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ON AN INTEGRAL INEQUALITY WITH A KERNEL SINGULAR IN TIME AND SPACE

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ABSTRACT. In this paper we deal with a nonlinear singular integral inequality which arises in the study of partial differential equations. The integral term is non local in time and space and the kernel involved is also singular in both the time and the space variable. The estimates we prove may be used to establish (global) existence and asymptotic behavior results for solutions of the corresponding problems.

Key words and phrases: Nonlinear integral inequality, Singular kernel.

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1. INTRODUCTION

We consider the following integral inequality

$$\varphi(t,x) \le k(t,x) + l(t,x)\int_\Omega \int_0^t \frac{F(s)\varphi^m(s,y)}{(t-s)^{1-\beta}\left|x-y\right|^{n-\alpha}}dyds,\ x\in\Omega,\ t>0, \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^n$ ($n \ge 1$) (bounded or possibly equal to $\mathbb{R}^n$), the functions $k(t,x)$, $l(t,x)$ and $F(t)$ are given positive continuous functions in t. The constants $0<\alpha<n$, $0<\beta<1$ and $m>1$ will be specified below.

This inequality arises in the theory of partial differential equations, for example, when treating the heat equation with a source of polynomial type

$$\begin{cases} u_t(t,x) = \Delta u(t,x) + u^m(t,x),\ x\in\mathbb{R}^n,\ t>0,\ m>1 \\ u(0,x) = u_0(x),\ x\in\mathbb{R}^n. \end{cases}$$

If we write the (weak) solution using the fundamental solution $G(t,x)$ of the heat equation

$$u_t(t,x) = \Delta u(t,x),$$

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namely,

$$u(t,x)=\int_{\mathbb{R}^n}G(t,x-y)u_0(y)dy+\int_0^t\int_{\mathbb{R}^n}G(t-s,x-y)u^m(s,y)dyds$$

and take into account the Solonnikov estimates of this fundamental solution (see [14] for instance), then one is led to an inequality of type (1.1).

The features of this inequality, which make it difficult to deal with, are the singularities of the kernel in both the space and time variables. It is also non integrable with respect to the time variable. The standard methods one can find in the literature (see the recent books by Bainov and Simeonov [1] and Pachpatte [12] and the references therein) concerning regular and/or summable kernels cannot be applied in our situation. Indeed, these methods are based on estimates involving the value of the kernels at zero and/or some L^p-norms of the kernels. In contrast, there are very few papers dealing explicitly with singular kernels similar to ours. Let us point out, however, some works concerning integral equations with singularities in time. In Henry [4, Lemmas 7.1.1 and 7.1.2], a similar inequality to (1.1) with only the integral with respect to time, namely

$$\psi(t)\le a(t)+b\int_0^t(t-s)^{\beta-1}s^{\gamma-1}\psi(s)ds,\ \beta>0,\ \gamma>0$$

i.e. the linear case ($m=1$) has been treated. The case $m>1$ has been considered by Medved in [9], [10]. More precisely, the following inequality

$$\psi(t)\le a(t)+b(t)\int_0^t(t-s)^{\beta-1}s^{\gamma-1}F(s)\psi^m(s)ds,\ \beta>0,\gamma>0$$

was discussed. The result was used to prove a global existence and an exponential decay result for a parabolic Cauchy problem with a source of power type and a time dependent coefficient, namely

$$\begin{cases} u_t+Au=f(t,u),\ u\in X,\\ u(0)=u_0\in X\end{cases}$$

with $\|f(t,u)\|\le t^{\kappa}\eta(t)\|u\|_{\alpha}^m$, $m>1,\ \kappa\ge 0$, where A is a sectorial operator (see [4]) and $\|\cdot\|_{\alpha}$ stands for the norm of the fractional space X^{α} associated to the operator A (see also [11]). This, in turn, has been improved and extended to integro-differential equations and functional differential equations by M. Kirane and N.-E. Tatar in [6] (see also N.-E. Tatar [13] and S. Mazouzi and N.-E. Tatar [7, 8] for more general abstract semilinear evolution problems).

Here, we shall combine the techniques in [9, 10], based on the application of Lemma 2.1 and the use of Lemma 2.3 below, with the Hardy-Littlewood-Sobolev inequality (see Lemma 2.2) to prove our result. We will give sufficient conditions yielding boundedness by continuous functions, exponential decay and polynomial decay of solutions to the integral inequality (1.1).

The paper is organized as follows. In the next section we prepare some notation and lemmas needed in the proofs of our results. Section 3 contains the statement and proof of our result and two corollaries giving sufficient conditions for the exponential decay and the polynomial decay. Finally we point out that our results hold (*a fortiori*) for *weakly* singular kernels in time.

2. PRELIMINARIES

In this section we introduce some material necessary for our results. We will use the usual L^p-space with its norm $\|\cdot\|_p$.

Lemma 2.1. *(Young inequality) We have, for* $\frac{1}{r} = \frac{1}{\rho} + \frac{1}{q}$*, the inequality*

$$\|fg\|_r \leq \|f\|_\rho \cdot \|g\|_q .$$

Lemma 2.2. *Let* $\alpha \in [0,1)$ *and* $\beta \in \mathbb{R}$*. There exists a positive constant* $C = C(\alpha,\beta)$ *such that*

$$\int_0^t s^{-\alpha} e^{\beta s} ds \leq \begin{cases} Ce^{\beta t}, & \text{if } \beta > 0; \\ C(t+1), & \text{if } \beta = 0; \\ C, & \text{if } \beta < 0. \end{cases}$$

Lemma 2.3. *(Hardy-Littlewood-Sobolev inequality)*

Let $u \in L^p(\mathbb{R}^n)$ *(*$p > 1$*),* $0 < \gamma < n$ *and* $\frac{\gamma}{n} > 1 - \frac{1}{p}$*, then* $(1/|x|^\gamma) * u \in L^q(\mathbb{R}^n)$ *with* $\frac{1}{q} = \frac{\gamma}{n} + \frac{1}{p} - 1$*. Also the mapping from* $u \in L^p(\mathbb{R}^n)$ *into* $(1/|x|^\gamma) * u \in L^q(\mathbb{R}^n)$ *is continuous.*

See [5, Theorem 4.5.3, p. 117].

Lemma 2.4. *Let* $a(t)$*,* $b(t)$*,* $K(t)$*,* $\psi(t)$ *be nonnegative, continuous functions on the interval* $I = (0,T)$ *(*$0 < T \leq \infty$*),* $\omega : (0,\infty) \to \mathbb{R}$ *be a continuous, nonnegative and nondecreasing function,* $\omega(0) = 0$*,* $\omega(u) > 0$ *for* $u > 0$ *and let* $A(t) = \max_{0\leq s\leq t} a(s)$*,* $B(t) = \max_{0\leq s\leq t} b(s)$*. Assume that*

$$\psi(t) \leq a(t) + b(t) \int_0^t K(s)\omega(\psi(s))ds,\ t \in I.$$

Then

$$\psi(t) \leq W^{-1}\left[W(A(t)) + B(t)\int_0^t K(s)ds\right],\ t \in (0,T_1),$$

where

$$W(v) = \int_{v_0}^v \frac{d\sigma}{\omega(\sigma)}, v \geq v_0 > 0,$$

W^{-1} *is the inverse of* W *and* $T_1 > 0$ *is such that*

$$W(A(t)) + B(t)\int_0^t K(s)ds \in D(W^{-1})$$

for all $t \in (0,T_1)$*.*

See [2] (or [5]) for the proof.

Lemma 2.5. *If* δ*,* ν*,* $\tau > 0$ *and* $z > 0$*, then*

$$z^{1-\nu}\int_0^z (z-\zeta)^{\nu-1}\zeta^{\delta-1}e^{-\tau\zeta}d\zeta \leq M(\nu,\delta,\tau),$$

where $M(\nu,\delta,\tau) = \max(1,2^{1-\nu})\,\Gamma(\delta)\left(1+\frac{\delta}{\nu}\right)\tau^{-\delta}$.

See [6] for the proof of this lemma.

3. ESTIMATION

In this section we state and prove our result on boundedness and also present an exponential and a polynomial decay result.

Theorem 3.1. *Assume that the constants* α*,* β *and* m *are such that* $0 < \alpha < \beta n$*,* $0 < \beta < 1$ *and* $m > 1$*.*

(i) *If* $\Omega = \mathbb{R}^n$, *then for any* r *satisfying* $\max\left(\frac{(m-1)n}{\alpha}, \frac{m}{\beta}\right) < r < \frac{mn}{\alpha}$, *we have*

$$\|\varphi(t,x)\|_r \le U_{p,r,\rho}(t)$$

with

$$U_{p,r,\rho}(t) = 2^{\frac{m(p-1)}{r}} K(t)^{\frac{1}{p}} \times \left[1 - 2^{m(p-1)}(m-1)C_1^{p-1}C_2^p K(t)^{m-1} L(t) e^{\varepsilon pt} \int_0^t e^{-\varepsilon ps} F^p(s)ds\right]^{\frac{m}{(1-m)r}},$$

where $K(t) = \max_{0\le s\le t} \|k(s,\cdot)\|_r^p$, $L(t) = \max_{0\le s\le t} \|l(s,\cdot)\|_\rho^p$, $p = \frac{r}{m}$ *and* $\rho = \frac{nr}{\alpha r-(m-1)n}$ *for some* $\varepsilon > 0$. *Here* C_1 *and* C_2 *are the best constants in Lemma 2.2 and Lemma 2.3, respectively. The estimation is valid as long as*

$$K(t)^{m-1} L(t) e^{\varepsilon pt} \int_0^t e^{-\varepsilon ps} F^p(s)ds \le \frac{1}{2^{m(p-1)}}(m-1)C_1^{p-1}C_2^p. \tag{3.1}$$

(ii) *If* Ω *is bounded, then*

$$\|\varphi(t,x)\|_{\tilde r} \le U_{p,r,\rho}(t)$$

for any $\tilde r \le r$ *where* p, r *and* ρ *are as in (i). If moreover,* $r < \frac{n}{n\beta-\alpha}$ *(but not necessarily* $r > \frac{(m-1)n}{\alpha}$*) that is* $\frac{m}{\beta} < r < \min\left(\frac{mn}{\alpha}, \frac{n}{n\beta-\alpha}\right)$, *then this estimation holds for any* $\frac{1}{\beta} < p \le \frac{r}{m}$ *provided that* $\rho > \frac{nr}{n-(n\beta-\alpha)r}$.

Proof. (i) Suppose that $\Omega = \mathbb{R}^n$. By the Minkowski inequality and the Young inequality (Lemma 2.1), we have

$$\|\varphi(t,x)\|_r \le \|k(t,x)\|_r + \|l(t,x)\|_\rho \left\| \int_{\mathbb{R}^n}\int_0^t \frac{F(s)\varphi^m(s,y)}{(t-s)^{1-\beta}|x-y|^{n-\alpha}} ds dy \right\|_q \tag{3.2}$$

for r, ρ and q such that $\frac{1}{r} = \frac{1}{\rho} + \frac{1}{q}$.

Let p be such that $\frac{1}{p} = \frac{1}{q} + \frac{\alpha}{n}$ and p' its conjugate *i.e.* $\frac{1}{p} + \frac{1}{p'} = 1$. Using the Hölder inequality, we see that

$$\int_0^t (t-s)^{\beta-1} F(s)\varphi^m(s,y)ds \le \left(\int_0^t (t-s)^{(\beta-1)p'} e^{\varepsilon p' s} ds\right)^{\frac{1}{p'}} \left(\int_0^t e^{-\varepsilon ps} F^p(s)\varphi^{mp}(s,y)ds\right)^{\frac{1}{p}}, \tag{3.3}$$

for any positive constant ε. We have multiplied by $e^{\varepsilon s}\cdot e^{-\varepsilon s}$ before applying Hölder inequality.

Choosing $q = \frac{nr}{(nm-\alpha r)}$, we see that $p = \frac{r}{m}$. By our assumption on r it is easy to see that $q > 1$, $p > 1$ and $1 + (\beta-1)p' > 0$. Therefore, we may apply Lemma 2.2 to get

$$\int_0^t (t-s)^{(\beta-1)p'} e^{\varepsilon p' s} ds < C_1 e^{\varepsilon p' t}.$$

Hence, inequality (3.3) becomes

$$\int_0^t (t-s)^{\beta-1} F(s)\varphi^m(s,y)ds \le C_1^{\frac{1}{p'}} e^{\varepsilon t} \left(\int_0^t e^{-\varepsilon ps} F^p(s)\varphi^{mp}(s,y)ds\right)^{\frac{1}{p}}.$$

It follows that

$$\left\| \int_{\mathbb{R}^n} \int_0^t \frac{F(s)\varphi^m(s,y)}{(t-s)^{1-\beta}|x-y|^{n-\alpha}} ds dy \right\|_q \leq C_1^{\frac{1}{p'}} e^{\varepsilon t} \left\| \int_{\mathbb{R}^n} \left(\int_0^t e^{-\varepsilon ps} F^p(s) \varphi^{mp}(s,y) ds \right)^{\frac{1}{p}} \frac{dy}{|x-y|^{n-\alpha}} \right\|_q . \tag{3.4}$$

As $r < \frac{mn}{\alpha}$, we may apply the results in Lemma 2.3 to obtain

$$\left\| \int_{\mathbb{R}^n} \left(\int_0^t e^{-\varepsilon ps} F^p(s) \varphi^{mp}(s,y) ds \right)^{\frac{1}{p}} \frac{dy}{|x-y|^{n-\alpha}} \right\|_q \leq C_2 \left\| \left(\int_0^t e^{-\varepsilon ps} F^p(s) \varphi^{mp}(s,\cdot) ds \right)^{\frac{1}{p}} \right\|_p , \tag{3.5}$$

with p as above and C_2 is the best constant in the result of Lemma 2.3. From (3.2), (3.4) and (3.5) it appears that

$$\|\varphi(t,x)\|_r \leq \|k(t,x)\|_r + C_3 e^{\varepsilon t} \|l(t,x)\|_\rho \left\| \left(\int_0^t e^{-\varepsilon ps} F^p(s) \varphi^{mp}(s,\cdot) ds \right)^{\frac{1}{p}} \right\|_p$$

or

$$\|\varphi(t,x)\|_r \leq \|k(t,x)\|_r + C_3 e^{\varepsilon t} \|l(t,x)\|_\rho \left(\int_{\mathbb{R}^n} \int_0^t e^{-\varepsilon ps} F^p(s) \varphi^{mp}(s,x) ds dx \right)^{\frac{1}{p}} , \tag{3.6}$$

where $C_3 = C_1^{\frac{1}{p'}} C_2$. Inequality (3.6) can also be written as

$$\|\varphi(t,x)\|_r \leq \|k(t,x)\|_r + C_3 e^{\varepsilon t} \|l(t,x)\|_\rho \left(\int_0^t e^{-\varepsilon ps} F^p(s) \|\varphi(s,x)\|_{mp}^{mp} ds \right)^{\frac{1}{p}} .$$

Observe that by our choice of p we have $r = mp$. It follows that

$$\|\varphi(t,x)\|_r \leq \|k(t,x)\|_r + C_3 e^{\varepsilon t} \|l(t,x)\|_\rho \left(\int_0^t e^{-\varepsilon ps} F^p(s) \|\varphi(s,x)\|_r^r ds \right)^{\frac{1}{p}} . \tag{3.7}$$

Applying the algebraic inequality

$$(a+b)^p \leq 2^{p-1}(a^p + b^p), \ a,b \geq 0, \ p > 1,$$

we deduce from (3.7) that

$$\|\varphi(t,x)\|_r^p \leq 2^{p-1} \|k(t,x)\|_r^p + C_4 e^{\varepsilon pt} \|l(t,x)\|_\rho^p \int_0^t e^{-\varepsilon ps} F^p(s) \|\varphi(s,\cdot)\|_r^r ds, \tag{3.8}$$

where $C_4 = 2^{p-1} C_3^p$. Let us put $\psi(t) = \|\varphi(t,x)\|_r^{r/m}$, then (3.8) takes the form

$$\psi(t) \leq 2^{p-1} \|k(t,\cdot)\|_r^p + C_4 e^{\varepsilon pt} \|l(t,x)\|_\rho^p \int_0^t e^{-\varepsilon ps} F^p(s) \psi^m(s) ds.$$

By the Lemma 2.4 with $\omega(u) = u^m$, $W(v) = \frac{1}{1-m}(v^{1-m} - v_0^{1-m})$ and $W^{-1}(z) = \left[(1-m)z + v_0^{1-m}\right]^{\frac{1}{1-m}}$, we conclude that

$$\psi(t) \leq W^{-1}\left[W\left(2^{p-1}K(t)\right) + C_4 e^{\varepsilon pt} L(t) \int_0^t e^{-\varepsilon ps} F^p(s)ds\right]$$
$$\leq 2^{p-1}K(t)\left[1 - (m-1)C_4\left(2^{p-1}K(t)\right)^{m-1} e^{\varepsilon pt} L(t) \int_0^t e^{-\varepsilon ps} F^p(s)ds\right]^{\frac{1}{1-m}},$$

where $K(t)$ and $L(t)$ are as in the statement of the theorem.

(ii) If Ω is bounded, then the first part in the assertion (ii) of the theorem follows from the argument in (i) by an extension procedure and the application of the embedding $L^r(\Omega) \subset L^{\tilde{r}}(\Omega)$ for $\tilde{r} \leq r$. Now if $r < \frac{n}{n\beta-\alpha}$ we choose q such that $r < q < \frac{n}{n\beta-\alpha}$ and $\rho > \frac{nr}{n-(n\beta-\alpha)r}$. The Young inequality is therefore applicable. Also as $1 < q < \frac{np}{n-\alpha p}$, the Hardy-Littlewood-Sobolev inequality (Lemma 2.3) applies (see also [3, p. 660] when Ω is bounded).

□

In what follows, in order to simplify the statement of our next results, we define $v := r$ or $\tilde{r}$ according to the cases (i) or (ii) in Theorem 3.1, respectively.

Corollary 3.2. *Suppose that the hypotheses of Theorem 3.1 hold. Assume further that $k(t,x)$ and $l(t,x)$ decay exponentially in time, that is $k(t,x) \leq e^{-\tilde{k}t}\bar{k}(x)$ and $l(t,x) \leq e^{-\tilde{l}t}\bar{l}(x)$ for some positive constants $\tilde{k}$ and $\tilde{l}$. Then $\varphi(t,x)$ is also exponentially decaying to zero i.e.,*

$$\|\varphi(t,x)\|_v \leq C_5 e^{-\mu t},\ t > 0 \tag{3.9}$$

for some positive constants C_5 and μ provided that

$$\left\|\bar{k}(x)\right\|_r^{m-1}\left\|\bar{l}(x)\right\|_\rho \int_0^\infty F^p(s)ds \leq \frac{1}{2^{m(p-1)}}(m-1)C_6^{p-1}C_2^p,$$

where C_6 is the best constant in Lemma 2.2 (third estimation) and the other constants are as in (i) and (ii) of Theorem 3.1.

Proof. From the inequality (1.1) we have

$$\varphi(t,x) \leq e^{-\tilde{k}t}\bar{k}(x) + e^{-\tilde{l}t}\bar{l}(x)\int_\Omega\int_0^t \frac{F(s)\varphi^m(s,y)}{(t-s)^{1-\beta}|x-y|^{n-\alpha}}dsdy. \tag{3.10}$$

After multiplying by $e^{-m\mu t}\cdot e^{m\mu t}$, where $\mu = \min\{\tilde{k},\tilde{l}\}$, we use the Hölder inequality to get

$$\int_0^t (t-s)^{\beta-1}F(s)\varphi^m(s,y)ds$$
$$\leq \left(\int_0^t (t-s)^{(\beta-1)p'}e^{-mp'\mu t}ds\right)^{\frac{1}{p'}}\left(\int_0^t F^p(s)e^{mp\mu t}\varphi^{mp}(s,y)ds\right)^{\frac{1}{p}}.$$

As in the proof of Theorem 3.1, $0 < (1-\beta)p' < 1$. If C_6 is the best constant in Lemma 2.2, then we may write

$$\int_0^t (t-s)^{\beta-1}F(s)\varphi^m(s,y)ds \leq C_6\left(\int_0^t F^p(s)\varphi^{mp}(s,y)ds\right)^{\frac{1}{p}}. \tag{3.11}$$

From (3.10) and (3.11) it appears that

$$e^{\mu t}\varphi(t,x) \leq \bar{k}(x) + C_6\bar{l}(x)\int_\Omega \left(\int_0^t F^p(s)e^{mp\mu t}\varphi^{mp}(s,y)ds\right)^{\frac{1}{p}} \frac{dyds}{|x-y|^{n-\alpha}}.$$

Taking the L^r-norm and applying the Minkowski inequality and then the Young inequality (Lemma 2.1), we find

$$\begin{aligned} e^{\mu t}\left\|\varphi(t,x)\right\|_r &\leq \left\|\bar{k}(x)\right\|_r \\ &\quad + C_6\left\|\bar{l}(x)\right\|_\rho \left\|\int_\Omega \left(\int_0^t F^p(s)e^{mp\mu t}\varphi^{mp}(s,y)ds\right)^{\frac{1}{p}} \frac{dyds}{|x-y|^{n-\alpha}}\right\|_q, \end{aligned}$$

with $\frac{1}{r} = \frac{1}{\rho} + \frac{1}{q}$. Applying Lemma 2.3, we arrive at

$$e^{\mu t}\left\|\varphi(t,x)\right\|_r \leq \left\|\bar{k}(x)\right\|_r + C_2C_6\left\|\bar{l}(x)\right\|_\rho \left\|\left(\int_0^t F^p(s)e^{mp\mu t}\varphi^{mp}(s,y)ds\right)^{\frac{1}{p}}\right\|_p$$

or

$$e^{\mu t}\left\|\varphi(t,x)\right\|_r \leq \left\|\bar{k}(x)\right\|_r + C_2C_6\left\|\bar{l}(x)\right\|_\rho \left(\int_0^t F^p(s)e^{mp\mu t}\left\|\varphi(s,x)\right\|_r^{mp} ds\right)^{\frac{1}{p}}. \tag{3.12}$$

Taking both sides of (3.12) to the power p, we obtain

$$e^{\mu pt}\left\|\varphi(t,x)\right\|_r^p \leq 2^{p-1}\left\|\bar{k}(x)\right\|_r^p + C_7\left\|\bar{l}(x)\right\|_\rho^p \int_0^t F^p(s)e^{mp\mu t}\left\|\varphi(s,x)\right\|_r^{mp} ds. \tag{3.13}$$

Next, putting

$$\chi(t) := e^{\mu pt}\left\|\varphi(t,x)\right\|_r^p,$$

the inequality (3.13) may be written as

$$\chi(t) \leq 2^{p-1}\left\|\bar{k}(x)\right\|_r^p + C_7\left\|\bar{l}(x)\right\|_\rho^p \int_0^t F^p(s)\chi^m(s)ds.$$

The rest of the proof is essentially the same as that of Theorem 3.1. □

In the following corollary we consider the somewhat more general inequality

$$\varphi(t,x) \leq k(t,x) + l(t,x)\int_\Omega\int_0^t \frac{s^\delta F(s)\varphi^m(s,y)}{(t-s)^{1-\beta}|x-y|^{n-\alpha}}dyds,\ x\in\Omega,\ t>0 \tag{3.14}$$

for some specified δ.

Corollary 3.3. *Suppose that the hypotheses of Theorem 3.1 hold. Assume further that* $k(t,x) \leq t^{-\hat{k}}\bar{k}(x)$ *and* $1 + \delta p' - mp'\min\{\hat{k}, 1-\beta\} > 0$. *Then any* $\varphi(t,x)$ *satisfying (3.14) is also polynomially decaying to zero*

$$\left\|\varphi(t,x)\right\|_v \leq C_8t^{-\omega},\ C_8,\omega > 0$$

provided that

$$\left\|\bar{k}(x)\right\|_r^{m-1} L(t)\int_0^t e^{\varepsilon ps}F^p(s)ds \leq \frac{1}{2^{m(p-1)}}(m-1)C_9^{p-1}C_2^p$$

where C_9 *is the best constant in Lemma 2.5.*

Proof. Let us consider the inequality

$$\varphi(t,x) \leq t^{-\hat{k}}\bar{k}(x) + l(t,x)\int_\Omega \int_0^t \frac{s^\delta F(s)\varphi^m(s,y)}{(t-s)^{1-\beta}\left|x-y\right|^{n-\alpha}}dyds. \tag{3.15}$$

Multiplying by $s^{-m\min\{\hat{k},1-\beta\}}e^{-\varepsilon s} \cdot s^{m\min\{\hat{k},1-\beta\}}e^{\varepsilon s}$, we obtain

$$\begin{aligned}\int_0^t (t-s)^{\beta-1}s^\delta F(s)\varphi^m(s,y)ds \\ \leq \left(\int_0^t (t-s)^{(\beta-1)p'} s^{\delta p' - mp'\min\{\hat{k},1-\beta\}} e^{-\varepsilon p' s} ds\right)^{\frac{1}{p'}} \\ \times \left(\int_0^t s^{mp\min\{\hat{k},1-\beta\}} e^{p\varepsilon s} F^p(s)\varphi^{mp}(s,y)ds\right)^{\frac{1}{p}}.\end{aligned} \tag{3.16}$$

As $1 + \delta p' - mp'\min\{\hat{k},1-\beta\} > 0$, we may apply Lemma 2.5 to the first term in the right hand side of (3.16) to get

$$\begin{aligned}\int_0^t (t-s)^{\beta-1}s^\delta F(s)\varphi^m(s,y)ds \\ \leq Mt^{\beta-1}\left(\int_0^t s^{mp\min\{\hat{k},1-\beta\}} e^{p\varepsilon s} F^p(s)\varphi^{mp}(s,y)ds\right)^{\frac{1}{p}}.\end{aligned} \tag{3.17}$$

Using (3.17) we infer from inequality (3.15) that

$$\begin{aligned}t^{\min\{\hat{k},1-\beta\}}\varphi(t,x) \leq \bar{k}(x) \\ + Ml(t,x)\int_\Omega \left(\int_0^t s^{mp\min\{\hat{k},1-\beta\}} e^{p\varepsilon s} F^p(s)\varphi^{mp}(s,y)ds\right)^{\frac{1}{p}} \frac{dy}{\left|x-y\right|^{n-\alpha}}.\end{aligned}$$

Next, after using the Hardy-Littlewood-Sobolev inequality and defining

$$\phi(t,x) := t^{p\min\{\hat{k},1-\beta\}}\left\|\varphi(t,x)\right\|_r^p,$$

we proceed as in Theorem 3.1 to find a (uniform) bound for $\phi(t,x)$. □

Remark 3.4. The investigation of (1.1) with a *weakly singular* kernel in time, that is

$$\varphi(t,x) \leq k(t,x) + l(t,x)\int_\Omega \int_0^t \frac{e^{-\gamma(t-s)}F(s)\varphi^m(s,y)}{(t-s)^{1-\beta}\left|x-y\right|^{n-\alpha}}dyds,\ \gamma > 0$$

is simpler since this kernel is summable (in time).

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