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ON AN ε-BIRKHOFF ORTHOGONALITY

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Abstract

We define an approximate Birkhoff orthogonality relation in a normed space. We compare it with the one given by S.S. Dragomir and establish some properties of it. In particular, we show that in smooth spaces it is equivalent to the approximate orthogonality stemming from the semi-inner-product.

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Contents

1. Introduction

In an inner product space, with the standard orthogonality relation $\perp$, one can consider the approximate orthogonality defined by:

$$x \perp^{\varepsilon} y \quad \Leftrightarrow \quad |\langle x|y\rangle| \leq \varepsilon \|x\| \|y\| .$$

($|\cos(x, y)| \leq \varepsilon$ for $x, y \neq 0$).

The notion of orthogonality in an arbitrary normed space, with the norm not necessarily coming from an inner product, may be introduced in various ways. One of the possibilities is the following definition introduced by Birkhoff [1] (cf. also James [6]). Let X be a normed space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$; then for $x, y \in X$

$$x \perp_{\mathrm{B}} y \Longleftrightarrow \forall \lambda \in \mathbb{K} : \|x + \lambda y\| \geq \|x\| .$$

We call the relation $\perp_{\mathrm{B}}$, a *Birkhoff orthogonality* (often called a Birkhoff-James orthogonality).

Our aim is to define an approximate Birkhoff orthogonality generalizing the $\perp^{\varepsilon}$ one. Such a definition was given in [3]:

$$x \underset{\varepsilon}{\perp}_{\mathrm{B}} y \Longleftrightarrow \forall \lambda \in \mathbb{K} : \|x + \lambda y\| \geq (1 - \varepsilon) \|x\| . \tag{1.1}$$

We are going to give another definition of this concept.

2. Birkhoff Approximate Orthogonality

Let us define an *approximate Birkhoff orthogonality*. For $\varepsilon \in [0,1)$:

$$x \perp_B^{\varepsilon} y \iff \forall \lambda \in \mathbb{K} : \|x + \lambda y\|^2 \geq \|x\|^2 - 2\varepsilon \|x\| \|\lambda y\| . \tag{2.1}$$

If the above holds, we say that x is ε-Birkhoff orthogonal to y.

Note, that the relation $\perp_B^{\varepsilon}$ is *homogeneous*, i.e., $x \perp_B^{\varepsilon} y$ implies $\alpha x \perp_B^{\varepsilon} \beta y$ (for arbitrary $\alpha, \beta \in \mathbb{K}$). Indeed, for any $\lambda \in \mathbb{K}$ we have (excluding the obvious case $\alpha = 0$)

$$\begin{aligned} \|\alpha x + \lambda \beta y\|^2 &= |\alpha|^2 \left\| x + \lambda \frac{\beta}{\alpha} y \right\|^2 \\ &\geq |\alpha|^2 \left(\|x\|^2 - 2\varepsilon \|x\| \left\| \lambda \frac{\beta}{\alpha} y \right\| \right) \\ &= \|\alpha x\|^2 - 2\varepsilon \|\alpha x\| \|\lambda \beta y\|. \end{aligned}$$

Proposition 2.1. *If X is an inner product space then, for arbitrary $\varepsilon \in [0,1)$,*

$$x \perp^{\varepsilon} y \iff x \perp_B^{\varepsilon} y.$$

We omit the proof – a more general result will be proved later (Theorem 3.3). As a corollary, for $\varepsilon = 0$, we obtain the well known fact: $x \perp_B y \Leftrightarrow x \perp y$ (in an inner product space).

Let us modify slightly the definition of Dragomir (1.1). Replacing $1 - \varepsilon$ by $\sqrt{1 - \varepsilon^2}$ we obtain:

$$x \perp_D^{\varepsilon} y \iff \forall \lambda \in \mathbb{K} : \|x + \lambda y\| \geq \sqrt{1 - \varepsilon^2} \|x\| .$$

Thus $x\perp_{\mathrm{D}}^{\varepsilon}y \;\Leftrightarrow\; x\underset{\rho}{\perp}_{\mathrm{B}}y$ with $\rho=\rho(\varepsilon)=1-\sqrt{1-\varepsilon^2}$.

Then, for inner product spaces we have:

$$x\perp_{\mathrm{D}}^{\varepsilon}y \quad\Longleftrightarrow\quad x\perp^{\varepsilon}y$$

(see [3, Proposition 1]).

T. Szostok [10], considering a generalization of the sine function introduced, for a real normed space X, the mapping:

$$s(x,y)=\begin{cases}\inf_{\lambda\in\mathbb{R}}\frac{\|x+\lambda y\|}{\|x\|}, & \text{for } x\in X\setminus\{0\};\\ 1, & \text{for } x=0.\end{cases}$$

It is easily seen that $x\perp_{\mathrm{B}}y \;\Leftrightarrow\; s(x,y)=1$. It is also apparent that $x\perp_{\mathrm{D}}^{\varepsilon}y \;\Leftrightarrow\; s(x,y)\ge\sqrt{1-\varepsilon^2}$. Defining $c(x,y):=\pm\sqrt{1-s^2(x,y)}$ (generalized cosine) one gets $x\perp_{\mathrm{D}}^{\varepsilon}y \;\Leftrightarrow\; |c(x,y)|\le\varepsilon$.

Let us compare the approximate orthogonalities $\perp_{\mathrm{D}}^{\varepsilon}$ and $\perp_{\mathrm{B}}^{\varepsilon}$. In an inner product space both of them are equal to ε-orthogonality $\perp^{\varepsilon}$. Thus one may ask if they are equal in an arbitrary normed space. This is not true. Moreover, neither $\perp_{\mathrm{B}}^{\varepsilon}\subset\perp_{\mathrm{D}}^{\varepsilon}$ nor $\perp_{\mathrm{D}}^{\varepsilon}\subset\perp_{\mathrm{B}}^{\varepsilon}$ holds generally (i.e., for an arbitrary normed space and all $\varepsilon\in[0,1)$). For, consider $X=\mathbb{R}^2$ (over $\mathbb{R}$) equipped with the *maximum* norm $\|(x_1,x_2)\|:=\max\{|x_1|,|x_2|\}$. Now, let $x=(1,0)$, $y=(\frac{1}{2},1)$, $\varepsilon=\frac{1}{2}$. One can verify that $x\perp_{\mathrm{B}}^{\varepsilon}y$ (i.e., that $\left(\max\left\{\left|1+\frac{\lambda}{2}\right|,|\lambda|\right\}\right)^2\ge 1-|\lambda|$ holds for each $\lambda\in\mathbb{R}$) but not $x\perp_{\mathrm{D}}^{\varepsilon}y$ (take $\lambda=-\frac{2}{3}$). Thus $\perp_{\mathrm{B}}^{\varepsilon}\not\subset\perp_{\mathrm{D}}^{\varepsilon}$.

On the other hand, for $x=(1,\frac{1}{2})$, $y=(1,0)$, $\varepsilon=\frac{\sqrt{3}}{2}$ we have $(\max\{|1+\lambda|,\frac{1}{2}\})^2\ge 1-\left(\frac{\sqrt{3}}{2}\right)^2$, i.e., $x\perp_{\mathrm{D}}^{\varepsilon}y$ but not $x\perp_{\mathrm{B}}^{\varepsilon}y$ (consider, for example, $\lambda=\frac{\sqrt{3}}{2}-1$). Thus $\perp_{\mathrm{D}}^{\varepsilon}\not\subset\perp_{\mathrm{B}}^{\varepsilon}$.

See also Remark 1 for further comparison of $\perp_{\mathrm{B}}^{\varepsilon}$ and $\perp_{\mathrm{D}}^{\varepsilon}$.

3. Semi–inner–product (approximate) Orthogonality

Let X be a normed space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The norm in X need not come from an inner product. However, (cf. G. Lumer [7] and J.R. Giles [5]) there exists a mapping $[\cdot|\cdot] : X \times X \to \mathbb{K}$ satisfying the following properties:

(s1) $[\lambda x + \mu y|z] = \lambda\,[x|z] + \mu\,[y|z]\,,\;\; x, y, z \in X,\; \lambda, \mu \in \mathbb{K};$

(s2) $[x|\lambda y] = \overline{\lambda}\,[x|y]\,,\;\; x, y \in X,\; \lambda \in \mathbb{K};$

(s3) $[x|x] = \|x\|^2\,,\;\; x \in X;$

(s4) $|\,[x|y]\,| \le \|x\| \cdot \|y\|\,,\;\; x, y \in X.$

(Cf. also [4].) We will call each mapping $[\cdot|\cdot]$ satisfying (s1)–(s4) a *semi-inner-product* (s.i.p.) in a normed space X. Let us stress that we assume that a s.i.p. generates the given norm in X (i.e., (s3) is satisfied). Note, that there may exist infinitely many different semi-inner-products in X. There is a unique s.i.p. in X if and only if X is smooth (i.e., there is a unique supporting hyperplane at each point of the unit sphere S or, equivalently, the norm is Gâteaux differentiable on S – cf. [2, 4]). If X is an inner product space, the only s.i.p. on X is the inner-product itself ([7, Theorem 3]).

We say that s.i.p. is *continuous* iff $\mathrm{Re}\,[y|x+\lambda y] \to \mathrm{Re}\,[y|x]$ as $\mathbb{R} \ni \lambda \to 0$ for all $x, y \in S$. The continuity of s.i.p is equivalent to the smoothness of X (cf.

[5, Theorem 3] or [4]). It follows also in that case (see the proof of Theorem 3 in [5]):

$$\lim_{\substack{\lambda\to 0\\ \lambda\in\mathbb{R}}} \frac{\|x+\lambda y\|-1}{\lambda} = \operatorname{Re}\,[y|x]\,, \quad x,y\in S. \tag{3.1}$$

Extending previous notations we define *semi-orthogonality* and *approximate semi-orthogonality*:

$$x\perp_{\rm s} y \quad\Leftrightarrow\quad [y|x]=0;$$

$$x\perp_{\rm s}^{\varepsilon} y \quad\Leftrightarrow\quad |\,[y|x]\,| \le \varepsilon\,\|x\|\cdot\|y\|\,,$$

for $x,y\in X$ and $0\le\varepsilon<1$.

Obviously, for an inner–product space: $\perp_{\rm s}=\perp$ and $\perp_{\rm s}^{\varepsilon}=\perp^{\varepsilon}$.

Proposition 3.1. *For $x,y\in X$, if $x\perp_{\rm s}^{\varepsilon}y$, then $x\perp_{\rm B}^{\varepsilon}y$ (i.e., $\perp_{\rm s}^{\varepsilon}\subset\perp_{\rm B}^{\varepsilon}$).*

Proof. Suppose that $x\perp_{\rm s}^{\varepsilon}y$, i.e., $|\,[y|x]\,|\le\varepsilon\,\|x\|\cdot\|y\|$. Then, for some $\theta\in[0,1]$ and for some $\varphi\in[-\pi,\pi]$ we have:

$$[y|x]=\theta\varepsilon\,\|x\|\cdot\|y\|\cdot e^{i\varphi}.$$

For arbitrary $\lambda\in\mathbb{K}$ we have:

$$\begin{aligned}\|x+\lambda y\|\cdot\|x\| &\ge |\,[x+\lambda y|x]\,|\\ &= \left|\|x\|^2+\lambda\,[y|x]\right|\\ &= \left|\|x\|^2+\theta\varepsilon\,\|x\|\cdot\|y\|\cdot\lambda\cdot e^{i\varphi}\right|\end{aligned}$$

whence

$$\begin{aligned}\|x+\lambda y\| &\geq \left|\|x\|+\theta\varepsilon\|y\|\cdot\lambda\cdot e^{i\varphi}\right| \\ &= \left|\|x\|+\theta\varepsilon\|y\|\operatorname{Re}\left(\lambda e^{i\varphi}\right)+i\theta\varepsilon\|y\|\operatorname{Im}\left(\lambda e^{i\varphi}\right)\right|.\end{aligned}$$

Therefore

$$\begin{aligned}\|x+\lambda y\|^2 &\geq \left(\|x\|+\theta\varepsilon\|y\|\operatorname{Re}\left(\lambda e^{i\varphi}\right)\right)^2+\left(\theta\varepsilon\|y\|\operatorname{Im}\left(\lambda e^{i\varphi}\right)\right)^2 \\ &= \|x\|^2+2\theta\varepsilon\|x\|\|y\|\operatorname{Re}\left(\lambda e^{i\varphi}\right) \\ &\qquad +\theta^2\varepsilon^2\|y\|^2\left(\left(\operatorname{Re}\left(\lambda e^{i\varphi}\right)\right)^2+\left(\operatorname{Im}\left(\lambda e^{i\varphi}\right)\right)^2\right) \\ &= \|x\|^2+2\theta\varepsilon\|x\|\|y\|\operatorname{Re}\left(\lambda e^{i\varphi}\right)+\theta^2\varepsilon^2\|\lambda y\|^2 \\ &\geq \|x\|^2+2\theta\varepsilon\|x\|\|y\|\operatorname{Re}\left(\lambda e^{i\varphi}\right) \\ &\geq \|x\|^2+2\theta\varepsilon\|x\|\|y\|\left(-\left|\lambda e^{i\varphi}\right|\right) \\ &= \|x\|^2-2\theta\varepsilon\|x\|\|\lambda y\| \\ &\geq \|x\|^2-2\varepsilon\|x\|\|\lambda y\|,\end{aligned}$$

i.e., $x\perp_{\mathrm{B}}^{\varepsilon}y$. □

Since $|\,[y|x]\,|\leq\|x\|\,\|y\|$, i.e., $x\perp_{\mathrm{s}}^{1}y$ for arbitrary x,y, the above result gives also $x\perp_{\mathrm{B}}^{1}y$ for all x,y. That is the reason we restrict ε to the interval $[0,1)$.

Proposition 3.2. *If X is a continuous s.i.p. space and $\varepsilon\in[0,1)$, then $\perp_{\mathrm{B}}^{\varepsilon}\subset\perp_{\mathrm{s}}^{\varepsilon}$.*

Proof. Suppose that $x\perp_{\mathrm{B}}^{\varepsilon}y$. Because of the homogeneity of relations $\perp_{\mathrm{B}}^{\varepsilon}$ and $\perp_{\mathrm{s}}^{\varepsilon}$ we may assume, without loss of generality, that $x,y\in S$. Then, for arbitrary

$\lambda \in \mathbb{K}$ we have:

$$0 \le \|x + \lambda y\|^2 - 1 + 2\varepsilon |\lambda| = [x|x + \lambda y] + [\lambda y|x + \lambda y] - 1 + 2\varepsilon |\lambda| .$$

Therefore

$$\begin{aligned} 0 &\le \operatorname{Re} [x|x + \lambda y] + \operatorname{Re} [\lambda y|x + \lambda y] - 1 + 2\varepsilon |\lambda| \\ &\le |[x|x + \lambda y]| + \operatorname{Re} [\lambda y|x + \lambda y] - 1 + 2\varepsilon |\lambda| \\ &\le \|x + \lambda y\| + \operatorname{Re} [\lambda y|x + \lambda y] - 1 + 2\varepsilon |\lambda| \end{aligned}$$

whence

$$\operatorname{Re} [\lambda y|x + \lambda y] + \|x + \lambda y\| - 1 \ge -2\varepsilon |\lambda| , \quad \text{for all } \lambda \in \mathbb{K}. \tag{3.2}$$

Let $\lambda_0 \in \mathbb{K} \setminus \{0\}$, $n \in \mathbb{N}$ and $\lambda = \frac{\lambda_0}{n}$. Then from (3.2) we have

$$\operatorname{Re} \left[\frac{\lambda_0}{n} y|x + \frac{\lambda_0}{n} y \right] + \left\| x + \frac{\lambda_0}{n} y \right\| - 1 \ge -2\varepsilon \frac{|\lambda_0|}{n};$$

$$\operatorname{Re} \left[\frac{\lambda_0}{|\lambda_0|} y|x + \frac{|\lambda_0|}{n} \frac{\lambda_0}{|\lambda_0|} y \right] + \frac{\left\| x + \frac{|\lambda_0|}{n} \frac{\lambda_0}{|\lambda_0|} y \right\| - 1}{\frac{|\lambda_0|}{n}} \ge -2\varepsilon.$$

Putting $y' := \frac{\lambda_0}{|\lambda_0|} y \in S$, $\xi_n := \frac{|\lambda_0|}{n} \in \mathbb{R}$ ($\xi_n \to 0$ as $n \to \infty$) we obtain from the above inequality

$$\operatorname{Re} [y'|x + \xi_n y'] + \frac{\|x + \xi_n y'\| - 1}{\xi_n} \ge -2\varepsilon.$$

Letting $n \to \infty$, using continuity of the s.i.p. and (3.1)

$$\mathrm{Re}\,[y'|x] + \mathrm{Re}\,[y'|x] \geq -2\varepsilon$$

whence

$$\mathrm{Re}\,[\lambda_0 y|x] \geq -\varepsilon|\lambda_0|.$$

Putting $-\lambda_0$ in the place of λ_0 we obtain $\mathrm{Re}\,[\lambda_0 y|x] \leq \varepsilon|\lambda_0|$ whence

$$|\mathrm{Re}\,[\lambda_0 y|x]| \leq \varepsilon|\lambda_0| \;\text{ for arbitrary } \lambda_0 \in \mathbb{K}.$$

Now, taking $\lambda_0 = \overline{[y|x]}$ we get

$$\left|\mathrm{Re}\left[\overline{[y|x]}y|x\right]\right| \leq \varepsilon|\,[y|x]\,|$$

whence $|\,[y|x]\,|^2 \leq \varepsilon|\,[y|x]\,|$ and finally $|\,[y|x]\,| \leq \varepsilon$, i.e., $x \perp^{\varepsilon}_{\mathrm{s}} y$. □

Without the additional continuity assumption, the inclusion $\perp^{\varepsilon}_{\mathrm{B}} \subset \perp^{\varepsilon}_{\mathrm{s}}$ need not hold.

Example 3.1. *Consider the space* l^1 *(with the norm* $\|x\| = \sum_{i=1}^{\infty} |x_i|$ *for* $x = (x_1, x_2, \dots) \in l^1$*). Define*

$$[x|y] := \|y\| \sum_{\substack{i=1 \\ y_i \neq 0}}^{\infty} \frac{x_i \overline{y_i}}{|y_i|}, \quad x, y \in l^1$$

— a semi-inner-product in l^1. Let $\varepsilon \in [0, \sqrt{2}-1)$ and let $x = (1,0,0,\ldots)$, $y = (1,1,\varepsilon,0,\ldots)$. Then, for an arbitrary $\lambda \in \mathbb{K}$:

$$\begin{aligned}\|x+\lambda y\|^2 - \|x\|^2 + 2\varepsilon \|x\| \, \|\lambda y\| &= (|1+\lambda| + |\lambda| + |\lambda\varepsilon|)^2 - 1 + 2\varepsilon(2+\varepsilon)\,|\lambda| \\ &\geq (1+|\lambda|\,\varepsilon)^2 - 1 + 2\varepsilon(2+\varepsilon)\,|\lambda| \\ &= 2\varepsilon(3+\varepsilon)\,|\lambda| + |\lambda|^2\,\varepsilon^2 \\ &\geq 0,\end{aligned}$$

i.e., $x \perp_{\mathrm{B}}^{\varepsilon} y$ (in fact, $x \perp_{\mathrm{B}} y$). On the other hand,

$$[y|x] = 1 = \frac{1}{2+\varepsilon}\,\|x\|\,\|y\| > \varepsilon\,\|x\|\,\|y\|$$

whence $\neg(x \perp_{\mathrm{s}}^{\varepsilon} y)$. In particular, for $\varepsilon = 0$, this shows that $\perp_{\mathrm{B}} \not\subset \perp_{\mathrm{s}}$ (cf. [4, 8, 9]).

From the last two propositions we have:

Theorem 3.3. *If X is a continuous s.i.p. space, then*

$$\perp_{\mathrm{B}}^{\varepsilon} = \perp_{\mathrm{s}}^{\varepsilon}.$$

Moreover we obtain, for $\varepsilon = 0$, (cf. [5, Theorem 2])

Corollary 3.4. *If X is a continuous s.i.p. space, then*

$$\perp_{\mathrm{B}} = \perp_{\mathrm{s}}.$$

Conversely, $\perp_{\mathrm{B}} \subset \perp_{\mathrm{s}}$ implies continuity of s.i.p. (smoothness) – cf. [4] and [8].

4. Some Remarks

Remark 1. *Dragomir [3, Definition 5] introduces the following concept: The s.i.p. $[\cdot|\cdot]$ is of (APP)-type if there exists a mapping $\eta : [0,1) \to [0,1)$ such that $\eta(\varepsilon) = 0 \ \Leftrightarrow \ \varepsilon = 0$ and $x\perp_{\mathrm{D}}^{\eta(\varepsilon)} y$ implies $x\perp_{\mathrm{s}}^{\varepsilon} y$ for all $\varepsilon \in [0,1)$. It follows from Proposition 3.1 that in that case we have also*

$$x\perp_{\mathrm{D}}^{\eta(\varepsilon)} y \ \Rightarrow \ x\perp_{\mathrm{B}}^{\varepsilon} y \tag{4.1}$$

for all $\varepsilon \in [0,1)$.

It follows from [3, Lemma 1] that for a closed, proper linear subspace G of a normed space X and for an arbitrary $\varepsilon \in (0,1)$, the set $G^{\perp_{\mathrm{D}}^{\varepsilon}}$ of all vectors $\perp_{\mathrm{D}}^{\varepsilon}$-orthogonal to G is nonzero. Using (4.1) we get

$$G^{\perp_{\mathrm{D}}^{\eta(\varepsilon)}} \subset G^{\perp_{\mathrm{B}}^{\varepsilon}}. \tag{4.2}$$

Therefore, we have

Lemma 4.1. *If X is a normed space with the s.i.p. $[\cdot|\cdot]$ of the (APP)-type, then for an arbitrary proper and closed linear subspace G and an arbitrary $\varepsilon \in [0,1)$ the set $G^{\perp_{\mathrm{B}}^{\varepsilon}}$ of all vectors ε-Birkhoff orthogonal to G is nonzero.*

We have also

Theorem 4.2. *If X is a normed space with the s.i.p. $[\cdot|\cdot]$ of the (APP)-type, then for an arbitrary closed linear subspace G and an arbitrary $\varepsilon \in [0,1)$ the following decomposition holds:*

$$X = G + G^{\perp_{\mathrm{B}}^{\varepsilon}}.$$

Proof. Fix G and $\varepsilon \in [0,1)$. It follows from [3, Theorem 3] that

$$X = G + G^{\perp_{\mathrm{D}}^{\eta(\varepsilon)}}.$$

Using (4.2) we get the assertion. □

The final example shows that the set of all ε-orthogonal vectors may be equal to the set of all orthogonal ones.

Example 4.1. *Consider again the space l^1 with the s.i.p. defined above. Let $e = (1,0,\ldots)$. Observe that vectors ε-orthogonal to e are, in fact, orthogonal to e:*

$$x \perp_{\mathrm{B}}^{\varepsilon} e \ \Rightarrow\ x \perp_{\mathrm{B}} e. \tag{4.3}$$

Indeed, let $\varepsilon \in [0,1)$ be fixed and let $x = (x_1, x_2, \ldots) \in l^1$ satisfy $x \perp_{\mathrm{B}}^{\varepsilon} e$. Because of the homogeneity of $\perp_{\mathrm{B}}^{\varepsilon}$ we may assume, without loss of generality, that $\|x\| = 1$ and $x_1 \geq 0$. Thus we have

$$\forall \lambda \in \mathbb{K} : \|x + \lambda e\|^2 \geq 1 - 2\varepsilon |\lambda| .$$

Therefore

$$\forall\, \lambda \in \mathbb{K} : (|x_1 + \lambda| + 1 - x_1)^2 \geq 1 - 2\varepsilon |\lambda| .$$

Suppose that $x_1 > 0$. Take $\lambda \in \mathbb{R}$ such that $\lambda < 0$, $\lambda > -x_1$ and $\lambda > -2(1-\varepsilon)$. Then we have

$$(x_1 + \lambda + 1 - x_1)^2 \geq 1 + 2\varepsilon\lambda,$$

which leads to $\lambda \leq -2(1-\varepsilon)$ *– a contradiction. Thus* $x_1 = 0$*, i.e,* $x = (0, x_2, x_3, \ldots)$ *and* $|x_2| + |x_3| + \cdots = 1$*. This yields, for arbitrary* $\lambda \in \mathbb{K}$*,*

$$\|x + \lambda e\| = |\lambda| + 1 \geq 1 = \|x\|\,,$$

i.e., $x \perp_{\text{B}} e$*. It follows from (4.3) that for* $G := \operatorname{lin} e$ *we have*

$$G^{\perp^{\varepsilon}_{\text{B}}} = G^{\perp_{\text{B}}}.$$

Note, that the implication $e \perp^{\varepsilon}_{\text{B}} x \Rightarrow e \perp_{\text{B}} x$ *is not true. Take for example* $x = \left(\frac{3}{4}, \frac{1}{4}, 0, \ldots\right)$*. Then* $[x|e] = \frac{3}{4}\|e\|\;\|x\|$*, i.e,* $e \perp^{\frac{3}{4}}_{\text{S}} x$*, whence (Proposition 3.1)* $e \perp^{\frac{3}{4}}_{\text{B}} x$*. On the other hand, for* $\lambda = -\frac{5}{3}$ *one has*

$$\|e + \lambda x\| = \frac{2}{3} < 1 = \|e\|,$$

i.e., $\neg(e \perp_{\text{B}} x)$*.*

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