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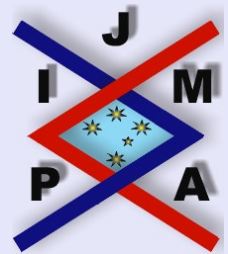
ON AN INEQUALITY INVOLVING POWER AND CONTRACTION OF MATRICES WITH AND WITHOUT TRACE

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Abstract

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Abstract

Let A and B be positive semidefinite matrices. Assuming that the eigenvalues of B are less than one, we prove the following trace inequalities

$$\mathrm{Tr}\left\{(BA^{\alpha}B)^{1/\alpha}\right\} \leq \mathrm{Tr}\left\{(BA^{\beta}B)^{1/\beta}\right\}$$

and

$$\mathrm{Tr}\left\{(BA^{\alpha}B)^{1/\alpha}\right\} \leq \mathrm{Tr}\left\{\left(B^{\alpha/\beta}A^{\alpha}B^{\alpha/\beta}\right)^{1/\alpha}\right\},$$

for all $0 < \alpha \leq \beta$. Moreover we prove that

$$\mathrm{Tr}\left\{\left(B^{\alpha/\beta}A^{\alpha}B^{\alpha/\beta}\right)^{1/\alpha}\right\} \leq \mathrm{Tr}\left\{\left(BA^{\beta}B\right)^{1/\beta}\right\},$$

for all $0 < \alpha \leq \beta$ and $0 < \alpha \leq 1$. Furthermore we prove that

$$(BA^{\alpha}B)^{1/\alpha} \leq (BA^{\beta}B)^{1/\beta}$$

in the cases (a) $1 \leq \alpha \leq \beta$ or (b) $\frac{1}{2} \leq \alpha \leq \beta$ and $\beta \geq 1$. Further we present counterexamples involving 2×2 matrices showing that the last inequality is, in general, violated in case that neither (a) nor (b) is fulfilled.
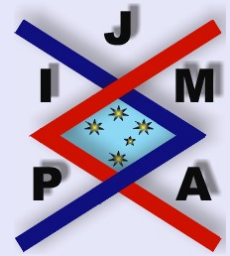
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On an Inequality Involving Power and Contraction of Matrices with and without Trace

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1. Introduction

Let M_n be the space of $n \times n$ complex matrices. We say that $A \in M_n$ is positive if A is Hermitian, that is $A^* = A$, and its eigenvalues $\lambda_i(A)$ ($i = 1, \ldots, n$) are nonnegative. A positive matrix A is denoted by $0 \leq A$ and we say that $A \leq B$ if $0 \leq B - A$. The identity matrix is denoted by I.

Our main result is the proof of the following inequalities involving the matrix $(BA^\alpha B)^{1/\alpha}$ with $\alpha > 0$:

Theorem 1.1. *Let be $A, B \in M_n$ with $0 \leq A$ and $0 \leq B \leq I$. Defining*

$$H(\alpha) := (BA^\alpha B)^{1/\alpha} \qquad and \qquad h(\alpha) := \mathrm{Tr}\left\{H(\alpha)\right\},$$

we prove the following operator and trace inequalities.

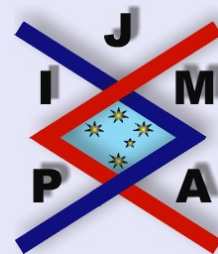*a) For all $1 \leq \alpha \leq \beta$ we have*

$$(1.1) \qquad\qquad H(\alpha) \leq H(\beta).$$

b') For all $1/2 \leq \alpha \leq 1$ and $\beta = 1$ we have

$$(1.2) \qquad\qquad H(\alpha) \leq H(\beta = 1).$$

b") For all $0 < \alpha < 1/2$ and $\beta = 1$ we can find matrices $A \geq 0$ and $0 \leq B \leq I$ such that

$$(1.3) \qquad\qquad H(\alpha) \not\leq H(\beta = 1).$$

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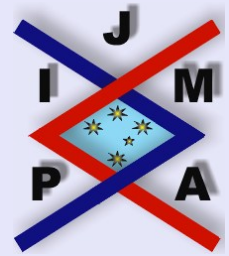
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b) Combining a) with b')

$$(1.4) \qquad H(\alpha) \leq H(\beta)$$

holds for all $1/2 \leq \alpha \leq \beta$ and $\beta \geq 1$.

c) For all $0 < \alpha \leq \beta$ we prove that

$$(1.5) \quad h(\alpha) \leq h(\beta), \quad that\ is, \quad \mathrm{Tr}\left\{(BA^{\alpha}B)^{1/\alpha}\right\} \leq \mathrm{Tr}\left\{(BA^{\beta}B)^{1/\beta}\right\} .$$

d) For all $0 < \alpha \leq \beta$ we have

$$(1.6) \qquad \mathrm{Tr}\left\{(BA^{\alpha}B)^{1/\alpha}\right\} \leq \mathrm{Tr}\left\{\left(B^{\alpha/\beta}A^{\alpha}B^{\alpha/\beta}\right)^{1/\alpha}\right\} .$$

e) For all $0 < \alpha \leq \beta$ and $0 < \alpha \leq 1$ we prove that

$$(1.7) \qquad \mathrm{Tr}\left\{\left(B^{\alpha/\beta}A^{\alpha}B^{\alpha/\beta}\right)^{1/\alpha}\right\} \leq \mathrm{Tr}\left\{(BA^{\beta}B)^{1/\beta}\right\} .$$

Remark 1. *The item a) is the main inequality of Theorem 1.1. As we will see, it is a direct consequence of the following result of F. Hansen [8]:*
"If f is an operator monotone function defined on the interval $[0, \infty)$, then $Kf(X)K^* \leq f(KXK^*)$ holds for every $X \geq 0$ and contraction K."
See also Lemma 2.1.

A proof of c) can be obtained combining a) with d) and e). More precisely, c) follows from a) in the case $\alpha \geq 1$ and from d) and e) in the case $0 < \alpha \leq 1$. The author would like to thank F. Hansen for indicating a simpler proof of c) which does not make use of d) and e). This simpler proof is presented below.

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We would like to state our discussion, motivation and background of Theorem 1.1 as follows.

A motivation to prove the inequality (1.5) is the application of the well-known trace inequality $|\text{Tr}\{X\}| \leq \text{Tr}\{|X|\}$, $X \in M_n$ for the particular case where $X = AB$ with $0 \leq A$ and $0 \leq B \leq I$. Here we use the definition $|X| := (X^*X)^{1/2}$. Applying this trace inequality we obtain $h(1) \leq h(2)$, because:

$$
(1.8) \qquad \begin{aligned}
h(1) &:= \text{Tr}\{BAB\} = \text{Tr}\{AB^2\} = \text{Tr}\{A^{1/2}B^2A^{1/2}\} \\
&\leq \text{Tr}\{A^{1/2}BA^{1/2}\} = \text{Tr}\{AB\} \\
&\leq \text{Tr}\{|AB|\} = \text{Tr}\{(BA^2B)^{1/2}\} =: h(2) \,,
\end{aligned}
$$

where in the first inequality of (1.8) we used that $0 \leq B^2 \leq B$ since $0 \leq B \leq I$. We also used that $|\text{Tr}\{AB\}| = \text{Tr}\{AB\}$ because $\text{Tr}\{AB\} = \text{Tr}\{A^{1/2}BA^{1/2}\} \geq 0$ since both A and B are positive matrices. Similar to (1.8), we can also show that $h\left(\frac{1}{2^{k+1}}\right) \leq h\left(\frac{1}{2^k}\right)$ for $k = 0, 1, 2, \ldots$.

Considering the special case $\alpha = 1/2$ and $\beta = 1$ of (1.2) we can easily prove that $H(1/2) \leq H(1)$, namely:

$$
\begin{aligned}
BAB - (BA^{1/2}B)^2 &= BA^{1/2}A^{1/2}B - BA^{1/2}B^2A^{1/2}B \\
&= BA^{1/2}(I - B^2)A^{1/2}B \geq 0 \,,
\end{aligned}
$$

because $0 \leq B \leq I$ and so $I - B^2 \geq 0$.

Now we put the inequalities presented in Theorem 1.1 in the context of known results. More precisely, we will derive two particular cases of (1.5) and (1.1) from [1], [3], [11], [10] and [6]. However, we need to impose some
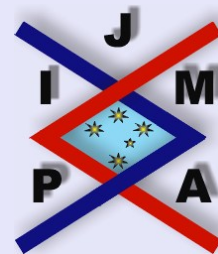
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restrictions on A and B in the hypothesis of theorem 1.1. These restrictions are 1) B is a projection and 2) $0 \leq B \leq A \leq I$.

1) Considering the restriction that $B = P$ is a projection we can show the following two particular cases of (1.5):

- The first particular case of (1.5) is

$$(1.9) \qquad h(1/k) \leq h(1) \qquad \text{for all} \quad k = 1, 2, 3, \ldots .$$

We can derive this trace inequality using the following result by Ando, Hiai and Okubo [1]:

"*For semidefinite matrices A, B the inequality*

$$\mathrm{Tr}\{A^{p_1} B^{q_1} \cdots A^{p_N} B^{q_N}\} \leq \mathrm{Tr}\{AB\}$$

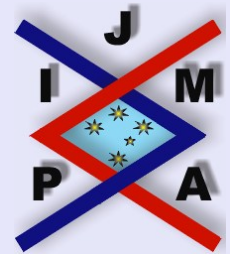*holds with p_i, $q_i \geq 0$ and $\sum_{i=1}^{N} p_i = \sum_{i=1}^{N} q_i = 1$.*"

Applying this result to $B = P$ and $p_i = q_i = 1/k$ we have

$$h(1) = \mathrm{Tr}\{PAP\} = \mathrm{Tr}\{AP\} \geq \mathrm{Tr}\left\{A^{\frac{1}{k}} P \cdots A^{\frac{1}{k}} P\right\}$$
$$= \mathrm{Tr}\left\{\left(PA^{\frac{1}{k}}P\right) \cdots \left(PA^{\frac{1}{k}}P\right)\right\}$$
$$= \mathrm{Tr}\{(PA^{1/k}P)^k\} = h(1/k) ,$$

which proves (1.9).

- The second particular case of (1.5) is

$$(1.10) \qquad h(\alpha) \leq h(1) \qquad \text{for all} \quad 0 < \alpha \leq 1 .$$

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This trace inequality can be derived from the Berezin-Lieb inequality ([3], [11]). To understand this, recall that the Berezin-Lieb inequality states that $\mathrm{Tr}\{f(PXP)\} \leq \mathrm{Tr}\{Pf(X)P\}$ *holds if P is a projection and f is a convex function on an interval containing the spectrum of X.* Now taking $X = A^\alpha$ and $f(\lambda) = \lambda^{1/\alpha}$ $(0 < \alpha \leq 1)$ we obtain (1.10), because

$$h(\alpha) = \mathrm{Tr}\left\{\left(PA^\alpha P\right)^{1/\alpha}\right\} \leq \mathrm{Tr}\{P(A^\alpha)^{1/\alpha}P\} = h(1).$$

2) Considering the restriction $0 \leq B \leq A \leq I$ we will show the following two particular cases of (1.1):

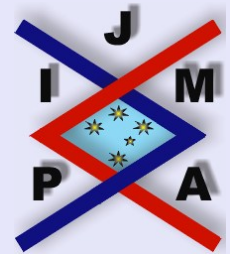- The first particular case of (1.1) is

  (1.11) $H(1) \leq H(2)$,   that is $BAB \leq (BA^2B)^{1/2}$.

Remark 2. *Although we have $BA^2B \leq (BA^2B)^{1/2}$ and $BA^2B \leq BAB$ (since $0 \leq A, B \leq I$) we cannot conclude from these two operator inequalities that $BAB \leq (BA^2B)^{1/2}$.*

- We can derive the operator inequality (1.11) based on the following result in Kamei [10] which is a variation of [5]:

  (1.12) $0 \leq B \leq A$ *assures*
  $$\left(B^{s/2}A^pB^{s/2}\right)^{\frac{1+s}{p+s}} \geq B^{s/2}AB^{s/2} \text{ for } p \geq 1 \text{ and } s \geq 0.$$

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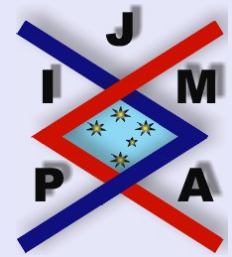
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To understand this, take $p = s = 2$ in (1.12), namely, we obtain

$$\left(BA^2B\right)^{\frac{3}{4}} \geq BAB.$$

On the other hand,

$$\left(BA^2B\right)^{\frac{1}{2}} \geq \left(BA^2B\right)^{\frac{3}{4}}$$

because $BA^2B \leq BIB \leq I$ since $0 \leq A, B \leq I$.

- The second particular case of (1.1) is a generalization of the first one. More precisely

(1.13) $H(1) \leq H(p)$, *that is,* $BAB \leq (BA^pB)^{1/p}$ *for all* $p \geq 1$.

We can obtain the above operator inequality using the following result (see [6]) which is also a variant of [5] and a more precise estimation than (1.12):

(1.14)   *The function* $F_r(p) = (B^r A^p B^r)^{\frac{1+2r}{p+2r}}$ *for* $p \geq 1$, $r \geq 0$

   is operator increasing as a function of p *whenever* $0 \leq B \leq A$.

Now the operator inequality $H(1) \leq H(p)$ follows from (1.14) setting $r = 1$, that is,

$$(BA^pB)^{\frac{3}{p+2}} = F_1(p) \geq F_1(1) = BAB.$$

On the other hand,

$$(BA^pB)^{\frac{1}{p}} \geq (BA^pB)^{\frac{3}{p+2}}$$

because $BA^pB \leq BIB \leq I$ $(0 \leq A, B \leq I)$ and $3/(p+2) \geq 1/p$ for $p \geq 1$.
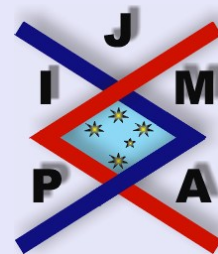
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We shall state the following couterexamples associated with Theorem 1.1.

Counterexamples. In order to show that we cannot generally drop "Tr" from the inequality (1.5) apart from the cases a) $1 \leq \alpha \leq \beta$ or b) $1/2 \leq \alpha \leq \beta$ and $\beta \geq 1$, consider the following concrete example of 2×2 matrices: Let be $B := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1/2 \end{smallmatrix}\right)$ and $A := 64P + Q$ with $P := 1/2 \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $Q := I - P$ orthogonal projections. Since we are working with 2×2 matrices, we observe that $\mathrm{Det}\left[H(\beta) - H(\alpha)\right] < 0$ implies $H(\beta) \not\geq H(\alpha)$. Based on this observation we calculate the following determinants:

1) $\mathrm{Det}\left[H(1) - H(1/3)\right] = -81/16 \approx -5.06 < 0$

2) $\mathrm{Det}\left[H(2/3) - H(1/2)\right] = 12 - \frac{63}{26}\sqrt{26} \approx -0.36 < 0$

3) $\mathrm{Det}\left[H(2/3) - H(1/3)\right] = 9 - \frac{2115}{832}\sqrt{26} \approx -3.96 < 0$

4) $\mathrm{Det}\left[H(1/3) - H(1/6)\right] = -9446625/2097152 \approx -4.5045 < 0$

5) $\mathrm{Det}\left[H(1/2) - H(1/3)\right] = -225/128 \approx -1.76 < 0$

6) $\mathrm{Det}\left[H(4/3) - H(1/3)\right] \approx -3.5 < 0$

and conclude that the respective affirmatives:

1) $H(\beta) \geq H(\alpha)$ holds for all $0 < \alpha < 1/2$ and $\beta = 1$

2) $H(\beta) \geq H(\alpha)$ holds for all $1/2 \leq \alpha < \beta < 1$

3) $H(\beta) \geq H(\alpha)$ holds for all $0 < \alpha < 1/2 < \beta < 1$

4) $H(\beta) \geq H(\alpha)$ holds for all $0 < \alpha < \beta < 1/2$

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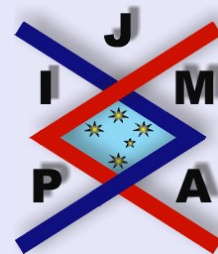
5) $H(\beta) \geq H(\alpha)$ holds for all $0 < \alpha < \beta = 1/2$

6) $H(\beta) \geq H(\alpha)$ holds for all $0 < \alpha < 1/2$ and $\beta > 1$

are **false**.

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2. Proof of Theorem 1.1

Definition 2.1. *We say that a real function f is operator concave on the interval I when for all real numbers $0 \leq \lambda \leq 1$,*

$$f((1 - \lambda)X + \lambda Y) \geq (1 - \lambda)f(X) + \lambda f(Y)$$

for every pair $X, Y \in M_n$ whose spectra lie in the interval I. Likewise we say that f is operator monotone when $f(X) \leq f(Y)$ for every pair $X, Y \in M_n$ with $X \leq Y$.

Lemma 2.1. *[Operator concavity, monotony and contractions, part of Theorems 2.1 and 2.5 of F. Hansen and G. K. Pedersen [9]]. Let $f : [0, \infty) \to [0, \infty)$ be a continuous function then the following conditions are equivalent:*

 *(i) f is operator concave on $[0, \infty)$.*

 (ii) f is operator monotone.

(iii) $Kf(X)K^* \leq f(KXK^*)$ for every contraction K (i.e. $\|K\| \leq 1$, where $\| \cdot \|$ is the operator norm) and for every matrix $X \geq 0$.

(iv) $Pf(X)P \leq f(PXP)$ for all projections P and matrices $X \geq 0$.

 A function f is called operator convex if the function $-f$ is operator concave.

 As an example of a contraction we have a matrix $B \in M_n$ with $0 \leq B \leq I$.

Lemma 2.2. *Let be $R, S \in M_n$ with $0 \leq R$ and $0 \leq S \leq I$ then the following estimate holds for all $\alpha > 0$*

$$(2.1) \qquad \mathrm{Tr}\left\{(SRS)^{1/\alpha}\right\} \leq \mathrm{Tr}\left\{R^{1/\alpha}\right\} .$$
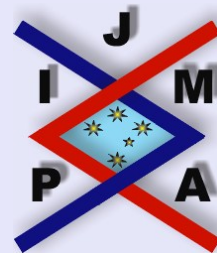
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In order to give a proof of Lemma 2.2 and c) of Theorem 1.1, we state the following Lemma 2.3 which is derived from *the minimax principle* for the sake of convenience for readers:

Lemma 2.3. *[[7], Lemma 1.1]. If A and B are $n \times n$ positive semidefinite matrices such that $A \geq B \geq 0$, then their eigenvalues of A and B are ordered as*

$$\lambda_j(A) \geq \lambda_j(B) \quad for \ j = 1, 2, \ldots, n.$$

Proof of Lemma 2.2. First we observe that matrices XY and YX have the same eigenvalues with the same multiplicities for $X, Y \in M_n$. Let be $0 \leq R$ and $0 \leq S \leq I$ and using this observation with $X = SR^{1/2}$ and $Y = R^{1/2}S$ we have
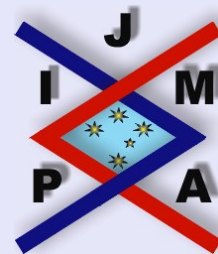
$$(2.2) \qquad \lambda_i(SRS) = \lambda_i(SR^{1/2}R^{1/2}S) = \lambda_i(R^{1/2}S^2R^{1/2})$$

for $i = 1, 2, \ldots, n$. Since $S^2 \leq S \leq I$ we have that $R^{1/2}S^2R^{1/2} \leq R^{1/2}R^{1/2} = R$. From the last operator inequality it follows from Lemma 2.3 that the eigenvalues of $R^{1/2}S^2R^{1/2}$ and R are ordered as

$$(2.3) \qquad 0 \leq \lambda_i(R^{1/2}S^2R^{1/2}) \leq \lambda_i(R)$$

for $i = 1, 2, \ldots, n$. From (2.2) and (2.3) we have for all $\alpha > 0$ that

$$\begin{aligned}
\mathrm{Tr}\{(SRS)^{1/\alpha}\} &= \sum_{i=1}^{n} \lambda_i(SRS)^{1/\alpha} \\
&= \sum_{i=1}^{n} \lambda_i(R^{1/2}S^2R^{1/2})^{1/\alpha} \leq \sum_{i=1}^{n} \lambda_i(R)^{1/\alpha} = \mathrm{Tr}\left\{R^{1/\alpha}\right\},
\end{aligned}$$

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which proves the Lemma 2.2. □

Proof of a) of (1.1) in Theorem 1.1. Defining $r := \alpha/\beta$ we have $0 < r \leq 1$ since $0 < \alpha \leq \beta$. Since the function $f(t) = t^r$, $0 \leq r \leq 1$ is operator concave (and monotone) on $[0, \infty)$ and the matrix B is a contraction we conclude by Lemma 2.1, setting $X = A^\beta$ and $K = B$ that

$$(2.4) \qquad BA^\alpha B = B(A^\beta)^r B \leq (BA^\beta B)^r .$$

holds for all $0 < \alpha \leq \beta$.

On the other hand using the fact that the function $f(t) = t^s$, $0 \leq s \leq 1$ is operator monotone and taking $s = 1/\alpha \leq 1$ (since $1 \leq \alpha$), it follows from (2.4) that

$$(2.5) \qquad (BA^\alpha B)^{1/\alpha} \leq (BA^\beta B)^{r/\alpha} = (BA^\beta B)^{1/\beta},$$

which proves a). □

Proof of b') and b'') in Theorem 1.1. In the case $1/2 \leq \alpha \leq 1$ and $\beta = 1$ we have $1 \leq r := 1/\alpha \leq 2$. Based on the fact that the function $f(t) = t^r$ is operator convex on $[0, \infty)$ if and only if $1 \leq r \leq 2$ (see [4] Theorem V.2.9) it follows by Lemma 2.1 setting $X := A^{1/r}$ with $r := 1/\alpha$ and $K := B$ that

$$\begin{aligned} H(\alpha) := (BA^\alpha B)^{1/\alpha} = \left(BA^{1/r}B\right)^r = (KXK^*)^r \\ \leq KX^r K^* = BAB := H(\beta = 1) , \end{aligned}$$

which proves b').
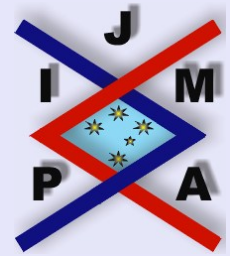
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In the case $0 < \alpha < 1/2$ and $\beta = 1$ we have $r = 1/\alpha > 2$ which means that the function $f(t) = t^r$ is not operator convex. It follows from Lemma 2.1 that we can find a matrix $X \geq 0$ and a projection P such that $(PXP)^{1/\alpha} \nleq PX^{1/\alpha}P$. Taking $A = X^{1/\alpha}$ and $B = P$ we have

$$H(\alpha) := (BA^\alpha B)^{1/\alpha} = (PXP)^{1/\alpha}$$
$$\nleq PX^{1/\alpha}P = BAB = H(\beta = 1) \,,$$

which proves b''). $\qquad\square$

Proof of c) in Theorem 1.1. First we recall that the operator inequality (2.4),

$$(2.6) \qquad BA^\alpha B \leq \left(BA^\beta B\right)^{\alpha/\beta} \,,$$

holds for all $0 < \alpha \leq \beta$. From (2.6) it follows from Lemma 2.3 that the eigenvalues of $BA^\alpha B$ and $\left(BA^\beta B\right)^{\alpha/\beta}$ are ordered as
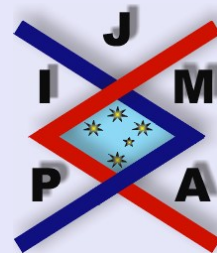
$$(2.7) \qquad \lambda_i(BA^\alpha B) \leq \lambda_i((BA^\beta B)^{\alpha/\beta})$$

for $i = 1, \ldots, n$.

From (2.6), (2.7) and since the function $f(t) = t^{1/\alpha}$ is increasing we obtain

$$\mathrm{Tr}\left\{(BA^\alpha B)^{1/\alpha}\right\} = \sum_{i=1}^n \lambda_i(BA^\alpha B)^{1/\alpha}$$
$$\leq \sum_{i=1}^n \lambda_i\left(\left(BA^\beta B\right)^{\alpha/\beta}\right)^{1/\alpha}$$
$$= \mathrm{Tr}\left\{(BA^\beta B)^{1/\beta}\right\} \,,$$

which proves c). $\qquad\square$

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Proof of d) in Theorem 1.1. Setting $r = \alpha/\beta$, $S := B^{1-r}$ and $R := B^r A^\alpha B^r$ we have $0 \le S \le I$ (because $0 < r \le 1$) and $0 \le R$. Applying the inequality (2.1) for this choice of R and S we obtain

$$\mathrm{Tr}\left\{(BA^\alpha B)^{1/\alpha}\right\} = \mathrm{Tr}\left\{\left(B^{1-r}\left(B^r A^\alpha B^r\right)B^{1-r}\right)^{1/\alpha}\right\}$$

$$(2.8) \qquad\qquad \le \mathrm{Tr}\left\{\left(B^r A^\alpha B^r\right)^{1/\alpha}\right\},$$

which proves d). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proof of e) in Theorem 1.1. First we note that we can express $\mathrm{Tr}\left\{\left(B^r A^\alpha B^r\right)^{1/\alpha}\right\}$ as a norm, namely:

$$(2.9) \qquad \mathrm{Tr}\left\{\left(B^r A^\alpha B^r\right)^{1/\alpha}\right\} = \left\|B^r A^\alpha B^r\right\|_{1/\alpha}^{1/\alpha} = \left\|B^r (A^\beta)^r B^r\right\|_{1/\alpha}^{1/\alpha},$$

where $\|\cdot\|_{1/\alpha}$ is the $1/\alpha$-trace norm which is an unitarily invariant norm (note that $1/\alpha \ge 1$ since in our hypothesis $0 < \alpha \le 1$).

On the other hand a result from [4] (Theorem IX.2.10) states that for every unitarily invariant norm $\|\|\cdot\|\|$ we have

$$(2.10) \qquad\qquad \|\|B^r A^r B^r\|\| \le \|\|(BAB)^r\|\|$$
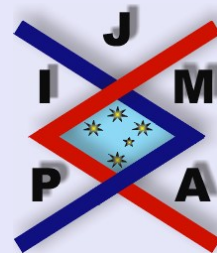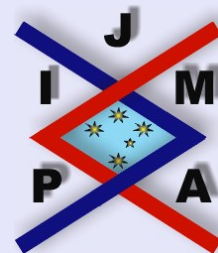
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for all $0 \leq r \leq 1$ if A and B are positive matrices. It follows from (2.10) that

$$
\begin{aligned}
\left\| B^r (A^\beta)^r B^r \right\|_{1/\alpha}^{1/\alpha} &\leq \left\| \left(B A^\beta B \right)^r \right\|_{1/\alpha}^{1/\alpha} \\
&= \mathrm{Tr} \left\{ \left| \left(B A^\beta B \right)^r \right|^{1/\alpha} \right\} \\
&= \mathrm{Tr} \left\{ \left(B A^\beta B \right)^{r/\alpha} \right\} \\
&= \mathrm{Tr}\{ (B A^\beta B)^{1/\beta} \}
\end{aligned}
$$

(2.11)

where we could drop the $|\cdot|$ within the trace in the above estimate because $B A^\beta B$ is a positive matrix. Now the proof of e) in Theorem 1.1 follows directly from (2.9) and (2.11). $\qquad \square$

The Hubbard model describes electrons in solids and poses several mathematical challenges (e.g. [12]). In this model, the extreme points of the set $\{B \in M_n | 0 \leq B \leq I\}$ correspond to many-electron functions (Slater Determinants). We searched for extreme points that minimize the functional energy of the model. Applying the trace inequality (1.5) for the case $\beta = 1$ V. Bach and the author could reproduce a special result from V. Bach and J. Poelchau [2] for the Hubbard model. Details on the Hubbard model and this application of the trace inequality on it go beyond the scope of this study and will be published separately.

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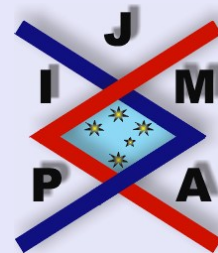Marcos V. Travaglia

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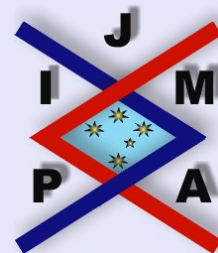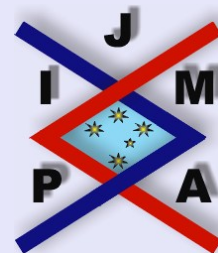Marcos V. Travaglia

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References

[1] T. ANDO, F. HIAI AND K. OKUBO, Trace inequalities for multiple products of two matrices, *Math. Ineq. Appl.,* **3** (2000), 307–318.

[2] V. BACH AND J. POELCHAU, Accuracy of the Hartree-Fock approximation for the Hubbard model, *J. Math. Phys.,* **38**(4) (1997), 2072–2083.

[3] F. BEREZIN, Convex functions of operators, *Mat. sb.,* **88** (1972), 268–276 (Russian).

[4] R. BHATIA, *Matrix Analysis, Graduate Texts in Mathematics 169*, Springer Verlag 1997.

[5] T. FURUTA, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$, *Proc. Amer. Math. Soc.,* **101** (1987), 85–88.

[6] T. FURUTA, Two operator functions with monotone property, *Proc. Amer. Math. Soc.,* **111**(2) (1991), 511–516.

[7] I.G. GOHBERG AND M.G. KREIN, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Transl. Math. Monographs, **18**, Amer. Math. Soc., 1969.

[8] F. HANSEN, An operator inequality, *Math. Ann.,* **246** (1980), 249–250.

[9] F. HANSEN AND G.K. PEDERSEN, Jensen's inequality for operators and Löwner's theorem, *Math. Ann.,* **258** (1982), 229–241.

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[10] E. KAMEI, A satellite to Furuta's inequality, *Math. Japon.,* **33** (1988), 883–886.

[11] E.H. LIEB, The classical limit of quantum spin systems, *Comm. Math. Phys.,* **31** (1973), 327–340.

[12] E.H. LIEB, *The Hubbard Model: Some Rigorous Results And Open Problems*, proceedings of the XIth International Congress of Mathematical Physics, Paris, 1994, edited by D. Iagolnitzer, International Press 1995, 392–412.

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