INTEGRAL MEANS FOR UNIFORMLY CONVEX AND STARLIKE FUNCTIONS ASSOCIATED WITH GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract: Making use of the generalized hypergeometric functions, we introduce some generalized class of $k-$uniformly convex and starlike functions and for this class, we settle the Silverman's conjecture for the integral means inequality. In particular, we obtain integral means inequalities for various classes of uniformly convex and uniformly starlike functions in the unit disc.

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open disc $U = \{z : z \in \mathcal{C},\ |z| < 1\}$. For functions $f \in A$ given by (1.1) and $g \in A$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \qquad z \in U. \tag{1.2}$$

For complex parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ $(\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m)$ the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$ {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \tag{1.3}$$

$$(l \leq m + 1;\ l, m \in N_0 := \mathbb{N} \cup \{0\}; z \in U)$$

where $\mathbb{N}$ denotes the set of all positive integers and $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \begin{cases} 1, & n = 0 \\ x(x+1)(x+2)\cdots(x+n-1), & n \in N. \end{cases} \tag{1.4}$$

The notation ${}_lF_m$ is quite useful for representing many well-known functions such as the exponential, the Binomial, the Bessel, the Laguerre polynomial, and others; for example see [5] and [17].

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For positive real values of $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ $(\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m)$, let $H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m) : A \to A$ be a linear operator defined by

$$[(H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m))(f)](z) := z\, {}_lF_m(\alpha_1, \alpha_2, \ldots \alpha_l; \beta_1, \beta_2 \ldots, \beta_m; z) * f(z)$$

$$= z + \sum_{n=2}^{\infty} \Gamma_n\, a_n z^n, \tag{1.5}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1} \ldots (\alpha_l)_{n-1}}{(n-1)!(\beta_1)_{n-1} \ldots (\beta_m)_{n-1}}\,. \tag{1.6}$$

For notational simplicity, we use a shorter notation $H^l_m[\alpha_1, \beta_1]$ for $H(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m)$ in the sequel.

The linear operator $H^l_m[\alpha_1, \beta_1]$ called the Dziok-Srivastava operator (see [7]), includes (as its special cases) various other linear operators introduced and studied by Bernardi [3], Carlson and Shaffer [6], Libera [10], Livingston [12], Owa [15], Ruscheweyh [21] and Srivastava-Owa [27].

For $\lambda \geq 0$, $0 \leq \gamma < 1$ and $k \geq 0$, we let $S^l_m(\lambda, \gamma, k)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\operatorname{Re}\left\{\frac{z(H^l_m[\alpha_1, \beta_1]f(z))' + \lambda z^2 (H^l_m[\alpha_1, \beta_1]f(z))''}{(1-\lambda)H^l_m[\alpha_1, \beta_1]f(z) + \lambda z (H^l_m[\alpha_1, \beta_1]f(z))'} - \gamma\right\} > k\left|\frac{z(H^l_m[\alpha_1, \beta_1]f(z))' + \lambda z^2 (H^l_m[\alpha_1, \beta_1]f(z))''}{(1-\lambda)H^l_m[\alpha_1, \beta_1]f(z) + \lambda z (H^l_m[\alpha_1, \beta_1]f(z))'} - 1\right|, \quad z \in U, \tag{1.7}$$

where $H^l_m[\alpha_1, \beta_1]f(z)$ is given by (1.5). We further let $TS^l_m(\lambda, \gamma, k) = S^l_m(\lambda, \gamma, k) \cap$

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T, where

$$(1.8) \qquad T := \left\{ f \in A : f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in U \right\}$$

is a subclass of A introduced and studied by Silverman [24].

In particular, for $0 \le \lambda < 1$, the class $TS_m^l(\lambda, \gamma, k)$ provides a transition from $k-$uniformly starlike functions to $k-$uniformly convex functions.

By suitably specializing the values of l, m, $\alpha_1, \alpha_2, \dots, \alpha_l$, $\beta_1, \beta_2, \dots, \beta_m$, λ, γ and k, the class $TS_m^l(\lambda, \gamma, k)$ reduces to the various subclasses introduced and studied in [1, 4, 13, 14, 20, 22, 23, 24, 28, 29]. As illustrations, we present some examples for the case when $\lambda = 0$.

Example 1.1. If $l = 2$ and $m = 1$ with $\alpha_1 = 1$, $\alpha_2 = 1$, $\beta_1 = 1$, then

$$(1.9) \quad TS_1^2(0, \gamma, k) \equiv UST(\gamma, k) := \left\{ f \in T : \operatorname{Re}\left\{ \frac{zf'(z)}{f(z)} - \gamma \right\} > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in U \right\}.$$

A function in $UST(\gamma, k)$ is called $k-$uniformly starlike of order γ, $0 \le \gamma < 1$. This class was introduced in [4]. We also note that the classes $UST(\gamma, 0)$ and $UST(0, 0)$ were first introduced in [24].

Example 1.2. If $l = 2$ and $m = 1$ with $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = 1$, then

$$(1.10) \quad TS_1^2(0, \gamma, k) \equiv UCT(\gamma, k) := \left\{ f \in T : \operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} - \gamma \right\} > k \left| \frac{zf''(z)}{f'(z)} \right|, \quad z \in U \right\}.$$

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A function in $UCT(\gamma, k)$ is called $k-$uniformly convex of order γ, $0 \leq \gamma < 1$. This class was introduced in [4]. We also observe that

$$UST(\gamma, 0) \equiv T^*(\gamma), \qquad UCT(\gamma, 0) \equiv C(\gamma)$$

are, respectively, well-known subclasses of starlike functions of order γ and convex functions of order γ. Indeed it follows from (1.9) and (1.10) that

$$f \in UCT(\gamma, k) \Leftrightarrow zf' \in UST(\gamma, k). \tag{1.11}$$

Example 1.3. If $l = 2$ and $m = 1$ with $\alpha_1 = \delta + 1\,(\delta \geq -1)$, $\alpha_2 = 1$, $\beta_1 = 1$, then

$$TS_1^2(0, \gamma, k) \equiv R_\delta(\gamma, k)$$
$$:= \left\{ f \in T : \mathrm{Re}\left(\frac{z(D^\delta f(z))'}{D^\delta f(z)} - \gamma \right) > k \left| \frac{z(D^\delta f(z))'}{D^\delta f(z)} - 1 \right|, \quad z \in U \right\},$$

where D^δ is called Ruscheweyh derivative of order δ $(\delta \geq -1)$ defined by

$$D^\delta f(z) := \frac{z}{(1-z)^{\delta+1}} * f(z) \equiv H_1^2(\delta + 1, 1; 1) f(z).$$

The class $R_\delta(\gamma, 0)$ was studied in [20, 22]. Earlier, this class was introduced and studied by the first author in [1, 2].

Example 1.4. If $l = 2$ and $m = 1$ with $\alpha_1 = c + 1(c > -1)$, $\alpha_2 = 1$, $\beta_1 = c + 2$, then

$$TS_1^2(0, \gamma, k) \equiv BT_c(\gamma, k)$$
$$:= \left\{ f \in T : \mathrm{Re}\left(\frac{z(J_c f(z))'}{J_c f(z)} - \gamma \right) > k \left| \frac{z(J_c f(z))'}{J_c f(z)} - 1 \right|, \quad z \in U \right\},$$

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where J_c is a Bernardi operator [3] defined by

$$J_c f(z) := \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt \equiv H_1^2(c+1, 1; c+2) f(z).$$

Note that the operator J_1 was studied earlier by Libera [10] and Livingston [12].

Example 1.5. If $l = 2$ and $m = 1$ with $\alpha_1 = a\,(a > 0)$, $\alpha_2 = 1$, $\beta_1 = c\,(c > 0)$, then

$$\begin{aligned} &TS_1^2(0, \gamma, k) \\ &\equiv LT_c^a(\gamma, k) \\ &:= \left\{ f \in T : \operatorname{Re}\left(\frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - \gamma \right) > k \left| \frac{z(L(a,c)f(z))'}{L(a,c)f(z)} - 1 \right|, \quad z \in U \right\}, \end{aligned}$$

where $L(a, c)$ is a well-known Carlson-Shaffer linear operator [6] defined by

$$L(a,c)f(z) := \left(\sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{k+1} \right) * f(z) \equiv H_1^2(a, 1; c) f(z).$$

The class $LT_c^a(\gamma, k)$ was introduced in [13].

We can construct similar examples for the case $l = 3$ and $m = 2$ with appropriate real values of the parameters by using the operator $H_2^3[\alpha_1, \beta_1]$, that is $H(\alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2)$ studied by Ponnusamy and Sabapathy [16].

We remark that the classes of uniformly convex and uniformly starlike functions were introduced by Goodman [8, 9] and later generalized by Ronning [18, 19] and others.

In [24], Silverman found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [25] and settled in [26], that

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^\eta d\theta \le \int_0^{2\pi} \left| f_2(re^{i\theta}) \right|^\eta d\theta,$$

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for all $f \in T$, $\eta > 0$ and $0 < r < 1$. In [26], he also proved his conjecture for the subclasses $T^*(\gamma)$ and $C(\gamma)$ of T.

In this note, we prove Silverman's conjecture for the functions in the family $TS_m^l(\lambda, \gamma, k)$. By taking appropriate choices of the parameters $l, m, \alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_m, \lambda, \gamma, k$, we obtain the integral means inequalities for several known as well as new subclasses of uniformly convex and uniformly starlike functions in U. In fact, these results also settle the Silverman's conjecture for several other subclasses of T.

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2. Lemmas and their Proofs

To prove our main results, we need the following lemmas.

Lemma 2.1. *If* γ *is a real number and* w *is a complex number , then* $\mathrm{Re}(w) \geq \gamma \Leftrightarrow |w + (1-\gamma)| - |w - (1+\gamma)| \geq 0.$

Lemma 2.2. *If* w *is a complex number and* γ, k *are real numbers, then*

$$\mathrm{Re}(w) \geq k|w-1| + \gamma \Leftrightarrow \mathrm{Re}\{w(1+ke^{i\theta}) - ke^{i\theta}\} \geq \gamma, \quad -\pi \leq \theta \leq \pi.$$

The proofs of Lemmas 2.1 and 2.2 are straight forward and so are omitted.

The basic tool of our investigation is the following lemma.

Lemma 2.3. *Let* $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, $k \geq 0$ *and suppose that the parameters* $\alpha_1, \dots, \alpha_l$ *and* $\beta_1, \dots, \beta_m$ *are positive real numbers. Then a function* f *belongs to the family* $TS^l_m(\lambda, \gamma, k)$ *if and only if*

$$\sum_{n=2}^{\infty}(1+n\lambda-\lambda)(n(1+k)-(\gamma+k))\Gamma_n\,|a_n| \leq 1-\gamma, \tag{2.1}$$

where

$$\Gamma_n = \frac{(\alpha_1)_{n-1}\dots(\alpha_l)_{n-1}}{(\beta_1)_{n-1}\dots(\beta_m)_{n-1}(n-1)!}. \tag{2.2}$$

Proof. Let a function f of the form $f(z) = z - \sum_{n=2}^{\infty}|a_n|z^n$ in T satisfy the condition (2.1). We will show that (1.7) is satisfied and so $f \in TS^l_m(\lambda, \gamma, k)$. Using Lemma 2.2, it is enough to show that

$$\mathrm{Re}\left\{\frac{z(H^l_m[\alpha_1,\beta_1]f(z))' + \lambda z^2(H^l_m[\alpha_1,\beta_1]f(z))''}{(1-\lambda)H^l_m[\alpha_1,\beta_1]f(z) + \lambda z(H^l_m[\alpha_1,\beta_1]f(z))'}(1+ke^{i\theta}) - ke^{i\theta}\right\} > \gamma, \tag{2.3}$$
$$-\pi \leq \theta \leq \pi.$$

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That is, $\operatorname{Re}\left\{\frac{A(z)}{B(z)}\right\} \geq \gamma$, where

$$\begin{aligned} A(z) &:= [z(H_m^l[\alpha_1,\beta_1]f(z))' + \lambda z^2(H_m^l[\alpha_1,\beta_1]f(z))'')](1+ke^{i\theta}) \\ &\qquad - ke^{i\theta}[(1-\lambda)H_m^l[\alpha_1,\beta_1]f(z) + \lambda z(H_m^l[\alpha_1,\beta_1]f(z))'] \\ &= z + \sum_{n=2}^{\infty}(1+\lambda n-\lambda)(ke^{i\theta}(n-1)+n)\Gamma_n|a_n|z^n, \\ B(z) &:= (1-\lambda)H_m^l[\alpha_1,\beta_1]f(z) + \lambda z(H_m^l[\alpha_1,\beta_1]f(z))' \\ &= z + \sum_{n=2}^{\infty}(1+\lambda n-\lambda)\Gamma_n|a_n|z^n. \end{aligned}$$

In view of Lemma 2.1, we only need to prove that

$$|A(z)+(1-\gamma)B(z)| - |A(z)-(1+\gamma)B(z)| \geq 0.$$

It is now easy to show that

$$\begin{aligned} &|A(z)+(1-\gamma)B(z)| - |A(z)-(1+\gamma)B(z)| \\ &\geq \left[2(1-\gamma) - 2\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[n(1+k)-(\gamma+k)]\Gamma_n|a_n|\right]|z| \\ &\geq 0, \end{aligned}$$

by the given condition (2.1). Conversely, suppose $f \in TS_m^l(\lambda,\gamma,k)$. Then by Lemma 2.2, we have (2.3).

Choosing the values of z on the positive real axis the inequality (2.3) reduces to

$$\operatorname{Re}\left\{\frac{(1-\gamma)-\sum_{n=2}^{\infty}(1+n\lambda-\lambda)(n-\gamma)\Gamma_n a_n z^{n-1} - ke^{i\theta}\sum_{n=2}^{\infty}(1+n\lambda-\lambda)(n-1)\Gamma_n a_n z^{n-1}}{1-\sum_{n=2}^{\infty}(1+n\lambda-\lambda)\Gamma_n a_n z^{n-1}}\right\} \geq 0.$$

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Since $\text{Re}(-e^{i\theta}) \geq -e^{i0} = -1$, the above inequality reduces to

$$\text{Re}\left\{\frac{(1-\gamma)-\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[n(k+1)-(\gamma+k)]\Gamma_n a_n r^{n-1}}{1-\sum_{n=2}^{\infty}(1+n\lambda-\lambda)\Gamma_n a_n r^{n-1}}\right\} \geq 0.$$

Letting $r \to 1^-$, by the mean value theorem we have desired inequality (2.1). □

Corollary 2.4. *If $f \in TS_m^l(\lambda,\gamma,k)$, then*

$$|a_n| \leq \frac{1-\gamma}{\Phi(\lambda,\gamma,k,n)}, \qquad 0 \leq \lambda \leq 1,\ 0 \leq \gamma < 1, k \geq 0,$$

where $\Phi(\lambda,\gamma,k,n) = (1+n\lambda-\lambda)[n(1+k)-(\gamma+k)]\Gamma_n$ and where Γ_n is given by (2.2).
Equality holds for the function

$$f(z) = z - \frac{(1-\gamma)}{\Phi(\lambda,\gamma,k,n)} z^n.$$

Lemma 2.5. *The extreme points of $TS_m^l(\lambda,\gamma,k)$ are*

$$(2.4)\qquad f_1(z) = z \quad \textit{and} \quad f_n(z) = z - \frac{(1-\gamma)}{\Phi(\lambda,\gamma,k,n)} z^n, \quad \textit{for} \quad n = 2,3,4,\ldots,$$

where $\Phi(\lambda,\gamma,k,n)$ is defined in Corollary 2.4.

The proof of the Lemma 2.5 is similar to the proof of the theorem on extreme points given in [24].

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For analytic functions g and h with $g(0) = h(0)$, g is said to be subordinate to h, denoted by $g \prec h$, if there exists an analytic function w such that $w(0) = 0$, $|w(z)| < 1$ and $g(z) = h(w(z))$, for all $z \in U$.

In 1925, Littlewood [11] proved the following subordination theorem.

Lemma 2.6. *If the functions f and g are analytic in U with $g \prec f$, then for $\eta > 0$, and $0 < r < 1$,*

$$\int_0^{2\pi} \left|g(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi} \left|f(re^{i\theta})\right|^\eta d\theta. \tag{2.5}$$

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3. Main Theorem

Applying Lemma 2.6, Lemma 2.3 and Lemma 2.5, we prove the following result.

Theorem 3.1. *Suppose* $f \in TS_m^l(\lambda, \gamma, k)$, $\eta > 0$, $0 \leq \lambda < 1$, $0 \leq \gamma < 1$, $k \geq 0$ *and* $f_2(z)$ *is defined by*

$$f_2(z) = z - \frac{1-\gamma}{\Phi(\lambda, \gamma, k, 2)} z^2,$$

where $\Phi(\lambda, \gamma, k, n)$ *is defined in Corollary 2.4. Then for* $z = re^{i\theta}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} |f(z)|^\eta \, d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta \, d\theta. \tag{3.1}$$

Proof. For $f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n$, (3.1) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \right|^\eta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{(1-\gamma)}{\Phi(\lambda, \gamma, k, 2)} z \right|^\eta d\theta.$$

By Lemma 2.6, it suffices to show that

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} \prec 1 - \frac{1-\gamma}{\Phi(\lambda, \gamma, k, 2)} z.$$

Setting

$$1 - \sum_{n=2}^{\infty} |a_n| z^{n-1} = 1 - \frac{1-\gamma}{\Phi(\lambda, \gamma, k, 2)} w(z), \tag{3.2}$$

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and using (2.1), we obtain

$$|w(z)| = \left| \sum_{n=2}^{\infty} \frac{\Phi(\lambda,\gamma,k,n)}{1-\gamma} |a_n| z^{n-1} \right|$$
$$\leq |z| \sum_{n=2}^{\infty} \frac{\Phi(\lambda,\gamma,k,n)}{1-\gamma} |a_n|$$
$$\leq |z|.$$

This completes the proof by Lemma 2.3. □

By taking different choices of l, m, $\alpha_1, \alpha_2, \ldots, \alpha_l$, $\beta_1, \beta_2, \ldots, \beta_m$, λ, γ and k in the above theorem, we can state the following integral means results for various subclasses studied earlier by several researchers.

In view of the Examples 1.1 to 1.5 in Section 1 and Theorem 3.1, we have following corollaries for the classes defined in these examples.

Corollary 3.2. *If $f \in UST(\gamma,k)$, $0 \leq \gamma < 1$, $k \geq 0$ and $\eta > 0$, then the assertion (3.1) holds true where*

$$f_2(z) = z - \frac{1-\gamma}{k+2-\gamma} z^2.$$

Remark 1. Fixing $k = 0$, Corollary 3.2 gives the integral means inequality for the class $T^*(\gamma)$ obtained in [26].

Corollary 3.3. *If $f \in UCT(\gamma,k)$, $0 \leq \gamma < 1$, $k \geq 0$ and $\eta > 0$, then the assertion (3.1) holds true where*

$$f_2(z) = z - \frac{1-\gamma}{2(k+2-\gamma)} z^2.$$

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Remark 2. Fixing $k = 0$, Corollary 3.3 gives the integral means inequality for the class $C(\gamma)$ obtained in [26]. Also, for $k = 1$, Corollary 3.3 yields the integral means inequality for the class UCT, studied in [28].

Corollary 3.4. *If $f \in R_\delta(\gamma, k)$, $\delta \geq -1$, $0 \leq \gamma < 1$, $k \geq 0$ and $\eta > 0$, then the assertion (3.1) holds true where*

$$f_2(z) = z - \frac{(1-\gamma)}{(\delta+1)(k+2-\gamma)} z^2 .$$

Corollary 3.5. *If $f \in BT_c(\gamma, k)$, $c > -1$, $0 \leq \gamma < 1$, $k \geq 0$ and $\eta > 0$, then the assertion (3.1) holds true where*

$$f_2(z) = z - \frac{(1-\gamma)(c+2)}{(c+1)(k+2-\gamma)} z^2 .$$

Corollary 3.6. *If $f \in LT_c^a(\gamma, k)$, $a > 0$, $c > 0$, $0 \leq \gamma < 1$, $k \geq 0$ and $\eta > 0$, then the assertion (3.1) holds true where*

$$f_2(z) = z - \frac{c(1-\gamma)}{a(k+2-\gamma)} z^2.$$

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