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REFINEMENTS OF CARLEMAN'S INEQUALITY

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Abstract

In this paper, we obtain a class of refined Carleman's Inequalities with the arithmetic-geometric mean inequality by decreasing their weight coefficient.

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Contents

1. Introduction

Let $\{a_n\}_{n=1}^{+\infty}$ be a non-negative sequence such that $0 \leq \sum_{n=1}^{+\infty} a_n < +\infty$, then, we have

$$\sum_{n=1}^{+\infty} (a_1 a_2 \dots a_n)^{1/n} \leq e \sum_{n=1}^{+\infty} a_n. \tag{1}$$

The equality in (1) holds if and only if $a_n = 0, n = 1, 2, \dots$. the coefficient e is optimal.

Inequality (1) is called Carleman's inequality. For details please refer to [1, 2]. The Carleman's inequality has found many applications in mathematics, and the study of the Carleman's inequality has a rich literature, for details, please refer to [3, 4]. Though the coefficient e is optimal, we can refine its weight coefficient. In this article we give a class of improved Carleman's inequalities by decreasing the weight coefficient with the arithmetic-geometric mean inequality.

2. Two Special Cases

In this section, we give two special cases of refined Carleman's inequality. First we prove two lemmas.

Lemma 2.1. *For $m = 1, 2, \dots,$ the inequality*

$$\left(1+\frac{1}{m}\right)^m \le e\left(1-\frac{1-2/e}{m}\right) \tag{2}$$

holds, where the constant $1 - \frac{2}{e} \approx 0.2642411$ is best possible.

Proof. Inequality

$$\left(1+\frac{1}{m}\right)^m \le e\left(1-\frac{\beta}{m}\right) \tag{3}$$

is equivalent to $\beta \le m - \frac{m}{e}\left(1+\frac{1}{m}\right)^m$.

Let $f(x) = \frac{1}{x} - \frac{1}{ex}(1+x)^{\frac{1}{x}}$, $x \in (0, 1]$.

It is obvious that the function f is decreasing on the interval $(0, 1]$. Consequently, $\beta = f(1) = 1 - \frac{2}{e}$ is the optimal value satisfying inequality (3), so (2) holds. The proof of Lemma 2.1 follows. □

Lemma 2.2. *For $m = 1, 2, \dots,$ the inequality*

$$\left(1+\frac{1}{m}\right)^m \le \frac{e}{\left(1+\frac{1}{m}\right)^{\frac{1}{\ln 2}-1}} \tag{4}$$

holds, where the constant $\frac{1}{\ln 2} - 1 \approx 0.442695$ is the best possible.

Proof. Inequality

$$(5)\qquad \left(1+\frac{1}{m}\right)^m \le \frac{e}{\left(1+\frac{1}{m}\right)^\alpha}$$

is equivalent to

$$\alpha \le \frac{1}{\ln\left(1+\frac{1}{m}\right)} - m.$$

Let

$$f(x) = \frac{1}{\ln(1+x)} - \frac{1}{x} \qquad x \in (0,1].$$

Since the function f is decreasing on the interval $(0,1]$, $\alpha = f(1) = \frac{1}{\ln 2} - 1$ is the optimal value satisfying inequality (5), and thus (4) holds. The proof of Lemma 2.2 follows. □

Theorem 2.3. *Let $\{a_n\}_{n=1}^{+\infty}$ be a non-negative sequence such that $0 \le \sum_{n=1}^{+\infty} a_n < +\infty$. Then the following inequalities hold:*

$$(6)\qquad \sum_{n=1}^{+\infty} (a_1 a_2 \dots a_n)^{1/n} \le e \sum_{m=1}^{+\infty} \left(1 - \frac{1-2/e}{m}\right) a_m,$$

and

$$(7)\qquad \sum_{n=1}^{+\infty} (a_1 a_2 \dots a_n)^{1/n} \le e \sum_{m=1}^{+\infty} \frac{a_m}{\left(1+\frac{1}{m}\right)^{\frac{1}{\ln 2}-1}}.$$

Proof. Let $c_i > 0$ $(i = 1, 2, \ldots)$. According to the arithmetic-geometric mean inequality, we have

$$(c_1 a_1 c_2 a_2 \cdots c_n a_n)^{1/n} \le \frac{1}{n} \sum_{m=1}^{n} c_m a_m.$$

Consequently,

$$\begin{aligned}
\sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} &= \sum_{n=1}^{+\infty} \left(\frac{c_1 a_1 c_2 a_2 \cdots c_n a_n}{c_1 c_2 \cdots c_n} \right)^{1/n} \\
&= \sum_{n=1}^{+\infty} (c_1 c_2 \cdots c_n)^{-1/n} (c_1 a_1 c_2 a_2 \cdots c_n a_n)^{1/n} \\
&\le \sum_{n=1}^{+\infty} (c_1 c_2 \cdots c_n)^{-1/n} \frac{1}{n} \sum_{m=1}^{n} c_m a_m \\
&= \sum_{m=1}^{+\infty} c_m a_m \sum_{n=m}^{+\infty} \frac{1}{n} (c_1 c_2 \cdots c_n)^{-1/n}.
\end{aligned}$$

Let $c_m = \frac{(m+1)^m}{m^{m-1}}$ $(m = 1, 2, \ldots)$. Then $c_1 c_2 \cdots c_n = (n+1)^n$, and

$$\sum_{n=m}^{+\infty} \frac{1}{n} (c_1 c_2 \cdots c_n)^{-1/n} = \sum_{n=m}^{+\infty} \frac{1}{n(n+1)} = \frac{1}{m}.$$

Therefore

$$\sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \le \sum_{m=1}^{+\infty} \frac{c_m}{m} a_m = \sum_{m=1}^{+\infty} \left(1 + \frac{1}{m}\right)^m a_m. \tag{8}$$

According to Lemmas 2.1 and 2.2, and substituting for $\left(1+\frac{1}{m}\right)^m$ of inequality (8), so (6) and (7) follow from Lemmas 2.1 and 2.2.

The proof is complete. $\square$

3. A Class of Refined Carleman's Inequalities

In this section we give a class of refined Carleman's inequalities. First we have the following inequality

Lemma 3.1. *For* $m = 1, 2, \dots,$ *the inequality*

$$\left(1+\frac{1}{m}\right)^m \le \frac{e\left(1-\frac{\beta}{m}\right)}{\left(1+\frac{1}{m}\right)^\alpha}, \tag{9}$$

holds, where $0 \le \alpha \le \frac{1}{\ln 2} - 1$, $0 \le \beta \le 1 - \frac{2}{e}$, *and* $e\beta + 2^{1+\alpha} = e$.

Proof. Inequality (9) is equivalent to

$$\beta \le m - \frac{m}{e}\left(1+\frac{1}{m}\right)^{m+\alpha}. \tag{10}$$

If

$$f(x) = \frac{1}{x} - \frac{1}{ex}(1+x)^{\frac{1}{x}+\alpha}, \; x \in (0,1], \; 0 \le \alpha \le \frac{1}{\ln 2} - 1,$$

then f is decreasing on interval $(0,1]$. Consequently, $\beta = f(1) = 1 - \frac{1}{e}2^{1+\alpha}$ is the optimal value satisfying inequality (10). Moreover, $0 \le \beta \le 1 - \frac{2}{e}$, and $e\beta + 2^{1+\alpha} = e$. So (9) holds, The proof is complete. □

Remark 3.1. *If* $\alpha = 0$, *then* $\beta = 1 - \frac{2}{e}$, *and we obtain Lemma 2.1; if* $\beta = 0$, *then* $\alpha = \frac{1}{\ln 2} - 1$, *and we obtain Lemma 2.2.*

Similar to Theorem 2.3, according to Lemma 3.1, we have

Theorem 3.2. *Let* $a_n \geq 0 \ \ (n = 1, 2, \dots),\ 0 \leq \sum_{n=1}^{+\infty} a_n < +\infty$, *then*

$$\sum_{n=1}^{+\infty} (a_1 a_2 \cdots a_n)^{1/n} \leq e \sum_{m=1}^{+\infty} \frac{\left(1 - \frac{\beta}{m}\right)}{\left(1 + \frac{1}{m}\right)^{\alpha}} a_m,$$

where α, β *satisfy* $0 \leq \alpha \leq \frac{1}{\ln 2} - 1$, $0 \leq \beta \leq 1 - \frac{2}{e}$, *and* $e\beta + 2^{1+\alpha} = e$.

Remark 3.2. *Theorem 2.3 gives two special cases of Theorem 3.2. If* $\alpha = 0$, $\beta = 1 - \frac{2}{e}$, *and* $\alpha = \frac{1}{\ln 2} - 1$, $\beta = 0$, *we can obtain (6) and (7) in Theorem 2.3 respectively.*

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