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SOME INEQUALITIES FOR THE DISPERSION OF A RANDOM VARIABLE WHOSE PDF IS DEFINED ON A FINITE INTERVAL

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ABSTRACT. Some inequalities for the dispersion of a random variable whose pdf is defined on a finite interval and applications are given.

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1. INTRODUCTION

In this note we obtain some inequalities for the dispersion of a continuous random variable X having the probability density function (p.d.f.) f defined on a finite interval $[a, b]$.

Tools used include: Korkine's identity, which plays a central role in the proof of Chebychev's integral inequality for synchronous mappings [24], Hölder's weighted inequality for double integrals and an integral identity connecting the variance $\sigma^2(X)$ and the expectation $E(X)$. Perturbed results are also obtained by using Grüss, Chebyshev and Lupaş inequalities. In Section 4, results from an identity involving a double integral are obtained for a variety of norms.

2. SOME INEQUALITIES FOR DISPERSION

Let $f : [a, b] \subset \mathbb{R} \to \mathbb{R}_+$ be the p.d.f. of the random variable X and

$$E(X) := \int_a^b t f(t)\, dt$$

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its *expectation* and

$$\sigma(X)=\left[\int_a^b (t-E(X))^2 f(t)\,dt\right]^{\frac{1}{2}}=\left[\int_a^b t^2 f(t)\,dt-[E(X)]^2\right]^{\frac{1}{2}}$$

its *dispersion* or *standard deviation.*

The following theorem holds.

Theorem 2.1. *With the above assumptions, we have*

$$0\le\sigma(X)\le\begin{cases}\frac{\sqrt{3}(b-a)^2}{6}\|f\|_\infty, & \textit{provided}\quad f\in L_\infty,[a,b];\\[2ex] \frac{\sqrt{2}(b-a)^{1+\frac{1}{q}}}{2[(q+1)(2q+1)]^{\frac{2}{q}}}\|f\|_p, & \textit{provided}\quad f\in L_p[a,b]\\ & \textit{and}\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \frac{\sqrt{2}(b-a)}{2}. & \end{cases}\tag{2.1}$$

Proof. Korkine's identity [24], is

$$\int_a^b p(t)\,dt\int_a^b p(t)g(t)h(t)\,dt-\int_a^b p(t)g(t)\,dt\cdot\int_a^b p(t)h(t)\,dt$$
$$=\frac{1}{2}\int_a^b\int_a^b p(t)p(s)(g(t)-g(s))(h(t)-h(s))\,dtds,\tag{2.2}$$

which holds for the measurable mappings $p,g,h:[a,b]\to\mathbb{R}$ for which the integrals involved in (2.2) exist and are finite. Choose in (2.2) $p(t)=f(t)$, $g(t)=h(t)=t-E(X)$, $t\in[a,b]$ to get

$$\sigma^2(X)=\frac{1}{2}\int_a^b\int_a^b f(t)f(s)(t-s)^2\,dtds\,.\tag{2.3}$$

It is obvious that

$$\begin{aligned}\int_a^b\int_a^b f(t)f(s)(t-s)^2\,dtds &\le \sup_{(t,s)\in[a,b]^2}|f(t)f(s)|\int_a^b\int_a^b (t-s)^2\,dtds\\ &= \frac{(b-a)^4}{6}\|f\|_\infty^2\end{aligned}\tag{2.4}$$

and then, by (2.3), we obtain the first part of (2.1).

For the second part, we apply Hölder's integral inequality for double integrals to obtain

$$\begin{aligned}\int_a^b\int_a^b f(t)f(s)(t-s)^2\,dtds &\le \left(\int_a^b\int_a^b f^p(t)f^p(s)\,dtds\right)^{\frac{1}{p}}\left(\int_a^b\int_a^b (t-s)^{2q}\,dtds\right)^{\frac{1}{q}}\\ &= \|f\|_p^2\left[\frac{(b-a)^{2q+2}}{(q+1)(2q+1)}\right]^{\frac{1}{q}},\end{aligned}$$

where $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$, and the second inequality in (2.1) is proved.

For the last part, observe that

$$\int_a^b\int_a^b f(t)f(s)(t-s)^2\,dtds\le\sup_{(t,s)\in[a,b]^2}(t-s)^2\int_a^b\int_a^b f(t)f(s)\,dtds=(b-a)^2$$

as

$$\int_a^b\int_a^b f(t)f(s)\,dtds=\int_a^b f(t)\,dt\int_a^b f(s)\,ds=1.$$

□

Using a finer argument, the last inequality in (2.1) can be improved as follows.

Theorem 2.2. *Under the above assumptions, we have*

$$0 \leq \sigma(X) \leq \frac{1}{2}(b-a). \tag{2.5}$$

Proof. We use the following Grüss type inequality:

$$0 \leq \frac{\int_a^b p(t)\, g^2(t)\, dt}{\int_a^b p(t)\, dt} - \left(\frac{\int_a^b p(t)\, g(t)\, dt}{\int_a^b p(t)\, dt}\right)^2 \leq \frac{1}{4}(M-m)^2, \tag{2.6}$$

provided that p, g are measurable on $[a,b]$ and all the integrals in (2.6) exist and are finite, $\int_a^b p(t)\, dt > 0$ and $m \leq g \leq M$ a.e. on $[a,b]$. For a proof of this inequality see [19].

Choose in (2.6), $p(t) = f(t)$, $g(t) = t - E(X)$, $t \in [a,b]$. Observe that in this case $m = a - E(X)$, $M = b - E(X)$ and then, by (2.6) we deduce (2.5). □

Remark 2.3. The same conclusion can be obtained for the choice $p(t) = f(t)$ and $g(t) = t$, $t \in [a,b]$.

The following result holds.

Theorem 2.4. *Let X be a random variable having the p.d.f. given by $f : [a,b] \subset \mathbb{R} \to \mathbb{R}_+$. Then for any $x \in [a,b]$ we have the inequality:*

$$\sigma^2(X) + (x - E(X))^2 \leq \begin{cases} (b-a)\left[\frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2\right]\|f\|_\infty, & \textit{provided } f \in L_\infty[a,b]; \\ \left[\frac{(b-x)^{2q+1}+(x-a)^{2q+1}}{2q+1}\right]^{\frac{1}{q}}\|f\|_p, & \textit{provided } f \in L_p[a,b],\ p>1, \textit{ and } \frac{1}{p}+\frac{1}{q}=1; \\ \left(\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right)^2. & \end{cases} \tag{2.7}$$

Proof. We observe that

$$\int_a^b (x-t)^2 f(t)\, dt = \int_a^b \left(x^2 - 2xt + t^2\right) f(t)\, dt = x^2 - 2xE(X) + \int_a^b t^2 f(t)\, dt \tag{2.8}$$

and as

$$\sigma^2(X) = \int_a^b t^2 f(t)\, dt - [E(X)]^2, \tag{2.9}$$

we get, by (2.8) and (2.9),

$$[x - E(X)]^2 + \sigma^2(X) = \int_a^b (x-t)^2 f(t)\, dt, \tag{2.10}$$

which is of interest in itself too.

We observe that

$$\begin{aligned}\int_a^b (x-t)^2 f(t)\,dt &\leq ess \sup_{t\in[a,b]} |f(t)| \int_a^b (x-t)^2\,dt \\ &= \|f\|_\infty \frac{(b-x)^3+(x-a)^3}{3} \\ &= (b-a)\|f\|_\infty \left[\frac{(b-a)^2}{12}+\left(x-\frac{a+b}{2}\right)^2\right]\end{aligned}$$

and the first inequality in (2.7) is proved.

For the second inequality, observe that by Hölder's integral inequality,

$$\begin{aligned}\int_a^b (x-t)^2 f(t)\,dt &\leq \left(\int_a^b f^p(t)\,dt\right)^{\frac{1}{p}} \left(\int_a^b (x-t)^{2q}\,dt\right)^{\frac{1}{q}} \\ &= \|f\|_p \left[\frac{(b-x)^{2q+1}+(x-a)^{2q+1}}{2q+1}\right]^{\frac{1}{q}},\end{aligned}$$

and the second inequality in (2.7) is established.

Finally, observe that,

$$\begin{aligned}\int_a^b (x-t)^2 f(t)\,dt &\leq \sup_{t\in[a,b]} (x-t)^2 \int_a^b f(t)\,dt \\ &= \max\left\{(x-a)^2,(b-x)^2\right\} \\ &= (\max\{x-a,b-x\})^2 \\ &= \left(\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right)^2,\end{aligned}$$

and the theorem is proved. □

The following corollaries are easily deduced.

Corollary 2.5. *With the above assumptions, we have*

(2.11)

$$0 \leq \sigma(X) \leq \begin{cases} (b-a)^{\frac{1}{2}} \left[\frac{(b-a)^2}{12}+\left(E(X)-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}} \|f\|_\infty^{\frac{1}{2}}, & \textit{provided } f\in L_\infty[a,b]; \\ \left[\frac{(b-E(X))^{2q+1}+(E(X)-a)^{2q+1}}{2q+1}\right]^{\frac{1}{2q}} \|f\|_p^{\frac{1}{2}}, & \textit{if } f\in L_p[a,b],\ p>1 \textit{ and } \frac{1}{p}+\frac{1}{q}=1; \\ \frac{b-a}{2}+\left|E(X)-\frac{a+b}{2}\right|. \end{cases}$$

Remark 2.6. The last inequality in (2.11) is worse than the inequality (2.5), obtained by a technique based on Grüss' inequality.

The best inequality we can get from (2.7) is that one for which $x=\frac{a+b}{2}$, and this applies for all the bounds since

$$\begin{aligned}\min_{x\in[a,b]} \left[\frac{(b-a)^2}{12}+\left(x-\frac{a+b}{2}\right)^2\right] &= \frac{(b-a)^2}{12}, \\ \min_{x\in[a,b]} \frac{(b-x)^{2q+1}+(x-a)^{2q+1}}{2q+1} &= \frac{(b-a)^{2q+1}}{2^{2q}(2q+1)},\end{aligned}$$

and

$$\min_{x\in[a,b]}\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]=\frac{b-a}{2}.$$

Consequently, we can state the following corollary as well.

Corollary 2.7. *With the above assumptions, we have the inequality:*

$$\begin{aligned} 0 &\leq \sigma^2(X)+\left[E(X)-\frac{a+b}{2}\right]^2 \\ &\leq \begin{cases} \frac{(b-a)^3}{12}\|f\|_\infty, & \text{provided } f\in L_\infty[a,b]; \\ \frac{(b-a)^{2q+1}}{4(2q+1)^{\frac{1}{q}}}\|f\|_p, & \text{if } f\in L_p[a,b],\ p>1, \\ & \text{and } \frac{1}{p}+\frac{1}{q}=1; \\ \frac{(b-a)^2}{4}. \end{cases} \end{aligned} \tag{2.12}$$

Remark 2.8. From the last inequality in (2.12), we obtain

$$0\leq\sigma^2(X)\leq(b-E(X))(E(X)-a)\leq\frac{1}{4}(b-a)^2, \tag{2.13}$$

which is an improvement on (2.5).

3. PERTURBED RESULTS USING GRÜSS TYPE INEQUALITIES

In 1935, G. Grüss (see for example [26]) proved the following integral inequality which gives an approximation for the integral of a product in terms of the product of the integrals.

Theorem 3.1. *Let $h,g:[a,b]\to\mathbb{R}$ be two integrable mappings such that $\phi\leq h(x)\leq\Phi$ and $\gamma\leq g(x)\leq\Gamma$ for all $x\in[a,b]$, where ϕ,Φ,γ,Γ are real numbers. Then,*

$$|T(h,g)|\leq\frac{1}{4}(\Phi-\phi)(\Gamma-\gamma), \tag{3.1}$$

where

$$T(h,g)=\frac{1}{b-a}\int_a^b h(x)g(x)\,dx-\frac{1}{b-a}\int_a^b h(x)\,dx\cdot\frac{1}{b-a}\int_a^b g(x)\,dx \tag{3.2}$$

and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ cannot be replaced by a smaller one.

For a simple proof of this as well as for extensions, generalisations, discrete variants and other associated material, see [25], and [1]-[21] where further references are given.

A 'premature' Grüss inequality is embodied in the following theorem which was proved in [23]. It provides a sharper bound than the above Grüss inequality.

Theorem 3.2. *Let h,g be integrable functions defined on $[a,b]$ and let $d\leq g(t)\leq D$. Then*

$$|T(h,g)|\leq\frac{D-d}{2}|T(h,h)|^{\frac{1}{2}}, \tag{3.3}$$

where $T(h,g)$ is as defined in (3.2).

Theorem 3.2 will now be used to provide a perturbed rule involving the variance and mean of a p.d.f.

3.1. **Perturbed Results Using 'Premature' Inequalities.** In this subsection we develop some perturbed results.

Theorem 3.3. *Let X be a random variable having the p.d.f. given by $f : [a,b] \subset \mathbb{R} \to \mathbb{R}_+$. Then for any $x \in [a,b]$ and $m \le f(x) \le M$ we have the inequality*

$$\begin{aligned} |P_V(x)| &:= \left| \sigma^2(X) + (x - E(X))^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2 \right| \\ &\le \frac{M-m}{2} \cdot \frac{(b-a)^2}{\sqrt{45}} \left[\left(\frac{b-a}{2}\right)^2 + 15\left(x - \frac{a+b}{2}\right)^2 \right]^{\frac{1}{2}} \\ &\le (M-m)\frac{(b-a)^3}{\sqrt{45}}. \end{aligned} \tag{3.4}$$

Proof. Applying the 'premature' Grüss result (3.3) by associating $g(t)$ with $f(t)$ and $h(t) = (x-t)^2$, gives, from (3.1)-(3.3)

$$\left| \int_a^b (x-t)^2 f(t)\,dt - \frac{1}{b-a}\int_a^b (x-t)^2\,dt \cdot \int_a^b f(t)\,dt \right| \le (b-a)\frac{M-m}{2}[T(h,h)]^{\frac{1}{2}}, \tag{3.5}$$

where from (3.2)

$$T(h,h) = \frac{1}{b-a}\int_a^b (x-t)^4\,dt - \left[\frac{1}{b-a}\int_a^b (x-t)^2\,dt\right]^2. \tag{3.6}$$

Now,

$$\frac{1}{b-a}\int_a^b (x-t)^2\,dt = \frac{(x-a)^3 + (b-x)^3}{3(b-a)} = \frac{1}{3}\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2 \tag{3.7}$$

and

$$\frac{1}{b-a}\int_a^b (x-t)^4\,dt = \frac{(x-a)^5 + (b-x)^5}{5(b-a)}$$

giving, for (3.6),

$$45T(h,h) = 9\left[\frac{(x-a)^5 + (b-x)^5}{b-a}\right] - 5\left[\frac{(x-a)^3 + (b-x)^3}{b-a}\right]^2. \tag{3.8}$$

Let $A = x - a$ and $B = b - x$ in (3.8) to give

$$\begin{aligned} 45T(h,h) &= 9\left(\frac{A^5 + B^5}{A+B}\right) - 5\left(\frac{A^3+B^3}{A+B}\right)^2 \\ &= 9\left[A^4 - A^3B + A^2B^2 - AB^3 + B^4\right] - 5\left[A^2 - AB + B^2\right]^2 \\ &= \left(4A^2 - 7AB + 4B^2\right)(A+B)^2 \\ &= \left[\left(\frac{A+B}{2}\right)^2 + 15\left(\frac{A-B}{2}\right)^2\right](A+B)^2. \end{aligned}$$

Using the facts that $A + B = b - a$ and $A - B = 2x - (a+b)$ gives

$$T(h,h) = \frac{(b-a)^2}{45}\left[\left(\frac{b-a}{2}\right)^2 + 15\left(x - \frac{a+b}{2}\right)^2\right] \tag{3.9}$$

and from (3.7)

$$\frac{1}{b-a}\int_a^b (x-t)^2\,dt = \frac{A^3+B^3}{3(A+B)} = \frac{1}{3}\left[A^2-AB+B^2\right] = \frac{1}{3}\left[\left(\frac{A+B}{2}\right)^2+3\left(\frac{A-B}{2}\right)^2\right],$$

giving

$$\frac{1}{b-a}\int_a^b (x-t)^2\,dt = \frac{(b-a)^2}{12}+\left(x-\frac{a+b}{2}\right)^2. \tag{3.10}$$

Hence, from (3.5), (3.9) (3.10) and (2.10), the first inequality in (3.4) results. The coarsest uniform bound is obtained by taking x at either end point. Thus the theorem is completely proved. □

Remark 3.4. The best inequality obtainable from (3.4) is at $x=\frac{a+b}{2}$ giving

$$\left|\sigma^2(X)+\left[E(X)-\frac{a+b}{2}\right]^2-\frac{(b-a)^2}{12}\right| \le \frac{M-m}{12}\frac{(b-a)^3}{\sqrt{5}}. \tag{3.11}$$

The result (3.11) is a tighter bound than that obtained in the first inequality of (2.12) since $0<M-m<2\|f\|_\infty$.

For a symmetric p.d.f. $E(X)=\frac{a+b}{2}$ and so the above results would give bounds on the variance.

The following results hold if the p.d.f $f(x)$ is differentiable, that is, for $f(x)$ absolutely continuous.

Theorem 3.5. *Let the conditions on Theorem 3.1 be satisfied. Further, suppose that f is differentiable and is such that*

$$\|f'\|_\infty := \sup_{t\in[a,b]}|f'(t)| < \infty.$$

Then

$$|P_V(x)| \le \frac{b-a}{\sqrt{12}}\|f'\|_\infty\cdot I(x), \tag{3.12}$$

where $P_V(x)$ is given by the left hand side of (3.4) and,

$$I(x) = \frac{(b-a)^2}{\sqrt{45}}\left[\left(\frac{b-a}{2}\right)^2+15\left(x-\frac{a+b}{2}\right)^2\right]^{\frac{1}{2}}. \tag{3.13}$$

Proof. Let $h,g:[a,b]\to\mathbb{R}$ be absolutely continuous and h',g' be bounded. Then Chebychev's inequality holds (see [23])

$$|T(h,g)| \le \frac{(b-a)^2}{12}\sup_{t\in[a,b]}|h'(t)|\cdot\sup_{t\in[a,b]}|g'(t)|.$$

Matić, Pečarić and Ujević [23] using a '*premature*' Grüss type argument proved that

$$|T(h,g)| \le \frac{(b-a)}{\sqrt{12}}\sup_{t\in[a,b]}|g'(t)|\sqrt{T(h,h)}. \tag{3.14}$$

Associating $f(\cdot)$ with $g(\cdot)$ and $(x-\cdot)^2$ with $h(\cdot)$ in (3.13) gives, from (3.5) and (3.9), $I(x)=(b-a)[T(h,h)]^{\frac{1}{2}}$, which simplifies to (3.13) and the theorem is proved. □

Theorem 3.6. *Let the conditions of Theorem 3.3 be satisfied. Further, suppose that f is locally absolutely continuous on (a,b) and let $f' \in L_2(a,b)$. Then*

$$|P_V(x)| \le \frac{b-a}{\pi} \|f'\|_2 \cdot I(x), \tag{3.15}$$

where $P_V(x)$ is the left hand side of (3.4) and $I(x)$ is as given in (3.13).

Proof. The following result was obtained by Lupaş (see [23]). For $h,g : (a,b) \to \mathbb{R}$ locally absolutely continuous on (a,b) and $h', g' \in L_2(a,b)$, then

$$|T(h,g)| \le \frac{(b-a)^2}{\pi^2} \|h'\|_2^{\dagger} \|g'\|_2^{\dagger},$$

where

$$\|k\|_2^{\dagger} := \left(\frac{1}{b-a} \int_a^b |k(t)|^2 \right)^{\frac{1}{2}} \quad \text{for } k \in L_2(a,b).$$

Matić, Pečarić and Ujević [23] further show that

$$|T(h,g)| \le \frac{b-a}{\pi} \|g'\|_2^{\dagger} \sqrt{T(h,h)}. \tag{3.16}$$

Associating $f(\cdot)$ with $g(\cdot)$ and $(x-\cdot)^2$ with h in (3.16) gives (3.15), where $I(x)$ is as found in (3.13), since from (3.5) and (3.9), $I(x) = (b-a)[T(h,h)]^{\frac{1}{2}}$. □

3.2. **Alternate Grüss Type Results for Inequalities Involving the Variance.** Let

$$S(h(x)) = h(x) - \mathcal{M}(h) \tag{3.17}$$

where

$$\mathcal{M}(h) = \frac{1}{b-a} \int_a^b h(u)\,du. \tag{3.18}$$

Then from (3.2),

$$T(h,g) = \mathcal{M}(hg) - \mathcal{M}(h)\mathcal{M}(g). \tag{3.19}$$

Dragomir and McAndrew [19] have shown, that

$$T(h,g) = T(S(h), S(g)) \tag{3.20}$$

and proceeded to obtain bounds for a trapezoidal rule. Identity (3.20) is now applied to obtain bounds for the variance.

Theorem 3.7. *Let X be a random variable having the p.d.f. $f : [a,b] \subset \mathbb{R} \to \mathbb{R}_+$. Then for any $x \in [a,b]$ the following inequality holds, namely,*

$$|P_V(x)| \le \frac{8}{3} \nu^3(x) \left\| f(\cdot) - \frac{1}{b-a} \right\|_\infty \quad \text{if } f \in L_\infty[a,b], \tag{3.21}$$

where $P_V(x)$ is as defined by the left hand side of (3.4), and $\nu = \nu(x) = \frac{1}{3}\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2$.

Proof. Using identity (3.20), associate with $h(\cdot)$, $(x-\cdot)^2$ and $f(\cdot)$ with $g(\cdot)$. Then

$$\begin{aligned} &\int_a^b (x-t)^2 f(t)\,dt - \mathcal{M}\left((x-\cdot)^2\right) \\ &\qquad = \int_a^b \left[(x-t)^2 - \mathcal{M}\left((x-\cdot)^2\right)\right] \left[f(t) - \frac{1}{b-a}\right] dt, \end{aligned} \tag{3.22}$$

where from (3.18),

$$\mathcal{M}\left((x-\cdot)^2\right)=\frac{1}{b-a}\int_a^b (x-t)^2\,dt=\frac{1}{3\,(b-a)}\left[(x-a)^3+(b-x)^3\right]$$

and so

$$3\mathcal{M}\left((x-\cdot)^2\right)=\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2. \tag{3.23}$$

Further, from (3.17),

$$S\left((x-\cdot)^2\right)=(x-t)^2-\mathcal{M}\left((x-\cdot)^2\right)$$

and so, on using (3.23)

$$S\left((x-\cdot)^2\right)=(x-t)^2-\frac{1}{3}\left(\frac{b-a}{2}\right)^2-\left(x-\frac{a+b}{2}\right)^2. \tag{3.24}$$

Now, from (3.22) and using (2.10), (3.23) and (3.24), the following identity is obtained

$$\begin{aligned}\sigma^2(X)+[x-E(X)]^2-\frac{1}{3}\left[\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2\right]& \\ =\int_a^b S\left((x-t)^2\right)\left(f(t)-\frac{1}{b-a}\right)dt,&\end{aligned} \tag{3.25}$$

where $S(\cdot)$ is as given by (3.24). Taking the modulus of (3.25) gives

$$|P_V(x)|=\left|\int_a^b S\left((x-t)^2\right)\left(f(t)-\frac{1}{b-a}\right)dt\right|. \tag{3.26}$$

Observe that under different assumptions with regard to the norms of the p.d.f. $f(x)$ we may obtain a variety of bounds.

For $f\in L_\infty[a,b]$ then

$$|P_V(x)|\le\left\|f(\cdot)-\frac{1}{b-a}\right\|_\infty\int_a^b\left|S\left((x-t)^2\right)\right|dt. \tag{3.27}$$

Now, let

$$S\left((x-t)^2\right)=(t-x)^2-\nu^2=(t-X_-)(t-X_+), \tag{3.28}$$

where

$$\nu^2=\mathcal{M}\left((x-\cdot)^2\right)=\frac{(x-a)^3+(b-x)^3}{3\,(b-a)}=\frac{1}{3}\left(\frac{b-a}{2}\right)^2+\left(x-\frac{a+b}{2}\right)^2, \tag{3.29}$$

and

$$X_-=x-\nu,\quad X_+=x+\nu. \tag{3.30}$$

Then,

$$H(t)=\int S\left((x-t)^2\right)dt=\int\left[(t-x)^2-\nu^2\right]dt=\frac{(t-x)^3}{3}-\nu^2t+k \tag{3.31}$$

and so from (3.31) and using (3.28) - (3.29) gives,

$$\begin{aligned}
(3.32)\qquad & \int_a^b \left|S\left((x-t)^2\right)\right| dt \\
&= H(X_-) - H(a) - [H(X_+) - H(X_-)] + [H(b) - H(X_+)] \\
&= 2[H(X_-) - H(X_+)] + H(b) - H(a) \\
&= 2\left\{-\frac{\nu^3}{3} - \nu^2 X_- - \frac{\nu^3}{3} + \nu^2 X_+\right\} + \frac{(b-x)^3}{3} - \nu^2 b + \frac{(x-a)^3}{3} + \nu^2 a \\
&= 2\left[2\nu^3 - \frac{2}{3}\nu^3\right] + \frac{(b-x)^3 + (x-a)^3}{3} - \nu^2(b-a) \\
&= \frac{8}{3}\nu^3.
\end{aligned}$$

Thus, substituting into (3.27), (3.26) and using (3.29) readily produces the result (3.21) and the theorem is proved. □

Remark 3.8. Other bounds may be obtained for $f \in L_p[a,b]$, $p \geq 1$ however obtaining explicit expressions for these bounds is somewhat intricate and will not be considered further here. They involve the calculation of

$$\sup_{t\in[a,b]} \left|(t-x)^2 - \nu^2\right| = \max\left\{\left|(x-a)^2 - \nu^2\right|, \nu^2, \left|(b-x)^2 - \nu^2\right|\right\}$$

for $f \in L_1[a,b]$ and

$$\left(\int_a^b \left|(t-x)^2 - \nu^2\right|^q dt\right)^{\frac{1}{q}}$$

for $f \in L_p[a,b]$, $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, where ν^2 is given by (3.29).

4. SOME INEQUALITIES FOR ABSOLUTELY CONTINUOUS P.D.F.S

We start with the following lemma which is interesting in itself.

Lemma 4.1. *Let X be a random variable whose probability density function $f : [a,b] \to \mathbb{R}_+$ is absolutely continuous on $[a,b]$. Then we have the identity*

$$\begin{aligned}
(4.1)\quad & \sigma^2(X) + [E(X) - x]^2 \\
&= \frac{(b-a)^2}{12} + \left(x - \frac{a+b}{2}\right)^2 + \frac{1}{b-a}\int_a^b\int_a^b (t-x)^2 p(t,s) f'(s)\,dsdt,
\end{aligned}$$

where the kernel $p : [a,b]^2 \to \mathbb{R}$ is given by

$$p(t,s) := \begin{cases} s-a, & \text{if } a \leq s \leq t \leq b, \\ s-b, & \text{if } a \leq t < s \leq b, \end{cases}$$

for all $x \in [a,b]$.

Proof. We use the identity (see (2.10))

$$(4.2)\qquad \sigma^2(X) + [E(X) - x]^2 = \int_a^b (x-t)^2 f(t)\,dt$$

for all $x \in [a,b]$.

On the other hand, we know that (see for example [22] for a simple proof using integration by parts)

$$f(t) = \frac{1}{b-a}\int_a^b f(s)\,ds + \frac{1}{b-a}\int_a^b p(t,s)\,f'(s)\,ds \tag{4.3}$$

for all $t \in [a,b]$.

Substituting (4.3) in (4.2) we obtain

$$\begin{aligned} &\sigma^2(X) + [E(X)-x]^2 \\ &= \int_a^b (t-x)^2 \left[\frac{1}{b-a}\int_a^b f(s)\,ds + \frac{1}{b-a}\int_a^b p(t,s)\,f'(s)\,ds\right] dt \\ &= \frac{1}{b-a}\cdot\frac{1}{3}\left[(x-a)^3+(b-x)^3\right] + \frac{1}{b-a}\int_a^b\int_a^b (t-x)^2\,p(t,s)\,f'(s)\,dsdt. \end{aligned} \tag{4.4}$$

Taking into account the fact that

$$\frac{1}{3}\left[(x-a)^3+(b-x)^3\right] = \frac{(b-a)^2}{12} + \left(x-\frac{a+b}{2}\right)^2, \quad x\in[a,b],$$

then, by (4.4) we deduce the desired result (4.1). □

The following inequality for P.D.F.s which are absolutely continuous and have the derivatives essentially bounded holds.

Theorem 4.2. *If* $f : [a,b] \to \mathbb{R}_+$ *is absolutely continuous on* $[a,b]$ *and* $f' \in L_\infty[a,b]$*, i.e.,* $\|f'\|_\infty := ess \sup\limits_{t\in[a,b]} |f'(t)| < \infty$*, then we have the inequality:*

$$\begin{aligned} &\left|\sigma^2(X) + [E(X)-x]^2 - \frac{(b-a)^2}{12} - \left(x-\frac{a+b}{2}\right)^2\right| \\ &\qquad\qquad \le \frac{(b-a)^2}{3}\left[\frac{(b-a)^2}{10} + \left(x-\frac{a+b}{2}\right)^2\right]\|f'\|_\infty \end{aligned} \tag{4.5}$$

for all $x \in [a,b]$.

Proof. Using Lemma 4.1, we have

$$\begin{aligned} &\left|\sigma^2(X) + [E(X)-x]^2 - \frac{(b-a)^2}{12} - \left(x-\frac{a+b}{2}\right)^2\right| \\ &= \frac{1}{b-a}\left|\int_a^b\int_a^b (t-x)^2\,p(t,s)\,f'(s)\,dsdt\right| \\ &\le \frac{1}{b-a}\int_a^b\int_a^b (t-x)^2\,|p(t,s)|\,|f'(s)|\,dsdt \\ &\le \frac{\|f'\|_\infty}{b-a}\int_a^b\int_a^b (t-x)^2\,|p(t,s)|\,dsdt. \end{aligned}$$

We have

$$\begin{aligned} I &:= \int_a^b \int_a^b (t-x)^2 |p(t,s)|\, ds dt \\ &= \int_a^b (t-x)^2 \left[\int_a^t (s-a)\, ds + \int_t^b (b-s)\, ds\right] dt \\ &= \int_a^b (t-x)^2 \left[\frac{(t-a)^2 + (b-t)^2}{2}\right] dt \\ &= \frac{1}{2}\left[\int_a^b (t-x)^2 (t-a)^2\, dt + \int_a^b (t-x)^2 (b-t)^2\, dt\right] \\ &= \frac{I_a + I_b}{2}. \end{aligned}$$

Let $A = x - a$, $B = b - x$ then

$$\begin{aligned} I_a &= \int_a^b (t-x)^2 (t-a)^2\, dt \\ &= \int_0^{b-a} \left(u^2 - 2Au + A^2\right) u^2 du \\ &= \frac{(b-a)^3}{3}\left[A^2 - \frac{3}{2}A(b-a) + \frac{3}{5}(b-a)^2\right] \end{aligned}$$

and

$$\begin{aligned} I_b &= \int_a^b (t-x)^2 (b-t)^2\, dt \\ &= \int_0^{b-a} \left(u^2 - 2Bu + B^2\right) u^2 du \\ &= \frac{(b-a)^3}{3}\left[B^2 - \frac{3}{2}B(b-a) + \frac{3}{5}(b-a)^2\right] \end{aligned}$$

Now,

$$\begin{aligned} \frac{I_a + I_b}{2} &= \frac{(b-a)^3}{3}\left[\frac{A^2+B^2}{2} - \frac{3}{4}(A+B)(b-a) + \frac{3}{5}(b-a)^2\right] \\ &= \frac{(b-a)^3}{3}\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2 - 3\frac{(b-a)^2}{20}\right] \\ &= \frac{(b-a)^3}{3}\left[\frac{(b-a)^2}{10} + \left(x - \frac{a+b}{2}\right)^2\right] \end{aligned}$$

and the theorem is proved. □

The best inequality we can get from (4.5) is embodied in the following corollary.

Corollary 4.3. *If f is as in Theorem 4.2, then we have*

$$\left|\sigma^2(X) + \left[E(X) - \frac{a+b}{2}\right]^2 - \frac{(b-a)^2}{12}\right| \le \frac{(b-a)^4}{30}\|f'\|_\infty . \tag{4.6}$$

We now analyze the case where f' is a Lebesgue $p-$integrable mapping with $p \in (1,\infty)$.

Remark 4.4. The results of Theorem 4.2 may be compared with those of Theorem 3.5. It may be shown that both bounds are convex and symmetric about $x = \frac{a+b}{2}$. Further, the bound given by the 'premature' Chebychev approach, namely from (3.12)-(3.13) is tighter than that obtained by the current approach (4.5) which may be shown from the following. Let these bounds be described by B_p and B_c so that, neglecting the common terms

$$B_p = \frac{b-a}{2\sqrt{15}} \left[\left(\frac{b-a}{2} \right)^2 + 15Y \right]^{\frac{1}{2}}$$

and

$$B_c = \frac{(b-a)^2}{100} + Y,$$

where

$$Y = \left(x - \frac{a+b}{2} \right)^2 .$$

It may be shown through some straightforward algebra that $B_c^2 - B_p^2 > 0$ for all $x \in [a,b]$ so that $B_c > B_p$.
The current development does however have the advantage that the identity (4.1) is satisfied, thus allowing bounds for $L_p[a,b]$, $p \geq 1$ rather than the infinity norm.

Theorem 4.5. *If $f : [a,b] \to \mathbb{R}_+$ is absolutely continuous on $[a,b]$ and $f' \in L_p$, i.e.,*

$$\|f'\|_p := \left(\int_a^b |f'(t)|^p \, dt \right)^{\frac{1}{p}} < \infty, \;\; p \in (1, \infty)$$

then we have the inequality

$$\left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2} \right)^2 \right| \leq \frac{\|f'\|_p}{(b-a)^{\frac{1}{p}} (q+1)^{\frac{1}{q}}} \left[(x-a)^{3q+2} \tilde{B}\left(\frac{b-a}{x-a}, 2q+1, q+2 \right) + (b-x)^{3q+2} \tilde{B}\left(\frac{b-a}{b-x}, 2q+1, q+2 \right) \right] \tag{4.7}$$

for all $x \in [a,b]$, when $\frac{1}{p} + \frac{1}{q} = 1$ and $\tilde{B}(\cdot,\cdot,\cdot)$ is the quasi incomplete Euler's Beta mapping:

$$\tilde{B}(z; \alpha, \beta) := \int_0^z (u-1)^{\alpha-1} u^{\beta-1} du, \;\; \alpha, \beta > 0, \; z \geq 1.$$

Proof. Using Lemma 4.1, we have, as in Theorem 4.2, that

$$\left| \sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2} \right)^2 \right| \leq \frac{1}{b-a} \int_a^b \int_a^b (t-x)^2 |p(t,s)| |f'(s)| \, ds dt. \tag{4.8}$$

Using Hölder's integral inequality for double integrals, we have

$$\begin{aligned} \int_a^b \int_a^b (t-x)^2 \left|p(t,s)\right| \left|f'(s)\right| dsdt & \\ \le \left(\int_a^b \int_a^b \left|f'(s)\right|^p dsdt\right)^{\frac{1}{p}} & \left(\int_a^b \int_a^b (t-x)^{2q} \left|p(t,s)\right|^q dsdt\right)^{\frac{1}{q}} \\ = (b-a)^{\frac{1}{p}} \left\|f'\right\|_p & \left(\int_a^b \int_a^b (t-x)^{2q} \left|p(t,s)\right|^q dsdt\right)^{\frac{1}{q}}, \end{aligned} \tag{4.9}$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

We have to compute the integral

$$\begin{aligned} D &:= \int_a^b \int_a^b (t-x)^{2q} \left|p(t,s)\right|^q dsdt \\ &= \int_a^b (t-x)^{2q} \left[\int_a^t (s-a)^q ds + \int_t^b (b-s)^q ds\right] dt \\ &= \int_a^b (t-x)^{2q} \left[\frac{(t-a)^{q+1} + (b-t)^{q+1}}{q+1}\right] dt \\ &= \frac{1}{q+1} \left[\int_a^b (t-x)^{2q} (t-a)^{q+1} dt + \int_a^b (t-x)^{2q} (b-t)^{q+1} dt\right]. \end{aligned} \tag{4.10}$$

Define

$$E := \int_a^b (t-x)^{2q} (t-a)^{q+1} dt. \tag{4.11}$$

If we consider the change of variable $t = (1-u)a + ux$, we have $t = a$ implies $u = 0$ and $t = b$ implies $u = \frac{b-a}{x-a}$, $dt = (x-a)du$ and then

$$\begin{aligned} E &= \int_0^{\frac{b-a}{x-a}} \left[(1-u)a + ux - x\right]^{2q} \left[(1-u)a + ux - a\right](x-a)du \\ &= (x-a)^{3q+2} \int_0^{\frac{b-a}{x-a}} (u-1)^{2q} u^{q+1} du \\ &= (x-a)^{3q+2} \tilde{B}\left(\frac{b-a}{x-a}, 2q+1, q+2\right). \end{aligned} \tag{4.12}$$

Define

$$F := \int_a^b (t-x)^{2q} (b-t)^{q+1} dt. \tag{4.13}$$

If we consider the change of variable $t = (1-v)b + vx$, we have $t = b$ implies $v = 0$, and $t = a$ implies $v = \frac{b-a}{b-x}$, $dt = (x-b)dv$ and then

$$\begin{aligned} F &= \int_{\frac{b-a}{b-x}}^0 \left[(1-v)b + vx - x\right]^{2q} \left[b - (1-v)b - vx\right]^{q+1} (x-b)dv \\ &= (b-x)^{3q+2} \int_0^{\frac{b-a}{b-x}} (v-1)^{2q} v^{q+1} dv \\ &= (b-x)^{3q+2} \tilde{B}\left(\frac{b-a}{b-x}, 2q+1, q+2\right). \end{aligned} \tag{4.14}$$

Now, using the inequalities (4.8)-(4.9) and the relations (4.10)-(4.14), since $D = \frac{1}{q+1}(E+F)$, we deduce the desired estimate (4.7). □

The following corollary is natural to be considered.

Corollary 4.6. *Let f be as in Theorem 4.5. Then, we have the inequality:*

$$\left|\sigma^2(X) + \left[E(X) - \frac{a+b}{2}\right]^2 - \frac{(b-a)^2}{12}\right| \leq \frac{\|f'\|_p (b-a)^{2+\frac{3}{q}}}{(q+1)^{\frac{1}{q}} 2^{3+\frac{2}{q}}} \left[B(2q+1, q+1) + \Psi(2q+1, q+2)\right]^{\frac{1}{q}}, \tag{4.15}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$ and $B(\cdot,\cdot)$ is Euler's Beta mapping and $\Psi(\alpha, \beta) := \int_0^1 u^{\alpha-1}(u+1)^{\beta-1} du$, $\alpha, \beta > 0$.

Proof. In (4.7) put $x = \frac{a+b}{2}$. The left side is clear. Now

$$\begin{aligned} \tilde{B}(2, 2q+1, q+2) &= \int_0^2 (u-1)^{2q} u^{q+1} du \\ &= \int_0^1 (u-1)^{2q} u^{q+1} du + \int_1^2 (u-1)^{2q} u^{q+1} du \\ &= B(2q+1, q+2) + \Psi(2q+1, q+2). \end{aligned}$$

The right hand side of (4.7) is thus:

$$\begin{aligned} &\frac{\|f'\|_p \left(\frac{b-a}{2}\right)^{\frac{3q+2}{q}}}{(b-a)^{\frac{1}{p}} (q+1)^{\frac{1}{q}}} \left[2B(2q+1, q+2) + 2\Psi(2q+1, q+2)\right]^{\frac{1}{q}} \\ &\qquad = \frac{\|f'\|_p (b-a)^{2+\frac{3}{q}}}{(q+1)^{\frac{1}{q}} 2^{3+\frac{2}{q}}} \left[B(2q+1, q+2) + \Psi(2q+1, q+2)\right]^{\frac{1}{q}} \end{aligned}$$

and the corollary is proved. □

Finally, if f is absolutely continuous, $f' \in L_1[a,b]$ and $\|f'\|_1 = \int_a^b |f'(t)| dt$, then we can state the following theorem.

Theorem 4.7. *If the p.d.f., $f : [a,b] \to \mathbb{R}_+$ is absolutely continuous on $[a,b]$, then*

$$\left|\sigma^2(X) + [E(X) - x]^2 - \frac{(b-a)^2}{12} - \left(x - \frac{a+b}{2}\right)^2\right| \leq \|f'\|_1 (b-a) \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^2 \tag{4.16}$$

for all $x \in [a,b]$.

Proof. As above, we can state that

$$\begin{aligned}
&\left|\sigma^2(X)+[E(X)-x]^2-\frac{(b-a)^2}{12}-\left(x-\frac{a+b}{2}\right)^2\right| \\
&\qquad\le \frac{1}{b-a}\int_a^b\int_a^b (t-x)^2\,|p(t,s)|\,|f'(s)|\,dsdt \\
&\qquad\le \sup_{(t,s)\in[a,b]^2}\left[(t-x)^2|p(t,s)|\right]\frac{1}{b-a}\int_a^b\int_a^b |f'(s)|\,dsdt \\
&\qquad= \|f'\|_1\,G
\end{aligned}$$

where

$$\begin{aligned}
G &:= \sup_{(t,s)\in[a,b]^2}\left[(t-x)^2|p(t,s)|\right] \\
&\le (b-a)\sup_{t\in[a,b]}(t-x)^2 \\
&= (b-a)\left[\max(x-a,b-x)\right]^2 \\
&= (b-a)\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^2,
\end{aligned}$$

and the theorem is proved. □

It is clear that the best inequality we can get from (4.16) is the one when $x=\frac{a+b}{2}$, giving the following corollary.

Corollary 4.8. *With the assumptions of Theorem 4.7, we have:*

$$\left|\sigma^2(X)+\left[E(X)-\frac{a+b}{2}\right]^2-\frac{(b-a)^2}{12}\right|\le\frac{(b-a)^3}{4}\|f'\|_1. \tag{4.17}$$

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