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A SURPRISING RESULT IN COMPARING ORTHOGONAL AND NONORTHOGONAL LINEAR EXPERIMENTS

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ABSTRACT. We demonstrate by example that within nonorthogonal linear experiments, a useful condition derived for comparing of the orthogonal ones not only fails but it may also lead to the reverse order.

Key words and phrases: Linear experiment, orthogonal/nonorthogonal experiment, single parameters, comparison of experiments.

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1. PRELIMINARIES

Any linear experiment is determined by the expectation $E(\mathbf{y})$ and the variance-covariance matrix $V(\mathbf{y})$ of the observation vector $\mathbf{y}$. In the standard case these two moments have the following representation:

$$E(\mathbf{y}) = \mathbf{X}\boldsymbol{\beta} \quad \text{and} \quad V(\mathbf{y}) = \sigma\mathbf{I}_n, \tag{1.1}$$

where $\mathbf{X}$ is a known $n \times p$ *design matrix* while $\boldsymbol{\beta} = (\beta_1, ..., \beta_p)'$ and σ are unknown parameters. To secure the identifiability of the parameters β_i's we assume that *rank*$(\mathbf{X}) = p$. Any standard linear experiment, being formally a structure of the form $(\mathbf{y},\mathbf{X}\boldsymbol{\beta},\sigma\mathbf{I}_n)$, will be denoted by $\mathcal{L}(\mathbf{X})$ and may be identified with its design matrix.

Now let us consider two linear experiments $\mathcal{L}_1 = \mathcal{L}(\mathbf{X}_1)$ and $\mathcal{L}_2 = \mathcal{L}(\mathbf{X}_2)$ with design matrices $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively, and with common parameters $\boldsymbol{\beta}$ and σ. In Stępniak [7], Stępniak and Torgersen [8] and Stępniak, Wang and Wu [9] the experiment $\mathcal{L}_1$ is said to be at least as good as $\mathcal{L}_2$ if for any parametric function $\varphi = \mathbf{c}'\boldsymbol{\beta}$ the variance of its Best Linear Unbiased Estimator (BLUE) in $\mathcal{L}_1$ is not greater than in $\mathcal{L}_2$. It was shown in the above papers that this relation among linear experiments reduces to the Loewner ordering for their information matrices $\mathbf{M}_1 = \mathbf{X}_1'\mathbf{X}_1$ and $\mathbf{M}_2 = \mathbf{X}_2'\mathbf{X}_2$. It appears that this ordering is very strong.

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Many authors, among others Kiefer [1] , Pukelsheim [4] Liski et al. [3], suggest some weaker criteria, among others of type A, D and E, based on some scalar functions of the information matrices. In this paper we focus on a reasonable criterion considered by Rao ([5, p. 236]).

Denote by $\mathcal{C}_p$ the class of all linear experiments with the same parameters $\boldsymbol{\beta}$ and σ, and by $\mathcal{O}_p$ its subclass containing orthogonal experiments only. Inspired by Rao we introduce the following definition.

Definition 1.1. We shall say that an experiment $\mathcal{L}_1$ belonging to $\mathcal{C}_p$ is *better than* $\mathcal{L}_2$ *with respect to the estimation of single parameters* (and write:$\mathcal{L}_1 \succ \mathcal{L}_2$) if for any β_i, $i = 1, ..., p$, its BLUE in $\mathcal{L}_1$ does not have greater variance than in $\mathcal{L}_2$ and less for some i.

One can easily state an algebraic criterion for comparing experiments within $\mathcal{O}_p$. The aim of this note is to reveal the fact that this criterion may lead to a reverse order outside this class.

2. ESTIMATION AND COMPARISON OF LINEAR EXPERIMENTS FOR SINGLE PARAMETERS

In this section we focus on estimation and comparison of linear experiments with respect to the estimation of single parameters β_i for all $i = 1, ..., p$. In this context a simple result provided by Scheffé ([6, Problem 1.5, p. 24]) will be useful. We shall state it in the form of a lemma.

Let $\mathcal{L} = \mathcal{L}(\mathbf{X})$ be a linear experiment of the form (1.1), where $\mathbf{X}$ is an $n \times p$ design matrix of rank p and let $\mathbf{x}_1, ..., \mathbf{x}_p$ be the columns of $\mathbf{X}$. For a given $\mathbf{x}_i$, $i = 1, ..., p$ denote by $\mathbf{P}_i$ the orthogonal projector onto the linear space generated by the remaining columns $\mathbf{x}_j$, $j \neq i$.

Lemma 2.1. *Under the above assumptions each parameter β_i in the experiment (1.1) is unbiasedly estimable and the variance of its BLUE may be presented in the form $\sigma(\mathbf{a}_i'\mathbf{a}_i)^{-1}$, where $\mathbf{a}_i = (\mathbf{I} - \mathbf{P}_i)\mathbf{x}_i$.*

In fact this lemma is a consequence of the well known Lehmann-Scheffé theorem on minimum variance unbiased estimation (cf. Lehmann and Scheffé [2]).

Now let us consider the class $\mathcal{O}_p$ of all orthogonal experiments, i.e. satisfying the condition $\mathbf{x}_i'\mathbf{x}_j = 0$ for $i \neq j$, with the same parameters $\boldsymbol{\beta}$ and σ. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be matrices with columns $\mathbf{x}_{1,1}, ..., \mathbf{x}_{1,p}$ and $\mathbf{x}_{2,1}, ..., \mathbf{x}_{2,p}$, respectively. The following theorem is a direct consequence of Lemma 2.1.

Theorem 2.2. *For any orthogonal experiments $\mathcal{L}_1 = \mathcal{L}(\mathbf{X}_1)$ and $\mathcal{L}_2 = \mathcal{L}(\mathbf{X}_2)$ belonging to the class $\mathcal{O}_p$ the first one is better than the second one for estimation of single parameters, i.e. $\mathcal{L}_1 \succ \mathcal{L}_2$, if and only if,*

$$\mathbf{x}_{1,i}'\mathbf{x}_{1,i} \geq \mathbf{x}_{2,i}'\mathbf{x}_{2,i} \quad \text{for } i = 1, ..., p \text{ with strict inequality for some } i. \tag{2.1}$$

Now we shall demonstrate by example that the ordering rule (2.1) may lead to unexpected results outside the class $\mathcal{O}_p$.

Example 2.1. Let $\mathbf{x}$ be an arbitrary n-column such that $\mathbf{x}'\mathbf{1}_n \neq \mathbf{0}$ and $\mathbf{x} \neq \lambda \mathbf{1}_n$ for any scalar λ. Consider two linear experiments $\mathcal{L}_1 = \mathcal{L}([\mathbf{1}_n, \mathbf{x}])$ and $\mathcal{L}_2 = \mathcal{L}([\mathbf{1}_n, (\mathbf{I}_n - \mathbf{P})\mathbf{x}])$ where $\mathbf{P} = \frac{1}{n}\mathbf{1}_n\mathbf{1}_n'$ is the orthogonal projector onto the one-dimensional linear space generated by $\mathbf{1}_n$. Since $\mathbf{x}'(\mathbf{I} - \mathbf{P})\mathbf{x} < \mathbf{x}'\mathbf{x}$, the condition (2.1) holds for $\mathbf{X}_1 = [\mathbf{1}_n, \mathbf{x}]$ and $\mathbf{X}_2 = [\mathbf{1}_n, (\mathbf{I}_n - \mathbf{P})\mathbf{x}]$. This may suggest that the experiment $\mathcal{L}_1$ is at least as good as $\mathcal{L}_2$ for estimation of the single parameters β_1 and β_2, i.e. that $\mathcal{L}(\mathbf{X}_1) \succ \mathcal{L}(\mathbf{X}_2)$.However, by Lemma 2.1, the variances of the BLUE's for β_2 in these two experiments are the same, while for β_1 the corresponding variance in $\mathcal{L}(\mathbf{X}_2)$ is less than in $\mathcal{L}(\mathbf{X}_1)$.

Conclusion. *In this example the condition (2.1) is met while $\mathcal{L}(\mathbf{X}_2) \succ \mathcal{L}(\mathbf{X}_1)$.*

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