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ON CERTAIN ROUGH INTEGRAL OPERATORS AND EXTRAPOLATION

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ABSTRACT. In this paper, we obtain sharp L^p estimates of two classes of maximal operators related to rough singular integrals and Marcinkiewicz integrals. These estimates will be used to obtain similar estimates for the related singular integrals and Marcinkiewicz integrals. By the virtue of these estimates and extrapolation we obtain the L^p boundedness of all the aforementioned operators under rather weak size conditions. Our results represent significant improvements as well as natural extensions of what was known previously.

Key words and phrases: Maximal operator, Rough kernel, Singular integral, Orlicz spaces, Block spaces, Extrapolation, L^p boundedness.

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1. INTRODUCTION AND MAIN RESULTS

Throughout this paper, let $\mathbf{R}^n$, $n \geq 2$, be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue surface measure $d\sigma$. Also, we let ξ' denote $\xi/|\xi|$ for $\xi \in \mathbf{R}^n \backslash \{0\}$ and p' denote the exponent conjugate to p, that is $1/p + 1/p' = 1$.

Let $K_{\Omega,h}(x) = \Omega(x')h(|x|)\,|x|^{-n}$, where $h : [0, \infty) \longrightarrow \mathbf{C}$ is a measurable function and Ω is a function defined on $\mathbf{S}^{n-1}$ with $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega\,(x)\,d\sigma\,(x) = 0. \tag{1.1}$$

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For $1 \leq \gamma \leq \infty$, let $\Delta_\gamma(\mathbf{R}_+)$ denote the collection of all measurable functions $h : [0, \infty) \longrightarrow \mathbf{C}$ satisfying $\sup_{R>0} \left(\frac{1}{R} \int_0^R |h(t)|^\gamma dt \right)^{1/\gamma} < \infty$.

Define the singular integral operator $S_{\Omega,h}$, the parametric Marcinkiewicz integral operator $\mathcal{M}_{\Omega,h}$ and their related maximal operators $S_\Omega^{(\gamma,*)}$ and $\mathcal{M}_\Omega^{(\gamma,*)}$ by

$$S_{\Omega,h} f(x) = \text{p.v.} \int_{\mathbf{R}^n} f(x-u) K_{\Omega,h}(u) du, \tag{1.2}$$

$$\mathcal{M}_{\Omega,h} f(x) = \left(\int_0^\infty \left| t^{-\rho} \int_{|u| \leq t} f(x-u) \, |u|^\rho \, K_{\Omega,h}(u) du \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \tag{1.3}$$

$$S_\Omega^{(\gamma,*)} f(x) = \sup_h |S_{\Omega,h} f(x)| \,, \tag{1.4}$$

$$\mathcal{M}_\Omega^{(\gamma,*)} f(x) = \sup_h |\mathcal{M}_{\Omega,h} f(x)| \,, \tag{1.5}$$

where $f \in \mathcal{S}(\mathbf{R}^n)$, $\rho = \sigma + i\tau$ ($\sigma, \tau \in \mathbf{R}$ with $\sigma > 0$) and the supremum is taken over the set of all radial functions h with $h \in L^\gamma(\mathbf{R}_+, dt/t)$ and $\|h\|_{L^\gamma(\mathbf{R}_+, dt/t)} \leq 1$.

If $h(t) \equiv 1$ and $\rho = 1$, we shall denote $S_{\Omega,h}$ by S_Ω and $\mathcal{M}_{\Omega,h}$ by $\mathcal{M}_\Omega$, which are respectively the classical singular integral operator of Calderón-Zygmund and the classical Marcinkiewicz integral operator of higher dimension.

The study of the mapping properties of $S_\Omega, \mathcal{M}_\Omega$ and their extensions has a long history. Readers are referred to [7], [3], [4], [16], [18], [5], [10], [22], [24], [25] and the references therein for applications and recent advances on the study of such operators.

Let us now recall some results which will be relevant to our current study. We start with the following results on singular integrals:

Theorem 1.1. *If Ω satisfies one of the following conditions, then $S_{\Omega,h}$ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

(a) *$\Omega \in L(\log L)(\mathbf{S}^{n-1})$ and $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma > 1$. Moreover, the condition $\Omega \in L(\log L)(\mathbf{S}^{n-1})$ is an optimal size condition for the L^p boundedness of S_Ω (see* [13] *in the case $h(t) \equiv 1$ and see* [10]*).*

(b) *$\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$ and $h \in L^\gamma(\mathbf{R}_+, dt/t)$ for some $\gamma > 1$ (see* [2] *and see* [8] *in the case $\gamma = 2$).*

(c) *$\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ for some $q > 1$ and $h \in \Delta_\gamma(\mathbf{R}_+)$ for some $\gamma > 1$. Moreover, the condition $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1})$ is an optimal size condition for the L^p boundedness of S_Ω (see* [4] *and* [5]*).*

(d) *$\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for some $q > 1$ and $h \in L^2(\mathbf{R}_+, dt/t)$ (*[9]*).*

Here $L(\log L)^\alpha(\mathbf{S}^{n-1})$ (for $\alpha > 0$) is the class of all measurable functions Ω on $\mathbf{S}^{n-1}$ which satisfy

$$\|\Omega\|_{L(\log L)^\alpha(\mathbf{S}^{n-1})} = \int_{\mathbf{S}^{n-1}} |\Omega(y)| \log^\alpha(2 + |\Omega(y)|) d\sigma(y) < \infty$$

and $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$, $\upsilon > -1$, is a special class of block spaces whose definition will be recalled in Section 2.

Theorem 1.2. *If Ω satisfies one of the following conditions, then $\mathcal{M}_{\Omega,h}$ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$.*

(a) $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ *and* $h = 1$. *Moreover, the exponent* $1/2$ *is the best possible (*[26], [7]*).*

(b) $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ *for some* $q > 1$ *and* $h = 1$. *Moreover, the condition* $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ *is an optimal size condition for the* L^2 *boundedness of* $\mathcal{M}_\Omega$ *(*[3]*).*

(c) $\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$, $h \in L^\gamma(\mathbf{R}_+, dt/t)$, $1 < \gamma \le 2$ *and* $\gamma' \le p < \infty$ *or* $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1}$, $h \in L^\gamma(\mathbf{R}_+, dt/t)$, $2 < \gamma < \infty$ *and* $2 \le p < \infty$ *(*[6]*).*

On the other hand, the study of the related maximal operator $S_\Omega^{(\gamma,*)}$ has attracted the attention of many authors. See for example, [14], [1], [5], [6], [8], [9], [27].

Theorem 1.3. *If Ω satisfies one of the following conditions, then $S_\Omega^{(\gamma,*)}$ is bounded on $L^p(\mathbf{R}^n)$.*

(a) $\Omega \in C(\mathbf{S}^{n-1})$, $(n\gamma)/(n\gamma - 1) < p < \infty$ *and* $1 \le \gamma \le 2$. *Moreover, the range of* p *is the best possible* [14].

(b) $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ *for some* $q > 1, \gamma = 2, 2 \le p < \infty$. *Moreover, the condition* $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ *is an optimal size condition for the* L^2 *boundedness of* $S_\Omega^{(2,*)}$ *to hold (*[1], [9]*).*

(c) $\Omega \in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1}), \gamma' \le p < \infty$. *Moreover, the exponent 1/2 is the best possible for the* L^2 *boundedness of* $S_\Omega^{(2,*)}$ *to hold (see* [8] *for* $\gamma = 2$ *and* [2] *for* $\gamma \ne 2$*).*

In view of the above results, the following question is very natural:

Problem 1.1. *Is there any analogue of Theorem 1.1(d) and Theorem 1.3(b) in the case $\gamma \ne 2$? Is there an analogy of Theorem 1.2(c)? Is there any room for improvement of the range of p in both Theorem 1.2(c) and Theorem 1.3(c)?*

The purpose of this paper is two-fold. First, we answer the above questions in the affirmative. Second, we present a unified approach different from the ones employed in previous papers (see for example, [1], [2], [6], [8], [9]) in dealing with the operators $S_{\Omega,h}$, $\mathcal{M}_{\Omega,h}$, $S_\Omega^{(\gamma,*)}$ and $\mathcal{M}_\Omega^{(\gamma,*)}$ when the kernel function Ω belongs to the block space $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ (for $\upsilon > -1$) or Ω belongs to the class $L(\log L)^\alpha(\mathbf{S}^{n-1})$ (for $\alpha > 0$). This approach will mainly rely on obtaining some delicate sharp L^p estimates and then applying an extrapolation argument.

Now, let us state our main results.

Theorem 1.4. *Suppose that $\Omega \in L^q(\mathbf{S}^{n-1})$ for some $1 < q \le \infty$. Then*

$$\left\| S_\Omega^{(\gamma,*)}(f) \right\|_{L^p(\mathbf{R}^n)} \le C_p \left(\frac{q}{q-1} \right)^{1/\gamma'} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^n)} \tag{1.6}$$

holds for $(n\alpha\gamma')/(\gamma' n + n\alpha - \gamma') < p < \infty$ and $1 \le \gamma \le 2$, where $\alpha = \max\{2, q'\}$. Moreover, the exponent $1/\gamma'$ is the best possible in the case $\gamma = 2$.

Theorem 1.5. *Let α and Ω be as in Theorem 1.4 and let $1 \le \gamma < \infty$. Then*

$$\left\| \mathcal{M}_\Omega^{(\gamma,*)}(f) \right\|_{L^p(\mathbf{R}^n)} \le C_p \left(\frac{q}{q-1} \right)^{1/\beta} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^n)} \tag{1.7}$$

holds for $(n\alpha\beta)/(\beta n + n\alpha - \beta) < p < \infty$, where $\beta = \max\{2, \gamma'\}$. Moreover, the exponent $1/\beta$ is the best possible in the case $\gamma = 2$.

Theorem 1.6. *Suppose that* $\Omega\in L^q(\mathbf{S}^{n-1})$, $1<q\le 2$, *and* $h\in L^\gamma(\mathbf{R}_+,dt/t)$ *for some* $1<\gamma\le\infty$. *Then*

$$\|S_{\Omega,h}(f)\|_{L^p(\mathbf{R}^n)}\le C_p\,(q-1)^{-1/\gamma'}\,\|h\|_{L^\gamma(\mathbf{R}_+,dt/t)}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\,\|f\|_{L^p(\mathbf{R}^n)} \tag{1.8}$$

for $1<p<\infty$ *and*

$$\|\mathcal{M}_{\Omega,h}(f)\|_{L^p(\mathbf{R}^n)}\le C_p\,(q-1)^{-1/\beta}\,\|h\|_{L^\gamma(\mathbf{R}_+,dt/t)}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\,\|f\|_{L^p(\mathbf{R}^n)} \tag{1.9}$$

for $(n\beta q')/(\beta n+nq'-\beta)<p<\infty$.

By the conclusions from Theorems 1.4 – 1.6 and applying an extrapolation method, we get the following results:

Theorem 1.7.

(a) *If* $\Omega\in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$ *and* $1<\gamma\le 2$, *the operator* $S_\Omega^{(\gamma,*)}$ *is bounded on* $L^p(\mathbf{R}^n)$ *for* $2\le p<\infty$;
(b) *If* $\Omega\in B_q^{(0,1/\gamma'-1)}(\mathbf{S}^{n-1})$ *and* $1<\gamma\le 2$, *the operator* $S_\Omega^{(\gamma,*)}$ *is bounded on* $L^p(\mathbf{R}^n)$ *for* $2\le p<\infty$;
(c) *If* $\Omega\in L(\log L)^{1/\gamma'}(\mathbf{S}^{n-1})$ *and* $h\in L^\gamma(\mathbf{R}_+,dt/t)$ *for some* $1<\gamma\le\infty$, *the operator* $S_{\Omega,h}$ *is bounded on* $L^p(\mathbf{R}^n)$ *for* $1<p<\infty$;
(d) *If* $B_q^{(0,1/\gamma'-1)}(\mathbf{S}^{n-1})$ *for some* $q>1$ *and* $h\in L^\gamma(\mathbf{R}_+,dt/t)$ *for some* $1<\gamma\le\infty$, *the operator* $S_{\Omega,h}$ *is bounded on* $L^p(\mathbf{R}^n)$ *for* $1<p<\infty$.

Theorem 1.8. *Let* $1<\gamma<\infty$ *and* $\beta=\max\{2,\gamma'\}$.

(a) *If* $\Omega\in L(\log L)^{1/\beta}(\mathbf{S}^{n-1})$ *and* $1<\gamma<\infty$, *the operator* $\mathcal{M}_\Omega^{(\gamma,*)}$ *is bounded on* $L^p(\mathbf{R}^n)$ *for* $2\le p<\infty$;
(b) *If* $\Omega\in B_q^{(0,1/\beta-1)}(\mathbf{S}^{n-1})$ *and* $1<\gamma<\infty$, *the operator* $\mathcal{M}_\Omega^{(\gamma,*)}$ *is bounded on* $L^p(\mathbf{R}^n)$ *for* $2\le p<\infty$.
(c) *If* $\Omega\in L(\log L)^{1/\beta}(\mathbf{S}^{n-1})$ *and* $h\in L^\gamma(\mathbf{R}_+,dt/t)$ *for some* $1<\gamma<\infty$, *the operator* $\mathcal{M}_{\Omega,h}$ *is bounded on* $L^p(\mathbf{R}^n)$ *for* $2\le p<\infty$;
(d) *If* $B_q^{(0,1/\beta-1)}(\mathbf{S}^{n-1})$ *for some* $q>1$ *and* $h\in L^\gamma(\mathbf{R}_+,dt/t)$ *for some* $1<\gamma<\infty$, *the operator* $\mathcal{M}_{\Omega,h}$ *is bounded on* $L^p(\mathbf{R}^n)$ *for* $2\le p<\infty$.

Remark 1.

(1) For any $q>1$, $0<\alpha<\beta$ and $-1<\upsilon$, the following inclusions hold and are proper:

$$L^q(\mathbf{S}^{n-1})\subset L(\log L)^{\beta}(\mathbf{S}^{n-1})\subset L(\log L)^{\alpha}(\mathbf{S}^{n-1}),$$
$$\bigcup_{r>1}L^r(\mathbf{S}^{n-1})\subset B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})\ \text{for any}\ -1<\upsilon,$$
$$B_q^{(0,\upsilon_2)}(\mathbf{S}^{n-1})\subset B_q^{(0,\upsilon_1)}(\mathbf{S}^{n-1})\ \text{for any}\ -1<\upsilon_1<\upsilon_2,$$
$$L^\gamma(\mathbf{R}_+,dt/t)\subset\Delta_\gamma(\mathbf{R}_+)\ \text{ for } 1\le\gamma<\infty.$$

The question regarding the relationship between $B_q^{(0,\upsilon-1)}$ and $L(\log L)^{\upsilon}$ over $\mathbf{S}^{n-1}$ (for $\upsilon>0$) remains open.

(2) The L^p boundedness of $S_\Omega^{(\gamma,*)}$ for $(n\alpha\gamma')/(\gamma' n+n\alpha-\gamma')<p<\infty$ and $\Omega\in L^q(\mathbf{S}^{n-1})$ was proved in [1] only in the case $\gamma=2$, but the importance of Theorem 1.4 lies in the fact that the estimate (1.6) in conjunction with an extrapolation argument will play a key role in obtaining all our results and will allow us to obtain the L^p boundedness of $S_\Omega^{(\gamma,*)}$ under optimal size conditions on Ω.

(3) We point out that it is still an open question whether the L^p boundedness of $S_\Omega^{(\gamma,*)}$ holds for $2 < \gamma < \infty$. In the case $\gamma = \infty$, the authors of [14] pointed out that the maximal operator $S_\Omega^{(\infty,*)}(f)$ is not bounded on all L^p spaces for $1 \le p \le \infty$. On the other hand, we notice that Theorem 1.5 gives that $\mathcal{M}_\Omega^{(\gamma,*)}$ is bounded on L^p for any $1 \le \gamma < \infty$.

(4) Theorem 1.7 (a)(c) and Theorem 1.8 (c) represent an improvement over the main results in [8] and improves the range of p in [2]. Also, Theorem 1.7 (b)(d) and Theorem 1.8 (d) represent an improvement over the main results in [9] and [1].

Throughout the rest of the paper the letter C will stand for a constant but not necessarily the same one in each occurrence.

2. DEFINITIONS AND SOME BASIC LEMMAS

Block spaces originated from the work of M.H. Taibleson and G. Weiss on the convergence of the Fourier series in connection with developments of the real Hardy spaces. Below we shall recall the definition of block spaces on $\mathbf{S}^{n-1}$. For further background information about the theory of spaces generated by blocks and its applications to harmonic analysis, one can consult the book [20]. In [20], Lu introduced the spaces $B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ with respect to the study of singular integral operators.

Definition 2.1. *A q-block on $\mathbf{S}^{n-1}$ is an L^q $(1 < q \le \infty)$ function $b(x)$ that satisfies*

(i) supp$(b) \subset I$;

(ii) $\|b\|_{L^q} \le |I|^{-1/q'}$ *, where $|\cdot|$ denotes the product measure on $\mathbf{S}^{n-1}$, and I is an interval on $\mathbf{S}^{n-1}$, i.e., $I = \{x \in \mathbf{S}^{n-1} : |x - x_0| < \alpha\}$ for some $\alpha > 0$ and $x_0 \in \mathbf{S}^{n-1}$.*

The block space $B_q^{(0,\upsilon)} = B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})$ is defined by

$$B_q^{(0,\upsilon)} = \left\{ \Omega \in L^1(\mathbf{S}^{n-1}) : \Omega = \sum_{\mu=1}^{\infty} \lambda_\mu b_\mu,\ M_q^{(0,\upsilon)}\left(\{\lambda_\mu\}\right) < \infty \right\},$$

where each λ_μ is a complex number; each b_μ is a q-block supported on an interval I_μ on $\mathbf{S}^{n-1}$, $\upsilon > -1$ and

$$M_q^{(0,\upsilon)}\left(\{\lambda_\mu\}\right) = \sum_{\mu=1}^{\infty} |\lambda_\mu| \left\{1 + \log^{(\upsilon+1)}\left(|I_\mu|^{-1}\right)\right\}.$$

Let

$$\|\Omega\|_{B_q^{(0,\upsilon)}(\mathbf{S}^{n-1})} = N_q^{(0,\upsilon)}(\Omega) = \inf\left\{M_q^{(0,\upsilon)}\left(\{\lambda_\mu\}\right)\right\},$$

where the infimum is taken over all q-block decompositions of Ω.

Definition 2.2. *For arbitrary $\theta \ge 2$ and $\Omega : \mathbf{S}^{n-1} \to \mathbf{R}$, we define the sequence of measures $\{\sigma_{\Omega,h,k} : k \in \mathbf{Z}\}$ and the corresponding maximal operator $\sigma^*_{\Omega,h,\theta}$* on $\mathbf{R}^n$ *by*

$$\int_{\mathbf{R}^n} f\, d\sigma_{\Omega,h,\theta,k} = \int_{\theta^k \le |u| < \theta^{k+1}} h(|u|) \frac{\Omega(u')}{|u|^n} f(u)\, du; \tag{2.1}$$

$$\sigma^*_{\Omega,h,\theta}(f) = \sup_{k\in\mathbf{Z}} \left| |\sigma_{\Omega,h,\theta,k}| * f \right|. \tag{2.2}$$

We shall need the following lemma which has its roots in [15] and [5]. A proof of this lemma can be obtained by the same proof (with only minor modifications) as that of Lemma 3.2 in [5]. We omit the details.

Lemma 2.1. *Let $\{\sigma_k : k \in \mathbf{Z}\}$ be a sequence of Borel measures on $\mathbf{R}^n$. Suppose that for some $a \geq 2$, α, $C > 0$, $B > 1$ and $p_0 \in (2, \infty)$ the following hold for $k \in \mathbf{Z}$, $\xi \in \mathbf{R}^n$ and for arbitrary functions $\{g_k\}$ on $\mathbf{R}^n$:*

(i) $|\hat{\sigma}_k(\xi)| \leq CB \left|a^k \xi\right|^{\pm \frac{\alpha}{\log(a)}}$;

(ii) $\left\| \left(\sum_{k\in\mathbf{Z}} |\sigma_k * g_k|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \leq CB \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^2 \right)^{\frac{1}{2}} \right\|_{p_0}$.

Then for $p_0' < p < p_0$, there exists a positive constant C_p independent of B such that

$$\left\| \sum_{k\in\mathbf{Z}} \sigma_k * f \right\|_p \leq C_p B \|f\|_p$$

holds for all f in $L^p(\mathbf{R}^n)$.

Lemma 2.2. *Let θ and $\sigma^*_{\Omega,h,\theta}$ be given as in Definition 2.2. Suppose $h \in L^\gamma(\mathbf{R}_+, dt/t)$ for some $1 < \gamma \leq \infty$ and $\Omega \in L^1(\mathbf{S}^{n-1})$. Then*

$$\left\|\sigma^*_{\Omega,h,\theta}(f)\right\|_p \leq C_p (\log\theta)^{1/\gamma'} \|h\|_{L^\gamma(\mathbf{R}_+,dt/t)} \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \|f\|_p \tag{2.3}$$

for $\gamma' < p \leq \infty$ and $f \in L^p(\mathbf{R}^n)$, where C_p is independent of Ω, θ and f.

Proof. By Hölder's inequality we have

$$|\sigma_{\Omega,h,\theta,k} * f(x)| \leq \|h\|_{L^\gamma(\mathbf{R}_+,dt/t)} \|\Omega\|^{1/\gamma}_{L^1(\mathbf{S}^{n-1})} \times \left(\int_{\theta^k}^{\theta^{k+1}} \int_{\mathbf{S}^{n-1}} |\Omega(y)| \, |f(x-ty)|^{\gamma'} d\sigma(y) \frac{dt}{t} \right)^{1/\gamma'}.$$

By Minkowski's inequality for integrals we get

$$\left\|\sigma^*_{\Omega,h,\theta}(f)\right\|_p \leq (\log\theta)^{1/\gamma'} \|h\|_{L^\gamma(\mathbf{R}_+,dt/t)} \|\Omega\|^{1/\gamma}_{L^1(\mathbf{S}^{n-1})} \times \left(\int_{\mathbf{S}^{n-1}} |\Omega(y)| \left\| M_y(|f|^{\gamma'}) \right\|_{p/\gamma'} d\sigma(y) \right)^{1/\gamma'},$$

where

$$M_y f(x) = \sup_{R>0} R^{-1} \int_0^R |f(x-sy)| \, ds$$

is the Hardy-Littlewood maximal function of f in the direction of y. By the fact that the operator M_y is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ with a bound independent of y, by the last inequality we get (2.3). Lemma 2.2 is thus proved. □

By following an argument similar to that in [17] and [2], we obtain the following:

Lemma 2.3. *Let θ and Ω be as in Lemma 2.2. Assume that $h \in L^\gamma(\mathbf{R}_+, dt/t)$ for some $\gamma \geq 2$. Then for $\gamma' < p < \infty$, there exists a positive constant C_p which is independent of θ such that*

$$\left\| \left(\sum_{k\in\mathbf{Z}} |\sigma_{\Omega,h,\theta,k} * g_k|^2 \right)^{\frac{1}{2}} \right\|_p \leq C_p (\log\theta)^{1/\gamma'} \|\Omega\|_{L^1(\mathbf{S}^{n-1})} \left\| \left(\sum_{k\in\mathbf{Z}} |g_k|^2 \right)^{\frac{1}{2}} \right\|_p$$

holds for arbitrary measurable functions $\{g_k\}$ on $\mathbf{R}^n$.

Now, we need the following simple lemma.

Lemma 2.4. *Let $q > 1$ and $\beta = \max\{2, q'\}$. Suppose that $\Omega \in L^q(\mathbf{S}^{n-1})$. Then for some positive constant C, we have*

$$\left|\int_{\mathbf{S}^{n-1}} \Omega(\xi) f(\xi) d\sigma(\xi)\right|^2 \leq \|\Omega\|_q^{\min\{2,q\}} \int_{\mathbf{S}^{n-1}} |\Omega(\xi)|^{\max\{0,2-q\}} |f(\xi)|^2 d\sigma(\xi) \tag{2.4}$$

and

$$\int_{\mathbf{S}^{n-1}} |\Omega(\xi)|^{\max\{0,2-q\}} |f(x-t\xi)| d\sigma(\xi) \leq C \|\Omega\|_q^{\max\{0,2-q\}} \left(\mathcal{M}_{Sph}\left(|f|^{\beta/2}\right)(x)\right)^{\frac{2}{\beta}} \tag{2.5}$$

for all positive real numbers t and $x \in \mathbf{R}^n$ and arbitrary functions f, where $\mathcal{M}_{Sph}$ is the spherical maximal operator defined by

$$\mathcal{M}_{Sph} f(x) = \sup_{t>0} \int_{\mathbf{S}^{n-1}} |f(x-tu)| d\sigma(u).$$

Proof. Let us first prove (2.4). First if $q \geq 2$, by Hölder's inequality we have

$$\begin{aligned} \left|\int_{\mathbf{S}^{n-1}} \Omega(\xi) f(\xi) d\sigma(\xi)\right|^2 &\leq \|\Omega\|_q^2 \left(\int_{\mathbf{S}^{n-1}} |f(\xi)|^{q'} d\sigma(\xi)\right)^{\frac{2}{q'}} \\ &\leq \|\Omega\|_q^2 \int_{\mathbf{S}^{n-1}} |f(\xi)|^2 d\sigma(\xi), \end{aligned}$$

which is the inequality (2.4) in the case $q \geq 2$. Next, if $1 < q < 2$, the inequality (2.4) follows by Schwarz's inequality.

To prove (2.5) we need again to consider two cases. First if $q \geq 2$, we notice that (2.5) is obvious. Next if $1 < q < 2$, (2.5) follows by Hölder's inequality on noticing that $\left(\frac{q}{2-q}\right)' = q'/2$. The lemma is thus proved. □

3. PROOF OF THE MAIN RESULTS

We begin with the proof of Theorem 1.4.

Proof. Since $L^q(\mathbf{S}^{n-1}) \subseteq L^2(\mathbf{S}^{n-1})$ for $q \geq 2$, Theorem 1.4 is proved once we prove that

$$\left\|S_\Omega^{(\gamma,*)}(f)\right\|_{L^p(\mathbf{R}^n)} \leq C_p (q-1)^{-1/\gamma'} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^n)} \tag{3.1}$$

holds for $1 < q \leq 2$, $(nq'\gamma')/(\gamma' n + nq' - \gamma') < p < \infty$ and $1 \leq \gamma \leq 2$.

We shall prove (3.1) by first proving (3.1) for the cases $\gamma = 1$ and $\gamma = 2$ and then we use the idea of interpolation to cover the case $1 < \gamma < 2$.

Proof of (3.1) for the case $\gamma = 1$. By duality we have

$$\begin{aligned} S_\Omega^{(1,*)} f(x) &= \left\|\int_{\mathbf{S}^{n-1}} f(x-tu)\Omega(u) d\sigma(u)\right\|_{L^\infty(\mathbf{R}_+, dt/t)} \\ &= \left\|\int_{\mathbf{S}^{n-1}} f(x-tu)\Omega(u) d\sigma(u)\right\|_{L^\infty(\mathbf{R}_+, dt)} \\ &\leq \sup_{t>0} \int_{\mathbf{S}^{n-1}} |f(x-tu)| |\Omega(u)| d\sigma(u). \end{aligned}$$

Using Hölder's inequality we get

$$S_\Omega^{(1,*)} f(x) \leq \|\Omega\|_q \left(\mathcal{M}_{Sph}\left(|f|^{q'}\right)(x)\right)^{1/q'}.$$

By the results of E.M. Stein [23] and J. Bourgain [12] we know that $\mathcal{M}_{Sph}(f)$ is bounded on $L^p(\mathbf{R}^n)$ for $n \geq 2$ and $p > n/(n-1)$. Thus by using the last inequality we get (3.1) for the case $\gamma = 1$.

Proof of (3.1) for the case $\gamma = 2$. By Hölder's inequality we obtain

$$S_{\Omega}^{(2,*)} f(x) \leq \left(\int_0^{\infty} \left| \int_{\mathbf{S}^{n-1}} \Omega(u) f(x - tu) d\sigma(u) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

Let $\theta = 2^{q'}$ and let $\{\varphi_k\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0, \infty)$ adapted to the interval $[\theta^{-k-1}, \theta^{-k+1}]$. Specifically, we require the following:

$$\varphi_k \in C^{\infty}, \quad 0 \leq \varphi_k \leq 1, \quad \sum_{k \in \mathbf{Z}} \varphi_k(t) = 1,$$
$$\text{supp } \varphi_k \subseteq [\theta^{-k-1}, \theta^{-k+1}], \quad \left| \frac{d^s \varphi_k(t)}{dt^s} \right| \leq \frac{C_s}{t^s},$$

where C_s is independent of the lacunary sequence $\{\theta^k : k \in \mathbf{Z}\}$. Define the multiplier operators T_k in $\mathbf{R}^n$ by $(\widehat{T_k f})(\xi) = \varphi_k(|\xi|)\hat{f}(\xi)$. Then for any $f \in \mathcal{S}(\mathbf{R}^n)$ and $k \in \mathbf{Z}$ we have $f(x) = \sum_{j \in \mathbf{Z}} (T_{k+j} f)(x)$. Thus, by Minkowski's inequality we have

$$\begin{aligned} S_{\Omega}^{(2,*)} f(x) &\leq \left(\sum_{k \in \mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \left| \sum_{j \in \mathbf{Z}} \int_{\mathbf{S}^{n-1}} \Omega(u) T_{k+j} f(x - tu) d\sigma(u) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \\ &\leq \sum_{j \in \mathbf{Z}} X_j f(x), \end{aligned}$$

where

$$X_j f(x) = \left(\sum_{k \in \mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega(u) T_{k+j} f(x - tu) d\sigma(u) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.$$

We notice that to prove (3.1) for the case $\gamma = 2$, it is enough to prove that the inequality

$$\|X_j(f)\|_p \leq C_p (q-1)^{-\frac{1}{2}} \|\Omega\|_q 2^{-\delta_p |j|} \|f\|_p \tag{3.2}$$

holds for $1 < q \leq 2$, $(nq'\gamma')/(n\gamma' + nq' - \gamma') < p < \infty$ and for some positive constants C_p and δ_p. This inequality can be proved by interpolation between a sharp L^2 estimate and a cruder L^p estimate. We prove (3.2) first for the case $p = 2$. By using Plancherel's theorem we have

$$\|X_j(f)\|_2^2 = \int_{\Delta_{k+l}} \sum_{k \in \mathbf{Z}} (H_k(\xi))^2 \left| \hat{f}(\xi) \right|^2 \frac{dt}{t} d\xi, \tag{3.3}$$

where $\Delta_k = \left\{ \xi \in \mathbf{R}^n : \theta^{-k-1} \leq |\xi| \leq \theta^{-k+1} \right\}$ and

$$H_k(\xi) = \left(\int_{\theta^k}^{\theta^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega(x) e^{-it\xi \cdot x} d\sigma(x) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}. \tag{3.4}$$

We claim that

$$|H_k(\xi)| \leq C (\log \theta)^{\frac{1}{2}} \|\Omega\|_q \min \left\{ \left| \theta^k \xi \right|^{-\frac{\lambda}{q'}}, \left| \theta^{k+1} \xi \right|^{\frac{\lambda}{q'}} \right\}, \tag{3.5}$$

where C is a constant independent of k, ξ and q.

Let us now turn to the proof of (3.5). First, by a change of variable and since

$$\left|\int_{\mathbf{S}^{n-1}} \Omega(x)e^{-i\theta^k t\xi\cdot x}d\sigma(x)\right|^2 = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \Omega(x)\overline{\Omega(y)}e^{-i\theta^k t\xi\cdot(x-y)}d\sigma(x)d\sigma(y),$$

we obtain

$$(H_k(\xi))^2 \le \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \Omega(x)\overline{\Omega(y)}\left(\int_1^\theta e^{-i\theta^k t\xi\cdot(x-y)}\frac{dt}{t}\right)d\sigma(x)d\sigma(y). \tag{3.6}$$

Using an integration by parts,

$$\left|\int_1^\theta e^{-i\theta^k t\xi\cdot(x-y)}\frac{dt}{t}\right| \le C(\log\theta)\left|\theta^k\xi\right|^{-1}\left|\xi'\cdot(x-y)\right|^{-1}.$$

By combining this estimate with the trivial estimate $\left|\int_1^\theta e^{-i\theta^k t\xi\cdot(x-y)}\frac{dt}{t}\right| \le (\log\theta)$, we get

$$\left|\int_1^\theta e^{-i\theta^k t\xi\cdot(x-y)}\frac{dt}{t}\right| \le C(\log\theta)\left|\theta^k\xi\right|^{-\alpha}\left|\xi'\cdot(x-y)\right|^{-\alpha} \quad \text{for any } 0<\alpha\le 1.$$

Thus, by the last inequality, (3.6) and Hölder's inequality we get

$$|H_k(\xi)| \le C(\log\theta)^{\frac{1}{2}}\|\Omega\|_q\left|\theta^k\xi\right|^{-\frac{\alpha}{2q'}}\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}\frac{d\sigma(x)d\sigma(y)}{\left|\xi'\cdot(x-y)\right|^{\alpha q'}}\right)^{\frac{1}{2q'}}.$$

By choosing α so that $\alpha q'<1$ we obtain that the last integral is bounded in $\xi'\in\mathbf{S}^{n-1}$ and hence

$$|H_k(\xi)| \le C(\log\theta)^{\frac{1}{2}}\|\Omega\|_q\left|\theta^k\xi\right|^{-\frac{\alpha}{2q'}}. \tag{3.7}$$

Secondly, by the cancellation conditions on Ω we obtain

$$\begin{aligned} H_k(\xi) &\le \left(\int_{\theta^k}^{\theta^{k+1}}\left(\int_{\mathbf{S}^{n-1}}|\Omega(x)|\left|e^{-it\xi\cdot x}-1\right|d\sigma(x)\right)^2\frac{dt}{t}\right)^{\frac{1}{2}} \\ &\le C(\log\theta)^{\frac{1}{2}}\|\Omega\|_1\left|\theta^{k+1}\xi\right|. \end{aligned}$$

By combining the last estimate with the estimate $|H_k(\xi)|\le C(\log\theta)^{1/2}\|\Omega\|_1$, we get

$$|H_k(\xi)| \le C(\log\theta)^{\frac{1}{2}}\|\Omega\|_1\left|\theta^{k+1}\xi\right|^{\frac{\alpha}{2q'}}. \tag{3.8}$$

By combining the estimates (3.7)–(3.8), we obtain (3.5).

Now, by (3.4) and (3.5) we obtain

$$\|X_j(f)\|_2 \le C2^{-\lambda|j|}(q-1)^{-\frac{1}{2}}\|\Omega\|_q\|f\|_2. \tag{3.9}$$

Let us now estimate $\|X_j(f)\|_p$ for $(nq'\gamma')/(\gamma'n+nq'-\gamma')<p<\infty$ with $p\ne 2$. Let us first consider the case $p>2$. By duality, there is a nonnegative function g in $L^{(p/2)'}(\mathbf{R}^n)$ with $\|g\|_{(p/2)'}\le 1$ such that

$$\|X_j(f)\|_p^2 = \int_{\mathbf{R}^n}\sum_{k\in\mathbf{Z}}\int_{\theta^k}^{\theta^{k+1}}\left|\int_{\mathbf{S}^{n-1}}\Omega(u)T_{k+j}f(x-tu)d\sigma(u)\right|^2\frac{dt}{t}g(x)dx.$$

By Hölder's inequality, Fubini's theorem and a change of variable we have

$$\begin{aligned}\|X_j(f)\|_p^2 &\leq \|\Omega\|_1 \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \int_{\mathbf{S}^{n-1}} |\Omega(u)| \left|T_{k+j}f(x-tu)\right|^2 g(x) d\sigma(u) \frac{dt}{t} dx \\ &\leq \|\Omega\|_1 \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_{\theta^k}^{\theta^{k+1}} \int_{\mathbf{S}^{n-1}} |\Omega(u)| \left|T_{k+j}f(x)\right|^2 g(x+tu) d\sigma(u) \frac{dt}{t} dx \\ &\leq C\|\Omega\|_1 \int_{\mathbf{R}^n} \left(\sum_{k\in\mathbf{Z}} \left|T_{k+j}f(x)\right|^2\right) \sigma^*_{\Omega,1,\theta}(\tilde{g})(-x)dx,\end{aligned}$$

where $\tilde{g}(x) = g(-x)$ and $\sigma^*_{\Omega,h,\theta}$ is defined as in (2.2). By the last inequality, Lemma 2.2 and using Littlewood-Paley theory ([24, p. 96]) we get

$$\begin{aligned}\|X_j(f)\|_p^2 &\leq C\|\Omega\|_1 \left\|\sigma^*_{\Omega,1,\theta}(\tilde{g})\right\|_{(p/2)'} \left\|\sum_{k\in\mathbf{Z}} \left|T_{k+j}f\right|^2\right\|_{(p/2)} \\ &\leq C_p(\log\theta)\|\Omega\|_1^2 \|f\|_p^2,\end{aligned}$$

which in turn easily implies

$$\|X_j(f)\|_p \leq C_p(q-1)^{-1/2} \|\Omega\|_q \|f\|_p \quad \text{for } 2<p<\infty. \tag{3.10}$$

By interpolating between (3.9) – (3.10) we get (3.2) for the case $2 \leq p < \infty$. Now, let us estimate $\|X_j(f)\|_p$ for the case $(nq'\gamma')/(\gamma' n + nq' - \gamma') < p < 2$. By a change of variable we get

$$X_j f(x) = \left(\sum_{k\in\mathbf{Z}} \int_1^\theta \left|\int_{\mathbf{S}^{n-1}} \Omega(u) T_{k+j}f(x-\theta^k tu) d\sigma(u)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}}.$$

By duality, there is a function $h = h_{k,j}(x,t)$ satisfying $\|h\| \leq 1$ and

$$h_{k,j}(x,t) \in L^{p'}\left(l^2\left[L^2\left([1,\theta], \frac{dt}{t}\right), k\right], dx\right)$$

such that

$$\begin{aligned}\|X_j(f)\|_p &= \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_1^\theta \int_{\mathbf{S}^{n-1}} \Omega(u) \left(T_{k+j}f\right)(x-\theta^k tu) h_{k,j}(x,t) d\sigma(u) \frac{dt}{t} dx \\ &= \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_1^\theta \int_{\mathbf{S}^{n-1}} \Omega(u) \left(T_{k+j}f\right)(x) h_{k,j}(x+\theta^k tu, t) d\sigma(u) \frac{dt}{t} dx.\end{aligned}$$

By Hölder's inequality and Littlewood-Paley theory we get

$$\begin{aligned}\|X_j(f)\|_p &\leq \left\|(L(h))^{\frac{1}{2}}\right\|_{p'} \left\|\left(\sum_{k\in\mathbf{Z}} \left|T_{k+j}f\right|^2\right)^{\frac{1}{2}}\right\|_p \\ &\leq C_p\|f\|_p \left\|(L(h))^{\frac{1}{2}}\right\|_{p'},\end{aligned} \tag{3.11}$$

where

$$L(h) = \sum_{k\in\mathbf{Z}} \left(\int_1^\theta \int_{\mathbf{S}^{n-1}} \Omega(u) h_{k,j}(x+\theta^k tu, t) d\sigma(u) \frac{dt}{t}\right)^2.$$

Since $p' > 2$ and $\left\|(L(h))^{1/2}\right\|_{p'} = \|L(h)\|_{p'/2}^{1/2}$, there is a nonnegative function $b \in L^{(p'/2)'}(\mathbf{R}^n)$ such that $\|b\|_{(p'/2)'} \leq 1$ and

$$\|L(h)\|_{p'/2} = \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \left(\int_1^\theta \int_{\mathbf{S}^{n-1}} \Omega(u) h_{k,j}(x+\theta^k tu, t) d\sigma(u) \frac{dt}{t}\right)^2 b(x)dx.$$

By the Schwarz inequality and Lemma 2.4, we obtain

$$\begin{aligned}
&\left(\int_1^\theta \int_{\mathbf{S}^{n-1}} \Omega(u) h_{k,j}(x+\theta^k tu, t) d\sigma(u) \frac{dt}{t}\right)^2 \\
&\leq C(\log\theta) \int_1^\theta \left(\int_{\mathbf{S}^{n-1}} \Omega(u) h_{k,j}(x+\theta^k tu, t) d\sigma(u)\right)^2 \frac{dt}{t} \\
&\leq C(\log\theta) \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^{\min\{2,q\}} \int_1^\theta \int_{\mathbf{S}^{n-1}} |\Omega(u)|^{\max\{0,2-q\}} \left|h_{k,j}(x+\theta^k tu, t)\right|^2 d\sigma(u) \frac{dt}{t}.
\end{aligned}$$

Therefore, by the last inequality, a change of variable, Fubini's theorem, Hölder's inequality, and Lemma 2.4 we get

$$\begin{aligned}
\|L(h)\|_{p'/2} &\leq C(\log\theta) \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^{\min\{2,q\}} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^{\max\{0,2-q\}} \\
&\quad \times \int_{\mathbf{R}^n} \left(\sum_{k\in\mathbf{Z}} \int_1^\theta |h_{k,j}(x,t)|^2 \frac{dt}{t}\right) \left(\mathcal{M}_{Sph}\left(\left|\tilde{b}\right|^{q'/2}\right)(-x)\right)^{\frac{2}{q'}} dx \\
&\leq C(\log\theta) \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2 \left\|\sum_{k\in\mathbf{Z}} \int_1^\theta |h_{k,j}(x,t)|^2 \frac{dt}{t}\right\|_{p'/2} \\
&\quad \times \left\|\left(\mathcal{M}_{Sph}\left(\left|\tilde{b}\right|^{q'/2}\right)\right)^{2/q'}\right\|_{(p'/2)'}.
\end{aligned}$$

By the condition on p we have $(2/q')(p'/2)' > n/(n-1)$ and hence by the L^p boundedness of $\mathcal{M}_{Sph}$ and the choice of b we obtain

$$\|L(h)\|_{p'/2} \leq C(q-1)^{-1} \|\Omega\|_{L^q(\mathbf{S}^{n-1})}^2.$$

Using the last inequality and (3.11) we have

$$\|X_j(f)\|_p \leq C_p(q-1)^{-1/2} \|\Omega\|_q \|f\|_p \qquad \text{for } (nq'\gamma')/(\gamma' n + nq' - \gamma') < p < 2. \tag{3.12}$$

Thus, by (3.9) and (3.12) and interpolation we get (3.2) for $(nq'\gamma')/(\gamma' n + nq' - \gamma') < p < 2$. This ends the proof of (3.1) for the case $\gamma = 2$.

Proof of (3.1) for the case $1 < \gamma < 2$.

We will use an idea that appeared in [19]. By duality we have

$$\left\|S_\Omega^{(\gamma,*)}(f)\right\|_{L^p(\mathbf{R}^n)} = \|F(f)\|_{L^p(L^{\gamma'}(\mathbf{R}_+,\frac{dt}{t}),\mathbf{R}^n,dx)},$$

where $F : L^p(\mathbf{R}^n) \to L^p(L^{\gamma'}(\mathbf{R}_+, \frac{dt}{t}), \mathbf{R}^n)$ is a linear operator defined by

$$F(f)(x;t) = \int_{\mathbf{S}^{n-1}} f(x-tu)\Omega(u)d\sigma(u).$$

From the inequalities (3.1) (for the case $\gamma = 2$) and (3.1) (for the case $\gamma = 1$), we interpret that

$$\|F(f)\|_{L^p(L^2(\mathbf{R}_+,\frac{dt}{t}),\mathbf{R}^n)} \leq C(q-1)^{-\frac{1}{2}} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_{L^p(\mathbf{R}^n)}$$

for $(2nq')/(2n+nq'-2) < p < \infty$ and

$$\|F(f)\|_{L^p(L^\infty(\mathbf{R}_+,\frac{dt}{t}),\mathbf{R}^n)} \le C\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\,\|f\|_{L^p(\mathbf{R}^n)} \tag{3.13}$$

for $q'n' \le p < \infty$. Applying the real interpolation theorem for Lebesgue mixed normed spaces to the above results (see [11]), we conclude that

$$\|F(f)\|_{L^p(L^{\gamma'}(\mathbf{R}_+,\frac{dt}{t}),\mathbf{R}^n)} \le C_p(q-1)^{-1/\gamma'}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1})}\,\|f\|_{L^p(\mathbf{R}^n)}$$

holds for $1 < q \le 2$, $(nq'\gamma')/(n\gamma'+nq'-\gamma') < p < \infty$ and $1 \le \gamma \le 2$. This completes the proof of Theorem 1.4. □

Next we shall present the proof of Theorem 1.5.

Proof. We need to consider two cases.

Case 1. $1 \le \gamma \le 2$. By an argument appearing in [6], we may, without loss of generality, in the definition of $\mathcal{M}_{\Omega,h}$, replace $|y| \le t$ with $\frac{1}{2}t \le |y| \le t$. By Minkowski's inequality and Fubini's theorem we get

$$\begin{aligned}
&\left(\int_0^\infty \left|\int_{\frac{1}{2}t\le|y|\le t} f(x-y)\frac{\Omega(y')}{|y|^{n-\rho}}h(|y|)dy\right|^2 \frac{dt}{t^{1+2\sigma}}\right)^{\frac{1}{2}}\\
&\le \left(\int_0^\infty \left(\int_0^\infty \left|\int_{\mathbf{S}^{n-1}} f(x-sy)\Omega(y)d\sigma(y)\right| |h(s)|\,\chi_{[\frac{1}{2}t,t]}(s)\frac{ds}{s^{1-\sigma}}\right)^2 \frac{dt}{t^{1+2\sigma}}\right)^{\frac{1}{2}}\\
&\le \int_0^\infty \left(\int_0^\infty \left|\int_{\mathbf{S}^{n-1}} f(x-sy)\Omega(y)d\sigma(y)\right|^2 |h(s)|^2\,\chi_{[\frac{1}{2}t,t]}(s)\frac{dt}{t^{1+2\sigma}}\right)^{\frac{1}{2}} \frac{ds}{s^{1-\sigma}}\\
&= \int_0^\infty \left|\int_{\mathbf{S}^{n-1}} f(x-sy)\Omega(y)d\sigma(y)\right| |h(s)| \left(\int_s^\infty \frac{dt}{t^{1+2\sigma}}\right)^{\frac{1}{2}} \frac{ds}{s^{1-\sigma}}\\
&= \frac{1}{\sqrt{2\sigma}}\int_0^\infty \left|\int_{\mathbf{S}^{n-1}} f(x-sy)\Omega(y)d\sigma(y)\right| |h(s)|\,\frac{ds}{s}.
\end{aligned}$$

Therefore, by duality we have

$$\mathcal{M}_\Omega^{(\gamma,*)}f(x) \le \frac{1}{\sqrt{2\sigma}}S_\Omega^{(\gamma,*)}f(x), \tag{3.14}$$

and hence (1.7) holds by (1.6) for $1 \le \gamma \le 2$.

Case 2. $2 < \gamma \le \infty$. By a change of variable and duality we have

$$\mathcal{M}_\Omega^{(\gamma,*)}f(x) \le \left(\int_0^\infty \left(\int_{1/2}^1 \left|\int_{\mathbf{S}^{n-1}} f(x-stu)\Omega\left(u\right)d\sigma(u)\right|^{\gamma'} \frac{ds}{s}\right)^{2/\gamma'} \frac{dt}{t}\right)^{\frac{1}{2}}.$$

Using the generalized Minkowski inequality and since $\gamma' < 2$, we get

$$\mathcal{M}_\Omega^{(\gamma,*)}f(x) \le \left(\int_{1/2}^1 |E_sf(x)|^{\gamma'}\,\frac{ds}{s}\right)^{1/\gamma'}, \tag{3.15}$$

where

$$E_sf(x) = \left(\int_0^\infty \left|\int_{\mathbf{S}^{n-1}} f(x-stu)\Omega\left(u\right)d\sigma(u)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}}.$$

Since $E_s f(x) = S_\Omega^{(2,*)} f(x)$, we obtain

$$\mathcal{M}_\Omega^{(\gamma,*)} f(x) \leq S_\Omega^{(2,*)} f(x), \tag{3.16}$$

and again (1.7) holds for the case $2 < \gamma \leq \infty$ by (1.6) in the case $\gamma = 2$. □

We now give the proof of Theorem 1.6.

Proof. For part (a), we need to consider two cases.

Case 1. $1 < \gamma \leq 2$.

By definition of $S_{\Omega,h} f(x)$ we have $S_{\Omega,h} f(x) = \lim_{\varepsilon \to 0} S_{\Omega,h}^{(\varepsilon)} f(x)$ for $f \in \mathcal{S}(\mathbf{R}^n)$, where $S_{\Omega,h}^{(\varepsilon)}$ is the truncated singular integral operator given by

$$S_{\Omega,h}^{(\varepsilon)} f(x) = \int_{|u|>\varepsilon} \frac{\Omega(u)}{|u|^n} h(|u|) f(x-u) du.$$

By Hölder's inequality and duality we have

$$\begin{aligned} \left| S_{\Omega,h}^{(\varepsilon)} f(x) \right| &\leq \int_\varepsilon^\infty |h(t)| \left| \int_{\mathbf{S}^{n-1}} f(x-tu) \Omega(u)\, d\sigma(u) \right| \frac{dt}{t} \\ &\leq \|h\|_{L^\gamma(\mathbf{R}_+, dt/t)} S_\Omega^{(*,\gamma)} f(x). \end{aligned}$$

Thus, by Theorem 1.4 we get

$$\left\| S_{\Omega,h}^{(\varepsilon)}(f) \right\|_p \leq C_p (q-1)^{-1/\gamma'} \|h\|_{L^\gamma(\mathbf{R}_+, dt/t)} \|\Omega\|_{L^q(\mathbf{S}^{n-1})} \|f\|_p \tag{3.17}$$

for $(nq'\gamma')/(n\gamma' + nq' - \gamma') < p < \infty$ and $1 < \gamma \leq 2$ and for some positive constant C_p independent of ε. In particular, (3.17) holds for $2 \leq p < \infty$ and $1 < \gamma \leq 2$. By duality (3.17) also holds for $1 < p \leq 2$ and $1 < \gamma \leq 2$. Using Fatou's lemma and (3.17) we get (1.8) for $1 < p < \infty$ and $1 < \gamma \leq 2$.

Case 2. $2 < \gamma \leq \infty$. Write $S_{\Omega,h}^{(\varepsilon)}(f) = \sum_{k \in \mathbf{Z}} \sigma_{\Omega,h,\theta,k} * f$. By Hölder's inequality we have

$$\begin{aligned} |\hat{\sigma}_{\Omega,h,\theta,k}(\xi)| &\leq \|h\|_{L^\gamma(\mathbf{R}_+, dt/t)} \left(\int_{\theta^k}^{\theta^{k+1}} \left| \int_{\mathbf{S}^{n-1}} \Omega(x) e^{-it\xi \cdot x} d\sigma(x) \right|^{\gamma'} \frac{dt}{t} \right)^{\frac{1}{\gamma'}} \\ &\leq (\log \theta)^{\frac{2-\gamma'}{2\gamma'}} \|h\|_{L^\gamma(\mathbf{R}_+, dt/t)} H_k(\xi), \end{aligned}$$

where $H_k(\xi)$ is defined as in (3.5). Thus, by the last inequality and (3.5) we obtain

$$|\hat{\sigma}_{\Omega,h,\theta,k}(\xi)| \leq C (\log \theta)^{1/\gamma'} \|\Omega\|_q \min \left\{ \left| \theta^k \xi \right|^{-\frac{\lambda}{q'}}, \left| \theta^{k+1} \xi \right|^{\frac{\lambda}{q'}} \right\}. \tag{3.18}$$

By Lemma 2.3, (3.18) and invoking Lemma 2.1 with $\theta = 2^{q'}$, $a_k = \theta^k$ we obtain (1.8) for $2 < \gamma \leq \infty$ and $\gamma' < p < \infty$ with C_p independent of ε. Since $\gamma' < 2$, we get (1.8) for $2 \leq p < \infty$ and $2 < \gamma \leq \infty$. By duality and Fatou's lemma we obtain (1.8) for $1 < p < \infty$ and $2 < \gamma \leq \infty$. This completes the proof of (1.8).

The proof of (1.9) follows immediately from Theorem 1.5 and

$$\mathcal{M}_{\Omega,h} f(x) \leq \mathcal{M}_\Omega^{(\gamma,*)} f(x)$$

for any $h \in L^\gamma(\mathbf{R}_+, dt/t)$, $1 < \gamma < \infty$. □

Proofs of Theorems 1.7 and 1.8. A proof of each of these theorems follows by Theorems 1.4 – 1.6 and an extrapolation argument. For more details, see [6], [21]. □

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