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ALGORITHMS FOR GENERAL MIXED QUASI VARIATIONAL INEQUALITIES

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ABSTRACT. In this paper, we use the auxiliary principle technique to suggest and analyze a predictor-corrector method for solving general mixed quasi variational inequalities. If the bifunction involving the mixed quasi variational inequalities is skew-symmetric , then it is shown that the convergence of the new method requires the partially relaxed strong monotonicity property of the operator, which is a weaker condition than cocoercivity. Since the general mixed quasi variational inequalities includes the classical quasi variational inequalities and complementarity problems as special cases, results obtained in this paper continue to hold for these problems. Our results can be viewed as an important extension of the previously known results for variational inequalities.

Key words and phrases: Variational inequalities, Auxiliary principle, Predictor-corrector method, Resolvent operator, Convergence.

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1. INTRODUCTION

In recent years, variational inequalities have been generalized and extended in many different directions using novel and innovative techniques to study wider classes of unrelated problems arising in optimal control, electrical networks, transportation, finance, economics, structural analysis, optimization and operations research in a general and unified framework, see [1] – [22] and the references therein. An important and useful generalization of variational inequalities is called the general mixed quasi variational inequality involving the nonlinear bifunction. For applications and numerical methods, see [1], [4], [6] – [8], [10], [11], [17], [18]. It is well known that due to the presence of the nonlinear bifunction, projection method and its variant forms including the Wiener-Hopf equations, descent methods cannot be extended to suggest iterative methods for solving the general mixed quasi variational inequalities. This fact has motivated researchers to develop other kinds of methods for solving the general mixed quasi variational inequalities. In particular, it has been shown that if the nonlinear bifunction is proper,

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convex and lower semicontinuous with respect to the first argument, then the general mixed quasi variational inequalities are equivalent to the fixed-point problems. This equivalence has been used to suggest and analyse some iterative methods for solving the general mixed quasi variational inequalities. In this approach, one has to evaluate the resolvent of the operator, which is itself a difficult problem. On the other hand, this technique cannot be extended for the nondifferentiable bifunction. To overcome these difficulties, we use the auxiliary principle technique. In recent years, this technique has been used to suggest and analyze various iterative methods for solving various classes of variational inequalities. It can be shown that several numerical methods including the projection and extragradient can be obtained as special cases from this technique, see [9, 17, 18, 19, 22] and references therein. In this paper, we again use the auxiliary principle to suggest a class of predictor-corrector methods for solving general mixed quasi variational inequalities. The convergence of these methods requires that the operator is partially relaxed strongly monotone, which is weaker than co-coercive. Consequently, we improve the convergence results of previously known methods, which can be obtained as special cases from our results.

2. PRELIMINARIES

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively. Let K be a nonempty closed convex set in H. Let $\varphi(\cdot,\cdot): H\times H\to R\cup\{+\infty\}$ be a nondifferentiable nonlinear bifunction.

For given nonlinear operators $T: H\to H$ and $g: H\longrightarrow H$, consider the problem of finding $u\in H$ such that

$$\langle Tu, g(v)-g(u)\rangle + \varphi(g(v),g(u)) - \varphi(g(u),g(u)) \geq 0, \qquad \text{for all } g(v)\in H. \tag{2.1}$$

The inequality of type (2.1) is called the general mixed quasi variational inequality. If the bifunction $\varphi(\cdot,\cdot)$ is proper, convex and lower-semicontinuous with respect to the first argument, then problem (2.1) is equivalent to finding $u\in H$ such that

$$0\in Tu+\partial\varphi(g(u),g(u)), \tag{2.2}$$

where $\partial\varphi(\cdot,\cdot)$ is the subdifferential of the bifunction $\varphi(\cdot,\cdot)$ with respect to the first argument, which is a maximal monotone operator. Problem (2.2) is known as finding the zero of the sum of the two (or more) maximal operators. It can be shown that a wide class of linear and nonlinear equilibrium problems arising in pure and applied sciences can be studied via the general mixed variational inequalities (2.1) and (2.2).

Special Cases

We remark that if $g\equiv I$, the identity operator, then problem (2.1) is equivalent to finding $u\in H$ such that

$$\langle Tu, v-u\rangle + \varphi(v,u) - \varphi(u,u) \geq 0, \qquad \text{for all } v\in H, \tag{2.3}$$

which are called the mixed quasi variational inequalities. For the applications and numerical methods of the mixed quasi variational inequalities, see [1], [4], [6] – [8], [10], [11], [17], [18].

We note that if $\varphi(\cdot,\cdot)$ is the indicator function of a closed convex-valued set $K(u)$ in H, that is,

$$\varphi(u,u)\equiv I_{K_u}(u) = \begin{cases} 0, & \text{if } u\in K(u) \\ +\infty, & \text{otherwise,} \end{cases}$$

then problem (2.1) is equivalent to finding $u\in H,\ g(u)\in K(u)$ such that

$$\langle Tu, g(v)-g(u)\rangle \geq 0, \qquad \text{for all } g(v)\in K(u). \tag{2.4}$$

Equation (2.4) is known as the general quasi variational inequality. For $K(u) \equiv K$, a convex set in H, problem (2.4) was introduced and studied by Noor [12] in 1988. It turned out that a wide class of odd-order and nonsymmetric free, unilateral, obstacle and equilibrium problems can be studied by the general quasi variational inequality, see [13] – [15], [19].

If $K^*(u) = \{u \in H : \langle u, v\rangle \geq 0, \text{for all } v \in K(u)\}$ is a polar cone of a convex-valued cone $K(u)$ in H and g is onto K, then problem (2.4) is equivalent to finding $u \in H$ such that

$$g(u) \in K(u), \quad Tu \in K^*(u), \text{and} \quad \langle Tu, g(u)\rangle = 0, \tag{2.5}$$

which is known as the general quasi complementarity problem. We note that if $g(u) = u - m(u)$, where m is a point -to-point mapping, then problem (2.5) is called the quasi(implicit) complementarity problem. For $g \equiv I$, problem (2.5) is known as the generalized complementarity problem. For the formulation and numerical methods of complementarity problems, see the references.

For $g \equiv I$, the identity operator, problem (2.4) collapses to: find $u \in K(u)$ such that

$$\langle Tu, v - u\rangle \geq 0, \qquad \text{for all } v \in K(u), \tag{2.6}$$

which is called the classical quasi variational inequality, see [2, 3, 14, 15, 19].

It is clear that problems (2.4) – (2.6) are special cases of the general mixed quasi variational inequality (2.1). In brief, for a suitable and appropriate choice of the operators T, g, $\varphi(\cdot,\cdot)$ and the space H, one can obtain a wide class of variational inequalities and complementarity problems. This clearly shows that problem (2.1) is quite general. Furthermore, problem (2.1) has important applications in various branches of pure and applied sciences.

We also need the following concepts.

Lemma 2.1. *For all* $u, v \in H$, *we have*

$$2\langle u, v\rangle = ||u + v||^2 - ||u||^2 - ||v||^2. \tag{2.7}$$

Definition 2.1. For all $u, v, z \in H$, an operator $T : H \to H$ is said to be:

(i) g-***partially relaxed strongly monotone***, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, g(z) - g(v)\rangle \geq -\alpha||g(u) - g(z)||^2$$

(ii) g-***cocoercive***, if there exists a constant $\mu > 0$ such that

$$\langle Tu - Tv, g(u) - g(v)\rangle \geq \mu\|Tu - Tv\|^2.$$

We remark that if $z = u$, then g-partially relaxed strongly monotone is exactly g-monotone of the operator T. It has been shown in [9] that g-cocoercivity implies g-partially relaxed strongly monotonicity. This shows that partially relaxed strongly monotonicity is a weaker condition than cocoercivity.

Definition 2.2. For all $u, v \in H$, the bifunction $\varphi(\cdot,\cdot)$ is said to be **skew-symmetric,** if

$$\varphi(u, u) - \varphi(u, v) - \varphi(v, u) + \varphi(v, v) \geq 0.$$

Note that if the bifunction $\varphi(\cdot,\cdot)$ is linear in both arguments, then it is nonnegative. This concept plays an important role in the convergence analysis of the predictor-corrector methods. For the properties and applications of the skew-symmetric bifunction, see Noor [18].

3. MAIN RESULTS

In this section, we suggest and analyze a new iterative method for solving the problem (2.1) by using the auxiliary principle technique of Glowinski, Lions and Tremolieres [5] as developed by Noor [9, 13, 14, 18].

For a given $u \in H$, consider the problem of finding a unique $w \in H$ satisfying the auxiliary general mixed quasi variational inequality

$$\langle \rho Tu + g(w) - g(u), g(v) - g(w) \rangle + \rho\varphi(g(v), g(u)) - \rho\varphi(g(u), g(u)) \geq 0, \quad \text{for all } v \in H, \tag{3.1}$$

where $\rho > 0$ is a constant.

We note that if $w = u$, then clearly w is a solution of the general mixed variational inequality (2.1). This observation enables us to suggest the following iterative method for solving the general mixed variational inequalities (2.1).

Algorithm 3.1. For a given $u_0 \in H$, compute the approximate solution u_{n+1} by the iterative scheme

$$\langle \rho Tw_n + g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1}) \rangle + \rho\varphi(g(v), g(u_{n+1})) - \rho\varphi(g(u_{n+1}), g(u_{n+1})) \geq 0, \quad \text{for all } v \in H \tag{3.2}$$

and

$$\langle \beta Tu_n + g(w_n) - g(u_n), g(v) - g(w_n) \rangle + \beta\varphi(g(v), g(w_n)) - \beta\varphi(g(w_n), g(w_n)) \geq 0, \quad \text{for all } v \in H, \tag{3.3}$$

where $\rho > 0$ and $\beta > 0$ are constants.

Note that if $g \equiv I$, the identity operator, then Algorithm 3.1 reduces to:

Algorithm 3.2. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\langle \rho Tw_n + u_{n+1} - w_n, v - u_{n+1} \rangle + \rho\varphi(v, u_{n+1}) - \rho\varphi(u_{n+1}, u_{n+1}) \geq 0, \quad \text{for all } v \in H,$$

and

$$\langle \beta Tu_n + w_n - u_n, v - w_n \rangle + \beta\varphi(v, w_n) - \beta\varphi(w_n, w_n) \geq 0, \quad \text{for all } v \in H.$$

For the convergence analysis of Algorithm 3.2, see [18].

If the bifunction $\varphi(\cdot, \cdot)$ is a proper, convex and lower-semicontinuous function with respect to the first argument, then Algorithm 3.1 collapses to:

Algorithm 3.3. For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$g(u_{n+1}) = J_{\varphi(u_{n+1})}[g(w_n) - \rho Tw_n],$$
$$g(w_n) = J_{\varphi(w_n)}[g(u_n) - \beta Tu_n], \quad n = 0, 1, 2, \ldots$$

where $J_\varphi(\cdot)$ is the resolvent operator associated with the maximal monotone operator $\partial\varphi(\cdot, \cdot)$, see [17, 18].

If the bifunction $\varphi(\cdot, \cdot)$ is the indicator function of a closed convex-valued set $K(u)$ in H, then Algorithm 3.1 reduces to the following method for solving general quasi variational inequalities (2.4) and appears to be new.

Algorithm 3.4. For a given $u_0 \in H$ such that $g(u_0) \in K(u_0)$, compute u_{n+1} by the iterative schemes

$$\langle \rho Tw_n + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1}) \rangle \geq 0, \quad \text{for all } g(v) \in K(u)$$

and

$$\langle \beta Tu_n + g(w_n) - g(u_n), g(v) - g(w_n) \rangle \geq 0, \quad \text{for all } g(v) \in K(u).$$

For a suitable choice of the operators T, g and the space H, one can obtain various new and known methods for solving variational inequalities.

For the convergence analysis of Algorithm 3.1, we need the following result, which is mainly due to Noor [9, 18]. We include its proof to convey an idea.

Lemma 3.1. *Let $\bar{u} \in H$ be the exact solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 3.1. If the operator $T : H \longrightarrow H$ is $g-$partially relaxed strongly monotone operator with constant $\alpha > 0$ and the bifunction $\varphi(\cdot,\cdot)$ is skew-symmetric, then*

$$||g(u_{n+1}) - g(\bar{u})||^2 \leq ||g(u_n) - g(\bar{u})||^2 - (1 - 2\rho\alpha)||g(u_{n+1}) - g(u_n)||^2. \tag{3.4}$$

Proof. Let $\bar{u} \in H$ be solution of (2.1). Then

$$\langle \rho T\bar{u}, g(v) - g(\bar{u})\rangle + \rho\varphi(g(v), g(\bar{u})) - \rho\varphi(g(\bar{u}), g(\bar{u})) \geq 0, \quad \text{for all } v \in H, \tag{3.5}$$

and

$$\langle \beta T\bar{u} g(v) - g(\bar{u})\rangle + \beta\varphi(g(v), g(\bar{u})) - \beta\varphi(g(\bar{u}), g(\bar{u})) \geq 0, \quad \text{for all } v \in H, \tag{3.6}$$

where $\rho > 0$ and $\beta > 0$ are constants.

Now taking $v = u_{n+1}$ in (3.5) and $v = \bar{u}$ in (3.2), we have

$$\langle \rho T\bar{u}, g(u_{n+1}) - g(\bar{u})\rangle + \rho\varphi(g(u_{n+1}), g(\bar{u})) - \rho\varphi(g(\bar{u}), g(\bar{u})) \geq 0 \tag{3.7}$$

and

$$\begin{aligned}\langle \rho Tw_n + g(u_{n+1}) - g(u_n), g(\bar{u}) - g(u_{n+1})\rangle + \rho\varphi(g(\bar{u}), g(u_{n+1}))& \\ - \rho\varphi(g(u_{n+1}), g(u_{n+1})) &\geq 0.\end{aligned} \tag{3.8}$$

Adding (3.7) and (3.8), we have

$$\begin{aligned}&\langle g(u_{n+1}) - g(u_n), g(\bar{u}) - g(u_{n+1})\rangle \\ &\quad \geq \rho\langle Tw_n - T\bar{u}, g(u_{n+1}) - g(\bar{u})\rangle + \rho\{\varphi(g(\bar{u}), g(\bar{u})) \\ &\qquad - \varphi(g(\bar{u}), g(u_{n+1})) - \varphi(g(u_{n+1}), g(\bar{u})) + \varphi(g(u_{n+1}), g(u_{n+1}))\} \\ &\quad \geq -\alpha\rho||g(u_{n+1}) - g(w_n)||^2, \end{aligned} \tag{3.9}$$

where we have used the fact that $N(\cdot,\cdot)$ is g-partially relaxed strongly monotone with constant $\alpha > 0$, and the fact that the bifunction $\varphi(\cdot,\cdot)$ is skew-symmetric.

Setting $u = g(\bar{u}) - g(u_{n+1})$ and $v = g(u_{n+1}) - g(u_n)$ in (2.7), we obtain

$$\begin{aligned}&\langle g(u_{n+1}) - g(u_n), g(\bar{u}) - g(u_{n+1})\rangle \\ &\qquad = \frac{1}{2}\{||g(\bar{u}) - g(u_n)||^2 - ||g(\bar{u}) - g(u_{n+1})||^2 - ||g(u_{n+1}) - g(u_n)||^2\}.\end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10), we have

$$||g(u_{n+1}) - g(\bar{u})||^2 \leq ||g(u_n) - g(\bar{u})||^2 - (1 - 2\alpha\rho)||g(u_{n+1}) - g(w_n)||^2. \tag{3.11}$$

Taking $v = \bar{u}$ in (3.3) and $v = w_n$ in (3.6), we have

$$\langle \beta T\bar{u}, g(w_n) - g(\bar{u})\rangle + \beta\varphi(g(w_n), g(\bar{u})) - \beta\varphi(g(\bar{u}), g(\bar{u})) \geq 0 \tag{3.12}$$

and

$$\begin{aligned}\langle \beta Tu_n + g(w_n) - g(u_n), g(\bar{u}) - g(w_n)\rangle& \\ + \beta\varphi(g(\bar{u}), g(w_n)) - \beta\varphi(g(w_n), g(w_n)) &\geq 0.\end{aligned} \tag{3.13}$$

Adding (3.12) and (3.13) and rearranging the terms, we have

$$\begin{aligned}\langle g(w_n) - g(u_n), g(\bar{u}) - g(w_n)\rangle &\geq \beta\langle Tu_n - T\bar{u}, g(w_n) - g(\bar{u})\rangle + \beta\{\varphi(g(\bar{u}), g(\bar{u})) \\ &\quad - \varphi(g(\bar{u}), g(w_n)) - \varphi(g(w_n), g(\bar{u})) + \varphi(g(w_n), g(w_n))\} \\ &\geq -\beta\alpha||g(u_n) - g(w_n)||^2, \end{aligned} \tag{3.14}$$

since $N(\cdot,\cdot)$ is a g-partially relaxed strongly monotone operator with constant $\alpha > 0$ and the bifunction $\varphi(\cdot,\cdot)$ is skew-symmetric.

Now taking $v = g(w_n) - g(u_n)$ and $u = g(\bar{u}) - g(w_n)$ in (2.7), (3.14) can be written as

$$\begin{aligned}||g(\bar{u}) - g(w_n)||^2 &\leq ||g(\bar{u}) - g(u_n)||^2 - (1 - 2\beta\alpha)||g(u_n) - g(w_n)||^2 \\ &\leq ||g(\bar{u})) - g(u_n)||^2, \quad \text{for } 0 < \beta < 1/2\alpha. \end{aligned} \tag{3.15}$$

Consider

$$\begin{aligned}||g(u_{n+1}) - g(w_n)||^2 &= ||g(u_{n+1}) - g(u_n) + g(u_n) - g(w_n)||^2 \\ &= ||g(u_{n+1}) - g(u_n)||^2 + ||g(u_n) - g(w_n)||^2 \\ &\quad + 2\langle g(u_{n+1}) - g(u_n), g(u_n) - g(w_n)\rangle. \end{aligned} \tag{3.16}$$

Combining (3.11), (3.15) and (3.16), we obtain

$$||g(u_{n+1}) - g(\bar{u})||^2 \leq ||g(u_n) - g(\bar{u})||^2 - (1 - 2\rho\alpha)||g(u_{n+1}) - g(u_n)||^2,$$

the required result. □

Theorem 3.2. *Let $g : H \longrightarrow H$ be invertible and $0 < \rho < \frac{1}{2\alpha}$. Let u_{n+1} be the approximate solution obtained from Algorithm 3.1 and $\bar{u} \in H$ be the exact solution of (2.1), then* $\lim_{n\to\infty} u_n = \bar{u}$.

Proof. Let $\bar{u} \in H$ be a solution of (2.1). Since $0 < \rho < \frac{1}{2\alpha}$. From (3.4), it follows that the sequence $\{||g(\bar{u}) - g(u_n)||\}$ is nonincreasing and consequently $\{u_n\}$ is bounded. Furthermore, we have

$$\sum_{n=0}^{\infty}(1 - 2\alpha\rho)||g(u_{n+1}) - g(u_n)||^2 \leq ||g(u_0) - g(\bar{u})||^2,$$

which implies that

$$\lim_{n\to\infty} ||g(u_{n+1}) - g(u_n)|| = 0. \tag{3.17}$$

Let $\hat{u}$ be the cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converge to $\hat{u} \in H$. Replacing w_n by u_{n_j} in (3.2) and (3.3), taking the limit $n_j \longrightarrow \infty$ and using (3.17), we have

$$\langle T\hat{u}, g(v) - g(\hat{u})\rangle + \varphi(g(v), g(\hat{u})) - \varphi(g(\hat{u}), g(\hat{u})) \geq 0, \quad \text{for all } v \in H,$$

which implies that $\hat{u}$ solves the general mixed variational inequality (2.1) and

$$\|g(u_{n+1}) - g(\bar{u})\|^2 \leq \|g(u_n) - g(\bar{u})\|^2.$$

Thus it follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and

$$\lim_{n\to\infty} g(u_n) = g(\hat{u}).$$

Since g is invertible,

$$\lim_{n\to\infty} (u_n) = \hat{u},$$

the required result. □

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