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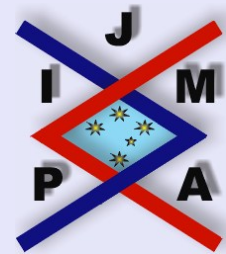
ON THE INEQUALITY OF P. TURÁN FOR LEGENDRE POLYNOMIALS

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Abstract

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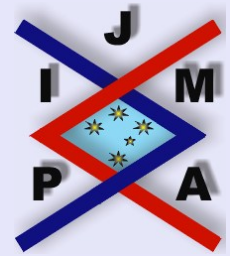


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Abstract

Our aim is to prove the inequalities

$$\frac{1-x^2}{n(n+1)} h_n \leq \left| \begin{array}{cc} P_n(x) & P_{n+1}(x) \\ P_{n-1}(x) & P_n(x) \end{array} \right| \leq \frac{1-x^2}{2}, \quad \forall x \in [-1, 1], \quad n = 1, 2, \dots,$$

where $h_n := \sum_{k=1}^n \frac{1}{k}$ and $(P_n)_{n=0}^{\infty}$ are the Legendre polynomials. At the same time, it is shown that the sequence having as general term

$$n(n+1) \left| \begin{array}{cc} P_n(x) & P_{n+1}(x) \\ P_{n-1}(x) & P_n(x) \end{array} \right|$$

is non-decreasing for $x \in [-1, 1]$.

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Key words: Orthogonal polynomials, Legendre polynomials, Turán Inequality, Positivity.

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1. Introduction

Let $(P_n)_{n=0}^{\infty}$ be the sequence of Legendre polynomials, that is

$$P_n(x) = \frac{1}{n!2^n} ((x^2 - 1)^n)^{(n)} = {}_2F_1 \left(-n, n + 1; 1; \frac{1 - x}{2} \right),$$

where

$${}_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \cdot \frac{z^k}{k!},$$

$$(a)_k := a(a + 1) \cdots (a + k - 1), \quad (a)_0 = 1.$$

Denote

$$\Delta_n(x) := \begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n-1}(x) & P_n(x) \end{vmatrix} = [P_n(x)]^2 - P_{n-1}(x)P_{n+1}(x).$$

Note that $P_n(1) = 1$, $P_n(-x) = (-1)^n P_n(x)$, i.e. $\Delta_n(1) = \Delta_n(-1) = 0$.

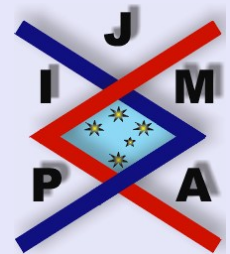
For instance

$$\Delta_1(x) = \frac{1 - x^2}{2}, \quad \Delta_2(x) = \frac{1 - x^4}{4}.$$

Paul Turán [3] has proved the following interesting inequality

$$(1.1) \quad \Delta_n(x) > 0, \quad \forall x \in (-1, 1), \quad n \in \{1, 2, \dots\}.$$

In [1] – [2] are given the following remarkable representations of $\Delta_n(x)$.



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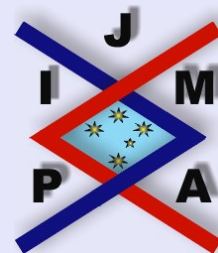
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Lemma 1.1 (A. Lupaş). Suppose $\varphi(x, t) := x^2 + t(1 - x^2)$ and $P_n(x_k) = 0$.
Then

$$(1.2) \quad \Delta_n(x) = \frac{1}{\pi n(n+1)} \int_{-1}^1 \frac{1 - P_n(\varphi(x, t))}{1 - t} \cdot \frac{dt}{\sqrt{1 - t^2}}$$

and

$$(1.3) \quad \Delta_n(x) = \frac{1 - x^2}{n(n+1)} \sum_{k=1}^n \left(\frac{P_n(x)}{x - x_k} \right)^2 (1 - xx_k) .$$



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2. Main Results

In this article our aim is to improve the Turán inequality (1.1).

Theorem 2.1. *If $x \in [-1, 1]$, $n \in \mathbb{N}$, $h_n := \sum_{k=1}^n \frac{1}{k}$, then*

$$(2.1) \quad \frac{1-x^2}{n(n+1)} h_n \leq \Delta_n(x) \leq \frac{1-x^2}{2}.$$

Proof. Let us denote $T_k(t) = \cos(k \cdot \arccost t)$, $\gamma_0 = \frac{1}{\pi}$, $\gamma_k = \frac{2}{\pi}$ for $k \geq 1$, and $\varphi(x, t) = x^2 + t(1 - x^2)$. According to addition formula for Legendre polynomials, we have

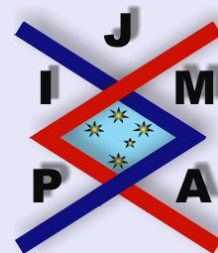
$$P_n(\varphi(x, t)) = \pi \sum_{k=0}^n \frac{(n-k)!}{(n+k)!} (1-x^2)^k [P_n^{(k)}(x)]^2 \gamma_k T_k(t).$$

If $t = 1$ we find

$$1 = \pi \sum_{k=0}^n \frac{(n-k)!}{(n+k)!} (1-x^2)^k [P_n^{(k)}(x)]^2 \gamma_k.$$

Therefore

$$\begin{aligned} \frac{1 - P_n(\varphi(x, t))}{1-t} &= 2 \sum_{k=1}^n \frac{(n-k)!}{(n+k)!} (1-x^2)^k [P_n^{(k)}(x)]^2 \frac{1 - T_k(t)}{1-t} \\ &= 2\pi \sum_{k=1}^n \frac{(n-k)!}{(n+k)!} (1-x^2)^k [P_n^{(k)}(x)]^2 \sum_{\nu=0}^k (k-\nu) \gamma_\nu T_\nu(t). \end{aligned}$$



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This shows us that

$$\begin{aligned} \max_{t \in [-1, 1]} \left\{ \frac{1 - P_n(\varphi(x, t))}{1 - t} \right\} &= \frac{1 - P_n(\varphi(x, t))}{1 - t} \Big|_{t=1} \\ &= (1 - x^2)P'_n(1) \\ &= \frac{n(n+1)}{2}(1 - x^2). \end{aligned}$$

Using the Lupaş identity (1.2) we obtain

$$\Delta_n(x) \leq \frac{1 - x^2}{2}, \quad (n \geq 1, \quad x \in [-1, 1]).$$

Taking into account the following well-known equalities

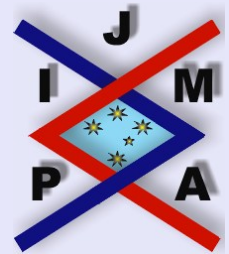
$$\begin{aligned} P_n(x) &= \frac{2n-1}{n}xP_{n-1}(x) - \frac{n-1}{n}P_{n-2}(x), \quad P_0(x) = 1, \quad P_1(x) = x, \\ (1 - x^2)P'_n(x) &= n(P_{n-1}(x) - xP_n(x)) = (n+1)(xP_n(x) - P_{n+1}(x)), \end{aligned}$$

we obtain

$$k(k+1)\Delta_k(x) - (k-1)k\Delta_{k-1}(x) = (1-x^2) [P'_k(x)P_{k-1}(x) - P_k(x)P'_{k-1}(x)].$$

The Christoffel-Darboux formula for Legendre polynomials enables us to write

$$k(k+1)\Delta_k(x) - (k-1)k\Delta_{k-1}(x) = \frac{1-x^2}{k} \sum_{j=0}^{k-1} (2j+1) [P_j(x)]^2, \quad k \geq 2.$$



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By summing for $k \in \{2, 3, \dots, n\}$ we give

$$n(n+1)\Delta_n(x) = (1-x^2)h_n + (1-x^2) \sum_{k=1}^{n-1} \frac{1}{k+1} \sum_{j=1}^k (2j+1) [P_j(x)]^2,$$

which implies $\Delta_n(x) \geq \frac{(1-x^2)h_n}{n(n+1)}$ for $x \in [-1, 1]$. □

Another remark regarding $\Delta_n(x)$ is the following :

Theorem 2.2. *The sequence $(n(n+1)\Delta_n(x))_{n=1}^{\infty}$, $x \in [-1, 1]$, is non-decreasing, i.e.*

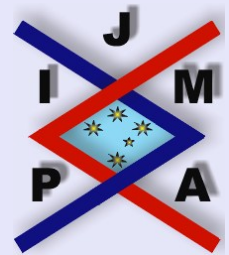
$$\Delta_n(x) \geq \frac{n-1}{n+1} \Delta_{n-1}(x), \quad x \in [-1, 1], \quad n \geq 2.$$

Proof. Let Π_m be the linear space of all polynomials, of degree $\leq m$, having real coefficients. Using a Lagrange-Hermite interpolation formula, every polynomial f from Π_{2n+1} with $f(-1) = f(1) = 0$ may be written as

$$(2.2) \quad f(x) = (1-x^2) \sum_{k=1}^n \left(\frac{P_n(x)}{P'_n(x_k)(x-x_k)} \right)^2 A_k(f; x),$$

where

$$A_k(f; x) = \frac{f(x_k) + (x-x_k)f'(x_k)}{1-x_k^2}.$$



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Let us observe that

$$(2.3) \quad \begin{aligned} P_{n-1}(x_k) &= \frac{1-x_k^2}{n} P'_n(x_k), & P_{n+1}(x_k) &= -\frac{1-x_k^2}{n+1} P'_n(x_k), \\ P_{n-2}(x_k) &= \frac{2n-1}{n(n-1)} x_k(1-x_k^2) P'_n(x_k), \\ P'_{n-1}(x_k) &= P'_{n+1}(x_k) = x_k P'_n(x_k). \end{aligned}$$

In (2.2) let us consider $f \in \Pi_{2n}$, where

$$f(x) = n(n+1)\Delta_n(x) - n(n-1)\Delta_{n-1}(x).$$

From (2.3) we find

$$f(x_k) = \frac{(1-x_k^2)^2}{n} [P'_n(x_k)]^2, \quad f'(x_k) = 0.$$

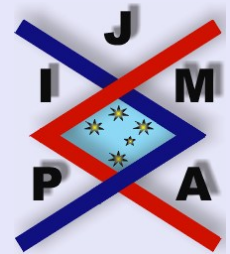
Because $A_k(f; x) = \frac{1-x_k^2}{n} [P'_n(x_k)]^2$, using (2.2) we give

$$f(x) = \frac{1-x^2}{n} \sum_{k=1}^n \left(\frac{P_n(x)}{x-x_k} \right)^2 (1-x_k^2) \geq 0, \quad x \in [-1, 1].$$

Therefore

$$(n+1)\Delta_n(x) - (n-1)\Delta_{n-1}(x) \geq 0 \quad \text{for } x \in [-1, 1].$$

□



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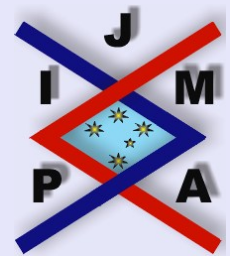
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