived the corresponding approximation order, a Berry-Esseen type result seems to be missing. Using an inequality for the asymmetry of distributions, which seems to be of its own interest, we prove, under the assumption  $E|X_i|^{2+\delta} < \infty$  for some  $\delta \in (0, 1]$  and  $N_n/n \rightarrow \tau$  (in an appropriate sense), a Berry-Esseen theorem for random summation.

*2000 Mathematics Subject Classification:* 60E15, 60F05, 60G40.

*Key words:* Random number of i.i.d. random variables, Central limit theorem for random sums, Asymmetry of distributions, Berry-Esseen theorem for random sums.

## Contents

1	<b>Introduction</b> .....	3
2	<b>An Inequality for the Asymmetry of Distributions</b> .....	6
3	<b>Some Further Inequalities</b> .....	15
4	<b>A Berry-Esseen Theorem for Random Sums</b> .....	18
	<b>References</b>	



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### An Inequality for the Asymmetry of Distributions and a Berry-Esseen Theorem for Random Summation

Hendrik Kläver and Norbert Schmitz

---

Title Page

Contents



Go Back

Close

Quit

Page 2 of 25

# 1. Introduction

One of the milestones of probability theory is the famous theorem of *Berry-Esseen* which gives uniform upper bounds for the deviation from the normal distribution in the Central Limit Theorem:

Let  $\{X_n, n \geq 1\}$  be independent random variables such that

$$EX_n = 0, \quad EX_n^2 =: \sigma_n^2, \quad s_n^2 := \sum_{i=1}^n \sigma_i^2 > 0,$$

$$\Gamma_n^{2+\delta} := \sum_{i=1}^n E|X_i|^{2+\delta} < \infty$$

for some  $\delta \in (0, 1]$  and  $S_n = \sum_{i=1}^n X_i$ ,  $n \geq 1$ . Then there exists a universal constant  $C_\delta$  such that

$$\sup_{x \in \mathbb{R}} \left| P \left( \frac{S_n}{s_n} \leq x \right) - \Phi(x) \right| \leq C_\delta \left( \frac{\Gamma_n}{s_n} \right)^{2+\delta},$$

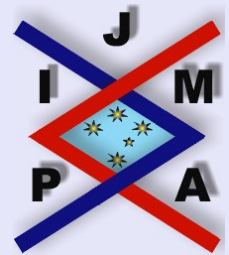
where  $\Phi$  denotes the cumulative distribution function of a  $\mathcal{N}(0, 1)$ - (normal) distribution (see e.g. Chow and Teicher [5, p. 299]).

For the special case of identical distributions this leads to:

Let  $\{X_n, n \geq 1\}$  be i.i.d. random variables with  $EX_n = 0$ ,  $EX_n^2 =: \sigma^2 > 0$ ,  $E|X_n|^{2+\delta} =: \gamma^{2+\delta} < \infty$  for some  $\delta \in (0, 1]$ . Then there exists a universal constant  $c_\delta$  such that

$$\sup_{x \in \mathbb{R}} \left| P \left( \frac{S_n}{\sigma\sqrt{n}} \leq x \right) - \Phi(x) \right| \leq \frac{c_\delta}{n^{\delta/2}} \left( \frac{\gamma}{\sigma} \right)^{2+\delta}.$$

Van Beek [2] showed that  $C_1 \leq 0.7975$ ; bounds for other values  $C_\delta$  are given by Tysiak [12], e.g.  $C_{0.8} \leq 0.812$ ;  $C_{0.6} \leq 0.863$ ,  $C_{0.4} \leq 0.950$ ,  $C_{0.2} \leq 1.076$ .



An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation

Hendrik Kläver and Norbert  
Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 3 of 25

On the other hand, there exist also central limit theorems for random summation, e.g. the theorem of *Blum, Hanson and Rosenblatt* [3] which generalizes previous results by Anscombe [1] and Renyi [11]:

Let  $\{X_n, n \geq 1\}$  be i.i.d. random variables with  $EX_n = 0$ ,  $\text{Var } X_n = 1$ ,  $S_n := \sum_{i=1}^n X_i$  and let  $\{N_n, n \geq 1\}$  be  $\mathbb{N}$ -valued random variables such that  $N_n/n \xrightarrow{P} U$  where  $U$  is a positive random variable. Then

$$P^{S_{N_n}/\sqrt{N_n}} \xrightarrow{d} \mathcal{N}(0, 1).$$

Therefore, the obvious question arises whether one can prove also Berry-Esseen type inequalities for random sums. A first result concerning the approximation order is due to *Landers and Rogge* [8] for random variables  $X_n$  with  $E|X_n|^3 < \infty$ ; this was generalized by *Callaert and Janssen* [4] to the case that  $E|X_n|^{2+\delta} < \infty$  for some  $\delta \in (0, 1]$ :

Let  $\{X_n, n \geq 1\}$  be i.i.d. random variables with  $EX_n = \mu$ ,  $\text{Var } X_n = \sigma^2 > 0$ , and  $E|X_n|^{2+\delta} < \infty$  for some  $\delta > 0$ . Let  $\{N_n, n \geq 1\}$  be  $\mathbb{N}$ -valued random variables,  $\{\varepsilon_n, n \geq 1\}$  positive real numbers with  $\varepsilon_n \xrightarrow{n \rightarrow \infty} 0$  where, for  $n$  large,  $n^{-\delta} \leq \varepsilon_n$  if  $\delta \in (0, 1]$  and  $n^{-1} \leq \varepsilon_n$  if  $\delta \geq 1$ . If there exists a  $\tau > 0$  such that

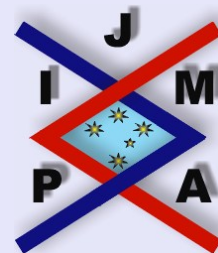
$$P\left(\left|\frac{N_n}{n\tau} - 1\right| > \varepsilon_n\right) = O(\sqrt{\varepsilon_n}),$$

then

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{\sum_{i=1}^{N_n}(X_i - \mu)}{\sigma\sqrt{n\tau}} \leq x\right) - \Phi(x) \right| = O(\sqrt{\varepsilon_n})$$

and

$$\sup_{x \in \mathbb{R}} \left| P\left(\frac{\sum_{i=1}^{N_n}(X_i - \mu)}{\sigma\sqrt{N_n}} \leq x\right) - \Phi(x) \right| = O(\sqrt{\varepsilon_n}).$$



**An Inequality for the Asymmetry of Distributions and a Berry-Esseen Theorem for Random Summation**

Hendrik Kläver and Norbert Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 4 of 25

Moreover, there exist several further results on convergence rates for random sums (see e.g. [7] and the papers cited there); as applications, sequential analysis, random walk problems, Monte Carlo methods and Markov chains are mentioned.

However, rates of convergence without any knowledge about the factors are of very limited importance for applications. Hence the aim of this paper is to prove a Berry-Esseen type result for random sums i.e. a uniform approximation with explicit constants. Obviously, due to the dependencies on the moments of the  $X_n$  as well as on the asymptotic behaviour of the sequence  $N_n$  such a result will necessarily be more complex than the original Berry-Esseen theorem.

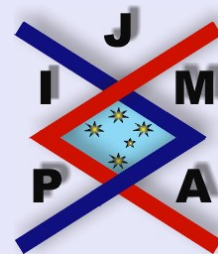
For the underlying random variables  $X_n$  we make the same assumption as in the special version of the Berry-Esseen theorem:

$$(M) : \begin{cases} X_n, n \geq 1, \text{ are i.i.d. random variables with } EX_n = 0, \\ Var X_n = 1 \text{ and } \gamma^{2+\delta} := E|X_n|^{2+\delta} < \infty \text{ for some } \delta \in (0, 1]. \end{cases}$$

Similarly as Landers and Rogge [8] or Callaert and Janssen [4] resp. we assume on the random indices

$$(R) : \begin{cases} N_n, n \geq 1, \text{ are integer-valued random variables and } \zeta_n, n \geq 1, \\ \text{real numbers with } \lim_{n \rightarrow \infty} \zeta_n = 0 \text{ such that there exist} \\ d, \tau > 0 \text{ with } P(|\frac{N_n}{n\tau} - 1| > \zeta_n) \leq d\sqrt{\zeta_n}. \end{cases}$$

As they are essential for applications, e.g. in sequential analysis, arbitrary dependencies between the indices and the summands are allowed.




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**An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation**

Hendrik Kläver and Norbert  
Schmitz

---

Title Page

Contents



Go Back

Close

Quit

Page 5 of 25

## 2. An Inequality for the Asymmetry of Distributions

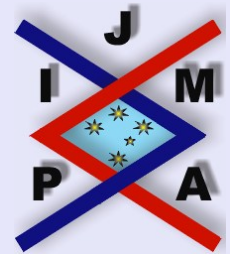
A main tool for deriving explicit constants for the rate of convergence is an inequality which seems to be of its own interest. For a smooth formulation we use (for the different values of the moment parameter  $\delta \in (0, 1]$ ) some (technical) notation: For  $\vartheta := 2/(1 + \delta)$  and  $y \geq 1$  let  $g_\delta(y)$  be defined by

$$g_\delta(y) := \begin{cases} 2y^2 - 1 + 2y\sqrt{y^2 - 1} & \text{if } \delta = 1 \\ \min \left\{ \begin{array}{l} 2^\vartheta y^{4\vartheta} - 1 + 2^{\frac{\vartheta+1}{2}} y^{2\vartheta} \sqrt{2^{\vartheta-1} y^{4\vartheta} - 1}, \\ 2y^{8\vartheta} - 1 + 2y^{4\vartheta} \sqrt{y^{8\vartheta} - 1} \end{array} \right\} & \text{if } \delta \in \left[ \frac{1}{3}, 1 \right) \\ \min \left\{ \begin{array}{l} 2^\vartheta y^{(2+2^k)\vartheta} - 1 + 2^{\frac{\vartheta+1}{2}} y^{(1+2^{k-1})\vartheta} \sqrt{2^{\vartheta-1} y^{(2+2^k)\vartheta} - 1}, \\ 2y^{(4+2^{k+1})\vartheta} - 1 + 2y^{(2+2^k)\vartheta} \sqrt{y^{(4+2^{k+1})\vartheta} - 1} \end{array} \right\} & \text{if } \delta \in \left( \frac{1}{2^{k+1}}, \frac{1}{2^{k-1}+1} \right], k \geq 2 \end{cases}$$

**Theorem 2.1.** *Let  $X$  be a random variable with  $EX = 0$ ,  $\text{Var } X = 1$  and  $\gamma^{2+\delta} := E|X|^{2+\delta} < \infty$  for some  $\delta \in (0, 1]$ . Then*

$$P(X < 0) \leq g_\delta(\gamma^{2+\delta})P(X > 0) \text{ and } P(X > 0) \leq g_\delta(\gamma^{2+\delta})P(X < 0).$$

*Proof.* Let  $X$  be a random variable as described above. Let  $X_+ = \max\{X, 0\}$ ,  $X_- = \min\{X, 0\}$ ,  $E(X_+) = E(X_-) = \alpha$  and  $P(X > 0) = p$ ,  $P(X = 0) = r$ ,  $P(X < 0) = 1 - r - p$ . Since  $E(X) = 0$ ,  $\text{Var}(X) = 1$  it is obvious that



An Inequality for the Asymmetry of Distributions and a Berry-Esseen Theorem for Random Summation

Hendrik Kläver and Norbert Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 6 of 25

$p, 1 - r - p > 0$ . As  $\alpha = pE(|X| | X > 0)$ , we have  $E(|X| | X > 0) = \frac{\alpha}{p}$ ; analogously  $E(|X| | X < 0) = \frac{\alpha}{1-r-p}$ .

Applying Jensen's inequality to the convex function  $f : [0, \infty) \rightarrow [0, \infty)$ ,  $f(x) = x^{\frac{3+\delta}{2}}$  yields

$$E(|X|^{\frac{3+\delta}{2}} | X > 0) \geq \left(\frac{\alpha}{p}\right)^{\frac{3+\delta}{2}} \quad \text{and} \quad E(|X|^{\frac{3+\delta}{2}} | X < 0) \geq \left(\frac{\alpha}{1-r-p}\right)^{\frac{3+\delta}{2}}.$$

Defining  $\beta_z := E|X|^z$  for  $z > 0$  we get

$$\begin{aligned} \beta_{\frac{3+\delta}{2}} &= pE\left(|X|^{\frac{3+\delta}{2}} | X > 0\right) + (1-r-p)E\left(|X|^{\frac{3+\delta}{2}} | X < 0\right) \\ &\geq \alpha^{\frac{3+\delta}{2}} \frac{(1-r-p)^{\frac{1+\delta}{2}} + p^{\frac{1+\delta}{2}}}{(p(1-r-p))^{\frac{1+\delta}{2}}} \end{aligned}$$

and, therefore,

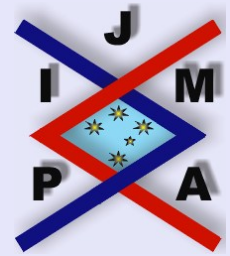
$$(i) \quad \alpha^{\frac{3+\delta}{2}} \leq \beta_{\frac{3+\delta}{2}} \frac{(p(1-r-p))^{\frac{1+\delta}{2}}}{(1-r-p)^{\frac{1+\delta}{2}} + p^{\frac{1+\delta}{2}}}$$

Since  $\gamma^{2+\delta} < \infty$ , we can apply the Cauchy-Schwarz-inequality to  $|X|^{1/2}$  and  $|X|^{1+\delta/2}$  and obtain

$$\left(E|X|^{\frac{3+\delta}{2}}\right)^2 \leq E|X| E|X|^{2+\delta} = 2\alpha\gamma^{2+\delta};$$

hence

$$(ii) \quad \alpha \geq \frac{\left(\beta_{\frac{3+\delta}{2}}\right)^2}{2\gamma^{2+\delta}}.$$



**An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation**

Hendrik Kläver and Norbert  
Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 7 of 25

Combining (i) and (ii) we obtain

$$\left( \frac{\left( \beta_{\frac{3+\delta}{2}} \right)^2}{2\gamma^{2+\delta}} \right)^{\frac{3+\delta}{2}} \leq \beta_{\frac{3+\delta}{2}} \frac{(p(1-r-p))^{\frac{1+\delta}{2}}}{p^{\frac{1+\delta}{2}} + (1-r-p)^{\frac{1+\delta}{2}}};$$

hence

$$x^2 \leq \frac{1}{4} - \frac{1}{1-r} \left( \frac{1}{2} \right)^{\vartheta+1} \frac{a_1}{a_2} \left( \left( \frac{1}{2} + x \right)^{\frac{1}{\vartheta}} + \left( \frac{1}{2} - x \right)^{\frac{1}{\vartheta}} \right)^{\vartheta}$$

with

$$x = \frac{p}{1-r} - \frac{1}{2}, \quad \vartheta = \frac{2}{1+\delta}, \quad a_1 = \left( \beta_{\frac{3+\delta}{2}} \right)^{\vartheta+2}, \quad a_2 = (\gamma^{2+\delta})^{\vartheta+1}.$$

Obviously  $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ ,  $\vartheta \in [1, 2)$ .

Since

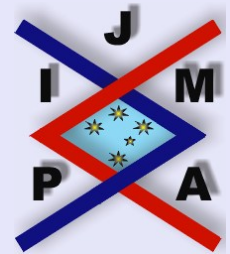
$$(iii) \quad x^\mu + y^\mu \geq (x+y)^\mu \quad \forall \mu \in (0, 1], \quad x, y \geq 0$$

and  $0 < 1-r \leq 1$  it follows altogether that

$$(iv) \quad |x| \leq \frac{1}{2} \sqrt{1 - \left( \frac{1}{2} \right)^{\vartheta-1} \frac{a_1}{a_2}}.$$

For large values of  $|x|$  this estimation is rather poor, so we notice, furthermore, that

$$\left( \left( \frac{1}{2} + x \right)^{\frac{1}{\vartheta}} + \left( \frac{1}{2} - x \right)^{\frac{1}{\vartheta}} \right)^{\vartheta} \geq 2^{\vartheta-1} (1 - 4x^2)^{1/2};$$



**An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation**

Hendrik Kläver and Norbert  
Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 8 of 25



hence

$$(v) \quad |x| \leq \frac{1}{2} \sqrt{1 - \left(\frac{a_1}{a_2}\right)^2}.$$

From (iv) and (v) it follows that

$$(vi) \quad \frac{p}{1-r-p} \leq 2^\vartheta \frac{a_2}{a_1} - 1 + 2^{\frac{\vartheta+1}{2}} \sqrt{\frac{a_2}{a_1} \left(2^{\vartheta-1} \frac{a_2}{a_1} - 1\right)}$$

and

$$(vii) \quad \frac{p}{1-r-p} \leq 2 \left(\frac{a_2}{a_1}\right)^2 - 1 + 2 \frac{a_2}{a_1} \sqrt{\left(\frac{a_2}{a_1}\right)^2 - 1}.$$

Now we estimate  $\beta_{\frac{3+\delta}{2}}$ . Due to Jensen's inequality we have  $\left(\beta_{\frac{3+\delta}{2}}\right)^{\frac{4}{3+\delta}} \leq \sigma^2 = 1$ . Let  $\delta = 1$ . Then  $\beta_{\frac{3+\delta}{2}} = \sigma^2 = 1$  and so  $\frac{a_2}{a_1} = (\gamma^3)^2$ . Let  $\delta \in \left[\frac{1}{3}, 1\right)$ . Then  $\frac{5-\delta}{2} \leq 2 + \delta$ , so  $\beta_{\frac{5-\delta}{2}}$  exists. Due to the Cauchy-Schwarz-inequality we have

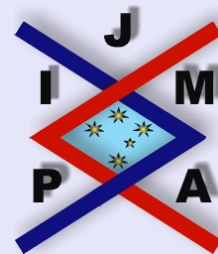
$$1 = (\sigma^2)^2 \leq \beta_{\frac{3+\delta}{2}} \beta_{\frac{5-\delta}{2}};$$

hence

$$\beta_{\frac{3+\delta}{2}} \geq \frac{1}{\beta_{\frac{5-\delta}{2}}} \geq \frac{1}{(\gamma^{2+\delta})^{\frac{5-\delta}{2(2+\delta)}}}.$$

altogether

$$\frac{a_2}{a_1} \leq (\gamma^{2+\delta})^{4\vartheta}.$$



**An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation**

Hendrik Kläver and Norbert  
Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 9 of 25

Let  $k \in \mathbb{N}$ ,  $k \geq 2$ . For  $\delta \geq \frac{1}{2^{k+1}}$  it follows that  $2 + \frac{1-\delta}{2^k} \leq 2 + \delta$ ; hence  $\beta_{2+\frac{1-\delta}{2^k}}$  exists. Due to the Cauchy-Schwarz-inequality we have

$$1 = (\sigma^2)^2 \leq \beta_{2+\frac{1-\delta}{2^k}} \beta_{2-\frac{1-\delta}{2^k}}$$

and

$$\left(\beta_{2-\frac{1-\delta}{2^{j+1}}}\right)^{2^j} = \left(\left(\beta_{2-\frac{1-\delta}{2^{j+1}}}\right)^2\right)^{2^{j-1}} \leq \left(\beta_{2-\frac{1-\delta}{2^j}} \beta_2\right)^{2^{j-1}}$$

for  $j \in \{1, \dots, k-1\}$ . This yields

$$\beta_{\frac{3+\delta}{2}} \geq \left(\beta_{2-\frac{1-\delta}{2^k}}\right)^{2^{k-1}} \geq \frac{1}{\left(\beta_{2+\frac{1-\delta}{2^k}}\right)^{2^{k-1}}} \geq \frac{1}{(\gamma^{2+\delta})^{\frac{2^{k+1}+1-\delta}{2(2+\delta)}}}.$$

Altogether we get  $\frac{a_2}{a_1} \leq (\gamma^{2+\delta})^{(2^k+2)\vartheta}$  for  $\delta \in \left[\frac{1}{2^{k+1}}, \frac{1}{2^{k-1}+1}\right)$ .

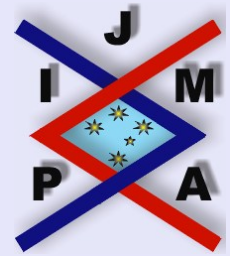
Combining this with (vi) and (vii) we obtain the assertion.  $\square$

**Remark 1.** For each  $\delta \in (0, 1]$  equality holds in Theorem 2.1 iff  $P^X = \frac{1}{2}(\varepsilon_1 + \varepsilon_{-1})$  (where  $\varepsilon_x$  denotes the Dirac measure in  $x$ ).

*Proof.* (i) Let  $X$  be a real random variable with  $P^X = \frac{1}{2}(\varepsilon_1 + \varepsilon_{-1})$ . Then  $E(X) = 0$ ,  $\text{Var}(X) = 1$ ,  $\gamma^{2+\delta} = 1$  and so  $g_\delta(\gamma^{2+\delta}) = 1$  for all  $\delta \in (0, 1]$ . Since  $P(X < 0) = P(X > 0) = \frac{1}{2}$  we get equality.

(ii) In Theorem 2.1 we have

$$\left(\frac{\beta_{\frac{3+\delta}{2}}^2}{2\gamma^{2+\delta}}\right)^{\frac{3+\delta}{2}} \leq \alpha^{\frac{3+\delta}{2}} \leq \frac{(p(1-r-p))^{\frac{1+\delta}{2}}}{(1-r-p)^{\frac{1+\delta}{2}} + p^{\frac{1+\delta}{2}}} \beta_{\frac{3+\delta}{2}}.$$



An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation

Hendrik Kläver and Norbert  
Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 10 of 25

In the first “ $\leq$ ” there is equality iff  $|X|^{1/2}$  and  $|X|^{1+\delta/2}$  are linearly dependent  $P$ -almost surely, i.e.  $P(|X| \in \{0, c\}) = 1$  for some  $c > 0$ . As  $E(X) = 0$ , we obtain  $P^X = p(\varepsilon_c + \varepsilon_{-c}) + (1-2p)\varepsilon_0$ . So the inequality is sharp iff  $g_\delta(\gamma^{2+\delta}) = 1$ . With  $1 = E(X^2) = 2c^2p$  we obtain

$$\gamma^{2+\delta} = 2pc^{2+\delta} = \frac{1}{(2p)^{\delta/2}}.$$

The functions  $h_\delta : (0, \frac{1}{2}] \rightarrow \mathbb{R}$ ,  $\delta \in (0, 1]$ , defined by  $h_\delta(p) = g_\delta\left(\frac{1}{(2p)^{\delta/2}}\right)$ , are strictly decreasing. Due to  $p \in (0, \frac{1}{2}]$  and  $h_\delta(\frac{1}{2}) = 1$  we get  $p = \frac{1}{2}$ ,  $r = 0$  and  $c = 1$ , therefore  $P^X = \frac{1}{2}(\varepsilon_1 + \varepsilon_{-1})$ .  $\square$

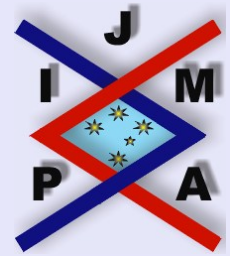
Since the Central Limit Theorem is concerned with sums of random variables (instead of single variables) we need a corresponding generalization of Theorem 2.1.

**Corollary 2.2.** *Under assumption (M),*

$$E\left(\left|\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i\right|^{2+\delta}\right) \leq \frac{\gamma^{2+\delta} - 1}{n^{\delta/2}} + n^{2-\delta/2}$$

*holds, and therefore,*

$$P\left(\sum_{i=1}^n X_i < 0\right) \leq g_\delta\left(\frac{\gamma^{2+\delta} - 1}{n^{\delta/2}} + n^{2-\delta/2}\right) P\left(\sum_{i=1}^n X_i > 0\right).$$



**An Inequality for the Asymmetry of Distributions and a Berry-Esseen Theorem for Random Summation**

Hendrik Kläver and Norbert Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 11 of 25

*Proof.*

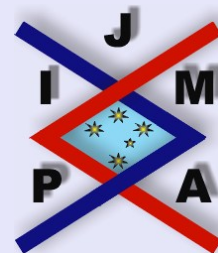
$$\begin{aligned} \left| \sum_{i=1}^n X_i \right|^{2+\delta} &= \left( \left| \sum_{i=1}^n X_i \right|^3 \right)^{\frac{2+\delta}{3}} \\ &\leq \left( \sum_{i=1}^n |X_i|^3 + 3 \sum_{\substack{k,i=1 \\ k \neq i}}^n |X_k^2 X_i| + \sum_{\substack{i,k,l=1 \\ i \neq k \neq l \neq i}}^n |X_i X_k X_l| \right)^{\frac{2+\delta}{3}} \\ &\leq \sum_{i=1}^n |X_i|^{2+\delta} + 3 \sum_{\substack{k,i=1 \\ k \neq i}}^n |X_k^2 X_i|^{\frac{2+\delta}{3}} + \sum_{\substack{i,k,l=1 \\ i \neq k \neq l \neq i}}^n |X_i X_k X_l|^{\frac{2+\delta}{3}}. \end{aligned}$$

Since the  $X_i$  are independent and, due to  $E|X_i|^2 = 1$  and Jensen's inequality,  $E|X_i|^\alpha \leq 1$  for  $\alpha \leq 2$ , we obtain

$$E \left| \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \right|^{2+\delta} \leq \frac{1}{n^{1+\delta/2}} (n\gamma^{2+\delta} + (n^3 - n)) = \frac{\gamma^{2+\delta} - 1}{n^{\delta/2}} + n^{2-\delta/2}.$$

From  $E \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \right) = 0$ ,  $E \left( \frac{1}{\sqrt{n}} \sum_{i=1}^n X_i \right)^2 = 1$  and Theorem 2.1 the assertion follows.  $\square$

We need a bound which does not depend on the number of summands. For this aim we use for  $n < \kappa := (4c_\delta \gamma^{2+\delta})^{2/\delta}$  the asymmetry inequality of Corollary 2.2 and for  $n \geq \kappa$  the Berry-Esseen bound (the choice of  $\kappa$  as boundary between “small” and “large”  $n$  is somewhat arbitrary).



**An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation**

Hendrik Kläver and Norbert  
Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 12 of 25

**Corollary 2.3.** Under Assumption (M)

$$P\left(\sum_{i=1}^n X_i < 0\right) \leq (f_\delta(\gamma^{2+\delta}) - 1)P\left(\sum_{i=1}^n X_i > 0\right),$$

where

$$f_\delta(y) := \max\left\{3, g_\delta(y), g_\delta\left(\frac{y-1}{4c_\delta}\right) + (4c_\delta y)^{\frac{4}{\delta}-1}\right\} + 1.$$

*Proof.* (i) Let  $n < \kappa$ . Then Corollary 2.2 yields

$$P\left(\sum_{i=1}^n X_i < 0\right) \leq g_\delta\left(\frac{\gamma^{2+\delta} - 1}{n^{\delta/2}} + n^{2-\delta/2}\right)P\left(\sum_{i=1}^n X_i > 0\right).$$

For the function  $h : [1, \infty) \rightarrow [1, \infty)$  defined by  $h(y) := \frac{\gamma^{2+\delta}-1}{y^{\delta/2}} + y^{2-\delta/2}$  we obtain

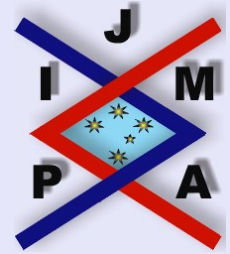
$$h''(y) = \frac{\delta}{2} \left(1 + \frac{\delta}{2}\right) (\gamma^{2+\delta} - 1)y^{-2-\delta/2} + \left(2 - \frac{\delta}{2}\right) \left(1 - \frac{\delta}{2}\right) y^{-\delta/2} > 0,$$

i.e.,  $h$  is convex. Hence the maximum of

$$h : \{1, \dots, \lfloor \kappa \rfloor\} \rightarrow [1, \infty)$$

is attained either for  $n = 1$  or for  $n = \lfloor \kappa \rfloor$ . Since  $g_\delta$  is strictly increasing this yields

$$g_\delta\left(\frac{\gamma^{2+\delta} - 1}{n^{\delta/2}} + n^{2-\delta/2}\right) \leq \max\left\{g_\delta(\gamma^{2+\delta}), g_\delta\left(\frac{\gamma^{2+\delta} - 1}{4c_\delta \gamma^{2+\delta}} + (4c_\delta \gamma^{2+\delta})^{4/\delta-1}\right)\right\}$$



An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation

Hendrik Kläver and Norbert  
Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 13 of 25

for all  $n < \kappa$ .

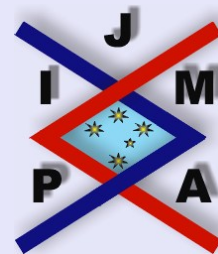
(ii) Let  $n \geq \kappa$ , i.e.  $n^{\delta/2} \geq 4c_\delta \gamma^{2+\delta}$ . Then the (special case of the) theorem of Berry-Esseen yields

$$\left| P \left( \sum_{i=1}^n X_i < 0 \right) - \frac{1}{2} \right| \leq c_\delta \frac{\gamma^{2+\delta}}{n^{\delta/2}} \leq \frac{1}{4}$$

and, therefore,

$$P \left( \sum_{i=1}^n X_i < 0 \right) / P \left( \sum_{i=1}^n X_i > 0 \right) \leq \frac{3/4}{1/4} = 3.$$

Combining (i) and (ii) we obtain the assertion. □



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**An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation**

Hendrik Kläver and Norbert  
Schmitz

---

Title Page

Contents



Go Back

Close

Quit

Page 14 of 25

### 3. Some Further Inequalities

The next inequality represents a quantification of Lemma 7 by Landers and Rogge [8].

**Lemma 3.1.** *Under Assumption (M)*

$$\begin{aligned}
 & P\left(\min_{p \leq n \leq q} \sum_{i=1}^n X_i \leq r\right) - P\left(\max_{p \leq n \leq q} \sum_{i=1}^n X_i \leq r\right) \\
 & \leq f_\delta(\gamma^{2+\delta}) \left( P\left(\sum_{i=1}^p X_i \leq r, \sum_{i=1}^q X_i \geq r\right) \right. \\
 & \quad \left. + P\left(\sum_{i=1}^p X_i \geq r, \sum_{i=1}^q X_i \leq r\right) \right)
 \end{aligned}$$

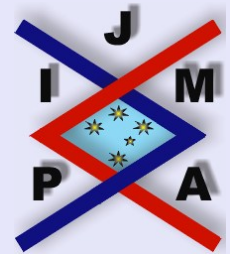
for all  $p, q \in \mathbb{N}$  s.t.  $p < q$  and for all  $r \in \mathbb{R}$ .

*Proof.* Using the same notation  $(A, \alpha, \beta, A_k)$  as Landers and Rogge [8, p. 280], we have to prove that

$$P(A) \leq f_\delta(\gamma^{2+\delta})(\alpha + \beta).$$

(i) First we show that  $P(A \cap \{\sum_{i=1}^p X_i \leq r\}) \leq f_\delta(\gamma^{2+\delta}) \cdot \alpha$ ; for this it is sufficient to prove that

$$P\left(A \cap \left\{\sum_{i=1}^p X_i \leq r\right\} \cap \left\{\sum_{i=1}^q X_i \leq r\right\}\right) \leq (f_\delta(\gamma^{2+\delta}) - 1) \cdot \alpha.$$



An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation

Hendrik Kläver and Norbert  
Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 15 of 25

But due to the independence of the  $A_k$  and  $\sum_{i=k+1}^q X_i$

$$\begin{aligned} P \left( A \cap \left\{ \sum_{i=1}^p X_i \leq r \right\} \cap \left\{ \sum_{i=1}^q X_i \leq r \right\} \right) \\ \leq \sum_{k=p+1}^{q-1} P(A_k) P \left( \sum_{i=k+1}^q X_i \leq 0 \right) \\ \leq (f_\delta(\gamma^{2+\delta}) - 1) \sum_{k=p+1}^{q-1} P(A_k) P \left( \sum_{i=k+1}^q X_i \geq 0 \right) \end{aligned}$$

according to Corollary 2.3

$$\leq (f_\delta(\gamma^{2+\delta}) - 1) \cdot \alpha.$$

(ii) Similarly, it follows that

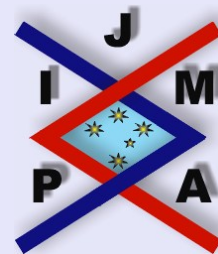
$$P \left( A \cap \left\{ \sum_{i=1}^k X_i > r \right\} \right) \leq f_\delta(\gamma^{2+\delta}) \cdot \beta.$$

(i) and (ii) yield the assertion.  $\square$

A thorough examination of the proof of Lemma 8 of Landers and Rogge [8] allows a generalization and quantification of their result:

**Lemma 3.2.** *Under Assumption (M),*

$$(i) P \left( \sum_{i=1}^n X_i \leq t, \sum_{i=1}^{n+k} X_i \geq t \right) \leq \frac{2c_\delta \gamma^{2+\delta}}{n^{\delta/2}} + \frac{1}{\sqrt{2\pi}} \sqrt{\frac{k}{n}}.$$



An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation

Hendrik Kläver and Norbert  
Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 16 of 25



$$(ii) P\left(\sum_{i=1}^n X_i \geq t, \sum_{i=1}^{n+k} X_i \leq t\right) \leq \frac{2c_\delta \gamma^{2+\delta}}{n^{\delta/2}} + \frac{1}{\sqrt{2\pi}} \sqrt{\frac{k}{n}}.$$

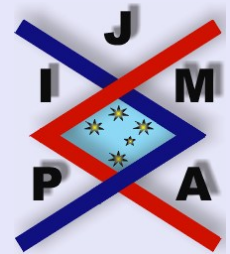
*Proof.* (ii) follows from (i) by replacing  $X_i$  by  $-X_i$ . Analogously to the proof of [8], Lemma 8, we obtain

$$\begin{aligned} P\left(\sum_{i=1}^n X_i \leq t, \sum_{i=1}^{n+k} X_i \geq t\right) &\leq \frac{2c_\delta \gamma^{2+\delta}}{n^{\delta/2}} + \int \left| \Phi\left(\frac{t}{\sqrt{n}}\right) - \Phi\left(\frac{t}{\sqrt{n}} - \frac{1}{\sqrt{n}} \sum_{i=1}^k x_i\right) \right| Q(dx_1, \dots, dx_k) \\ &\leq \frac{2c_\delta \gamma^{2+\delta}}{n^{\delta/2}} + \frac{1}{\sqrt{2\pi}} \sqrt{\frac{k}{n}} \int \left| \frac{1}{\sqrt{k}} \sum_{i=1}^k x_i \right| Q(dx_1, \dots, dx_k) \\ &\leq \frac{2c_\delta \gamma^{2+\delta}}{n^{\delta/2}} + \frac{1}{\sqrt{2\pi}} \sqrt{\frac{k}{n}}, \end{aligned}$$

since

$$E \left| \frac{1}{\sqrt{k}} \sum_{i=1}^k X_i \right| \leq \sqrt{E \left| \frac{1}{\sqrt{k}} \sum_{i=1}^k X_i \right|^2} = 1.$$

□



**An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation**

Hendrik Kläver and Norbert  
Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 17 of 25

## 4. A Berry-Esseen Theorem for Random Sums

As an important application of these inequalities we state our main result:

**Theorem 4.1 (Berry-Esseen type result for random sums).** *Let  $\{X_n, n \geq 1\}$  be i.i.d. random variables with  $EX_n = 0$ ,  $\text{Var } X_n = 1$  and  $\gamma^{2+\delta} := E|X_n|^{2+\delta} < \infty$  for some  $\delta \in (0, 1]$ ; let  $\{N_n, n \in \mathbb{N}\}$  be integer-valued random variables and  $\{\zeta_n, n \in \mathbb{N}\}$  real numbers with  $\lim_{n \rightarrow \infty} \zeta_n = 0$  such that there exist  $d, \tau > 0$  with*

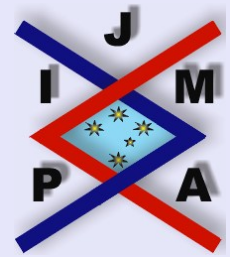
$$P\left(\left|\frac{N_n}{n\tau} - 1\right| > \zeta_n\right) \leq d\sqrt{\zeta_n}.$$

Then<sup>1</sup>

$$\begin{aligned} \text{(i)} \quad \sup_{t \in \mathbb{R}} \left| P\left(\frac{1}{\sqrt{n\tau}} \sum_{i=1}^{N_n} X_i \leq t\right) - \Phi(t) \right| \\ \leq c_\delta \gamma^{2+\delta} (1 + 2^{2+\delta/2} f_\delta(\gamma^{2+\delta})) \frac{1}{[n\tau]^{\delta/2}} + \frac{1}{\sqrt{2\pi e [n\tau]}} \\ + 2f_\delta(\gamma^{2+\delta}) \sqrt{\frac{2 + 1/\tau}{\pi}} \max\left\{\frac{1}{n}, \zeta_n\right\} + 2d\sqrt{\zeta_n}. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \sup_{t \in \mathbb{R}} \left| P\left(\frac{1}{\sqrt{N_n}} \sum_{i=1}^{N_n} X_i \leq t\right) - \Phi(t) \right| \\ \leq c_\delta \gamma^{2+\delta} (1 + 2^{2+\delta/2} f_\delta(\gamma^{2+\delta})) \frac{1}{[n\tau]^{\delta/2}} + \frac{1}{\sqrt{2\pi e [n\tau]}} \end{aligned}$$

<sup>1</sup> $[x] := \max\{n \in \mathbb{N} : n \leq x\}$ .



An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation

Hendrik Kläver and Norbert  
Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 18 of 25

$$+ 2f_\delta(\gamma^{2+\delta})\sqrt{\frac{2+1/\tau}{\pi} \max\left\{\frac{1}{n}, \zeta_n\right\}} + (3d+1)\sqrt{\zeta_n}$$

for all  $n \in \mathbb{N}$  s.t.  $\frac{n\tau}{2} - n\tau\zeta_n \geq 1$ .

Since  $\frac{n\tau}{2} - n\tau\zeta_n \xrightarrow{n \rightarrow \infty} \infty$  there exists an  $n_0$  s.t.  $\frac{n\tau}{2} - n\tau\zeta_n \geq 1$  for all  $n \geq n_0$ .

*Proof.* (i) Let  $n \in \mathbb{N}$  fulfill  $\frac{n\tau}{2} - n\tau\zeta_n \geq 1$  and define as Landers and Rogge [8, p. 271]

$$b_n(t) := t\sqrt{n\tau} \text{ and } I_n := \{k \in \mathbb{N} : \lfloor n\tau - n\tau\zeta_n \rfloor \leq k \leq \lfloor n\tau + n\tau\zeta_n \rfloor\}.$$

Due to the assumption on  $N_n$  we have

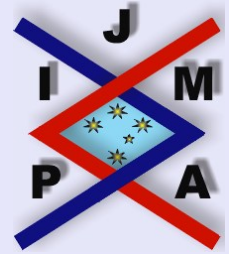
$$P(N_n \notin I_n) \leq d\sqrt{\zeta_n}.$$

For

$$A_n(t) := \left\{ \max_{k \in I_n} \sum_{i=1}^k X_i \leq b_n(t) \right\}, \quad B_n(t) := \left\{ \min_{k \in I_n} \sum_{i=1}^k X_i \leq b_n(t) \right\}$$

follows (see Landers and Rogge [8, p. 272]) for each  $t \in \mathbb{R}$

$$P(A_n(t) \cap \{N_n \in I_n\}) \leq P\left(\sum_{i=1}^{N_n} X_i \leq b_n(t), N_n \in I_n\right) \leq P(B_n(t))$$



**An Inequality for the Asymmetry of Distributions and a Berry-Esseen Theorem for Random Summation**

Hendrik Kläver and Norbert Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 19 of 25

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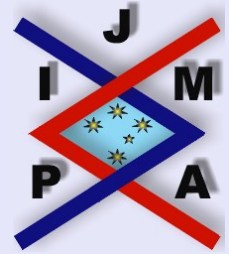
$$P(A_n(t)) \leq P\left(\sum_{i=1}^{\lfloor n\tau \rfloor} X_i \leq b_n(t)\right) \leq P(B_n(t)).$$

Using the Berry-Esseen theorem and the result (3.3) of Petrov [10, p. 114], we obtain

$$\begin{aligned} & \sup_{t \in \mathbb{R}} \left| P\left(\sum_{i=1}^{\lfloor n\tau \rfloor} X_i \leq b_n(t)\right) - \Phi(t) \right| \\ & \leq \sup_{t \in \mathbb{R}} \left| P\left(\frac{1}{\sqrt{\lfloor n\tau \rfloor}} \sum_{i=1}^{\lfloor n\tau \rfloor} X_i \leq t\sqrt{\frac{n\tau}{\lfloor n\tau \rfloor}}\right) - \Phi\left(t\sqrt{\frac{n\tau}{\lfloor n\tau \rfloor}}\right) \right| \\ & \quad + \sup_{t \in \mathbb{R}} \left| \Phi\left(t\sqrt{\frac{n\tau}{\lfloor n\tau \rfloor}}\right) - \Phi(t) \right| \\ & \leq \frac{c_\delta \gamma^{2+\delta}}{\lfloor n\tau \rfloor^{\delta/2}} + \frac{1}{\sqrt{2\pi e \lfloor n\tau \rfloor}}. \end{aligned}$$

For  $p(n) := \lfloor n\tau - n\tau\zeta_n \rfloor$ ,  $q(n) := \lfloor n\tau + n\tau\zeta_n \rfloor$  we obtain from Lemma 3.1

$$\begin{aligned} & P(B_n(t)) - P(A_n(t)) \\ & \leq f_\delta(\gamma^{2+\delta}) \cdot \left( P\left(\sum_{i=1}^{p(n)} X_i \leq b_n(t) \leq \sum_{i=1}^{q(n)} X_i \right) \right) \end{aligned}$$



**An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation**

Hendrik Kläver and Norbert  
Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 20 of 25

$$+P \left( \sum_{i=1}^{p(n)} X_i \geq b_n(t) \geq \sum_{i=1}^{q(n)} X_i \right).$$

According to Lemma 3.2 it follows that

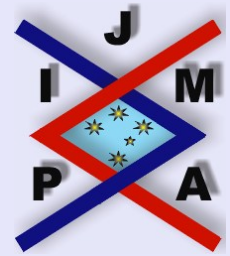
$$\begin{aligned} \sup_{t \in \mathbb{R}} P \left( \sum_{i=1}^{p(n)} X_i \leq b_n(t) \leq \sum_{i=1}^{q(n)} X_i \right) \\ \leq \frac{2c_\delta \gamma^{2+\delta}}{(p(n))^{\delta/2}} + \frac{1}{\sqrt{2\pi}} \sqrt{\frac{q(n) - p(n)}{p(n)}} \\ \leq \frac{2^{1+\delta/2} c_\delta \gamma^{2+\delta}}{(n\tau)^{\delta/2}} + \sqrt{\frac{2 + 1/\tau}{\pi} \max \left\{ \frac{1}{n}, \zeta_n \right\}}, \end{aligned}$$

since  $p(n) \geq n\tau - n\tau\zeta_n - 1 \geq n\tau/2$  and, therefore,

$$\sqrt{\frac{q(n) - p(n)}{p(n)}} \leq \sqrt{\frac{2n\tau\zeta_n + 1}{n\tau/2}} = \sqrt{4\zeta_n + \frac{2}{n\tau}};$$

analogously

$$\sup_{t \in T} P \left( \sum_{i=1}^{p(n)} X_i \geq b_n(t) \geq \sum_{i=1}^{q(n)} X_i \right) \leq \frac{2^{1+\delta/2} c_\delta \gamma^{2+\delta}}{(n\tau)^{\delta/2}} + \sqrt{\frac{2 + 1/\tau}{\pi} \max \left\{ \frac{1}{n}, \zeta_n \right\}}.$$



**An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation**

Hendrik Kläver and Norbert  
Schmitz

Title Page

Contents



Go Back

Close

Quit

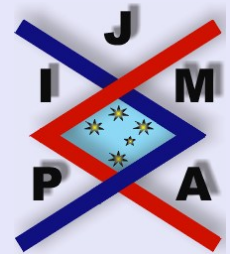
Page 21 of 25

Altogether we obtain

$$\begin{aligned}
 & \sup_{t \in \mathbb{R}} \left| P \left( \sum_{i=1}^{N_n} X_i \leq b_n(t) \right) - \Phi(t) \right| \\
 & \leq \sup_{t \in \mathbb{R}} \left| P \left( \sum_{i=1}^{N_n} X_i \leq b_n(t), N_n \in I_n \right) - \Phi(t) \right| + P(N_n \notin I_n) \\
 & \leq P(B_n(t)) - P(A_n(t)) + \frac{c_\delta \gamma^{2+\delta}}{[n\tau]^{\delta/2}} + \frac{1}{\sqrt{2\pi e [n\tau]}} + 2d\sqrt{\zeta_n} \\
 & \leq 2f_\delta (\gamma^{2+\delta}) \left( \frac{2^{1+\delta} c_\delta \gamma^{2+\delta}}{(n\tau)^{\delta/2}} + \sqrt{\frac{2+1/\tau}{\pi} \max \left\{ \frac{1}{n}, \zeta_n \right\}} \right) \\
 & \quad + \frac{c_\delta \gamma^{2+\delta}}{[n\tau]^{\delta/2}} + \frac{1}{\sqrt{2\pi e [n\tau]}} + 2d\sqrt{\zeta_n} \\
 & \leq c_\delta \gamma^{2+\delta} (1 + 2^{2+\delta/2} f_\delta (\gamma^{2+\delta})) \frac{1}{[n\tau]^{\delta/2}} + \frac{1}{\sqrt{2\pi e [n\tau]}} \\
 & \quad + 2f_\delta (\gamma^{2+\delta}) \sqrt{\frac{2+1/\tau}{\pi} \max \left\{ \frac{1}{n}, \zeta_n \right\}} + 2d\sqrt{\zeta_n}.
 \end{aligned}$$

(ii) Applying Lemma 1 of Michel and Pfanzagl [9] for

$$r = \zeta_n, \quad f = \frac{1}{\sqrt{n\tau}} \sum_{i=1}^{N_n} X_i, \quad g = \sqrt{\frac{N_n}{n\tau}}$$



**An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation**

Hendrik Kläver and Norbert  
Schmitz

Title Page

Contents



Go Back

Close

Quit

Page 22 of 25

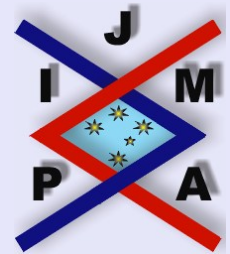
and using the fact that  $\left| \sqrt{\frac{N_n}{n\tau}} - 1 \right| > \sqrt{\zeta_n}$  implies  $\left| \frac{N_n}{n\tau} - 1 \right| > \zeta_n$ , hence

$$P \left( \left| \sqrt{\frac{N_n}{n\tau}} - 1 \right| > \sqrt{\zeta_n} \right) \leq P \left( \left| \frac{N_n}{n\tau} - 1 \right| > \zeta_n \right) \leq d\sqrt{\zeta_n},$$

we obtain from part (i)

$$\begin{aligned} \sup_{t \in T} \left| P \left( \frac{1}{\sqrt{N_n}} \sum_{i=1}^{N_n} X_i \leq t \right) - \Phi(t) \right| \\ \leq c_\delta \gamma^{2+\delta} (1 + 2^{2+\delta/2} f_\delta(\gamma^{2+\delta})) \frac{1}{[n\tau]^{\delta/2}} + \frac{1}{\sqrt{2\pi e [n\tau]}} \\ + 2f_\delta(\gamma^{2+\delta}) \sqrt{\frac{2 + 1/\tau}{\pi} \max \left\{ \frac{1}{n}, \zeta_n \right\}} + (3d + 1) \sqrt{\zeta_n}. \end{aligned}$$

□



**An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation**

Hendrik Kläver and Norbert  
Schmitz

Title Page

Contents



Go Back

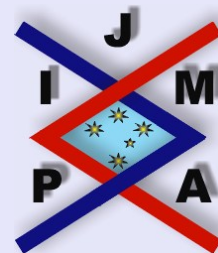
Close

Quit

Page 23 of 25

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An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation

Hendrik Kläver and Norbert  
Schmitz

---

Title Page

Contents



Go Back

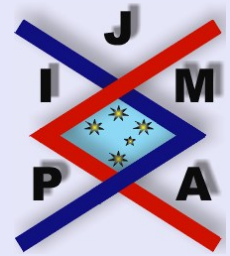
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Quit

Page 24 of 25



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An Inequality for the  
Asymmetry of Distributions and  
a Berry-Esseen Theorem for  
Random Summation

Hendrik Kläver and Norbert  
Schmitz

---

Title Page

Contents



Go Back

Close

Quit

Page 25 of 25