

ON HEISENBERG AND LOCAL UNCERTAINTY PRINCIPLES FOR THE q -DUNKL TRANSFORM

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Abstract: In this paper, we provide, for the q -Dunkl transform studied in [2], a Heisenberg uncertainty principle and two local uncertainty principles leading to a new Heisenberg-Weyl type inequality.



**Uncertainty Principles for
the q -Dunkl Transform**
Ahmed Fitouhi, Ferjani Nouri
and Sana Guesmi

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1. Introduction

In harmonic analysis, the uncertainty principle states that a function and its Fourier transform cannot be simultaneously sharply localized. A quantitative formulation of this fact is provided by the Heisenberg uncertainty principle, which asserts that every square integrable function f on \mathbb{R} verifies the following inequality

$$(1.1) \quad \left(\int_{-\infty}^{+\infty} x^2 |f(x)|^2 dx \right) \left(\int_{-\infty}^{+\infty} \lambda^2 |\widehat{f}(\lambda)|^2 d\lambda \right) \geq \frac{1}{4} \left(\int_{-\infty}^{+\infty} x^2 |f(x)|^2 dx \right)^2,$$

where

$$\widehat{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-i\lambda x} dx$$

is the classical Fourier transform.

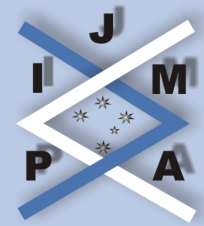
Generalizations of this result in both classical and quantum analysis have been treated and many versions of Heisenberg-Weyl type uncertainty inequalities were obtained for several generalized Fourier transforms (see [1], [14], [10]).

In [2], by the use of the q^2 -analogue differential operator studied in [11], Bettaibi et al. introduced a new q -analogue of the classical Dunkl operator and studied its related Fourier transform, which is a q -analogue of the classical Bessel-Dunkl one and called the q -Dunkl transform.

The aim of this paper is twofold: first, we prove a Heisenberg uncertainty principle for the q -Dunkl transform and next, we state for this transform two local uncertainty principles leading to a new q -Heisenberg-Weyl type inequality.

This paper is organized as follows: in Section 2, we present some preliminary notions and notations useful in the sequel. In Section 3, we recall some results and properties from the theory of the q -Dunkl operator and the q -Dunkl transform (see [2]). Section 4 is devoted to proving a Heisenberg uncertainty principle for the q -Dunkl transform and as consequences, we obtain Heisenberg uncertainty principles

for the q^2 -analogue Fourier transform [12, 11] and for the q -Bessel transform [2]. Finally, in Section 5, we state, for the q -Dunkl transform, two local uncertainty principles, which give a new Heisenberg-Weyl type inequality for the q -Dunkl transform.



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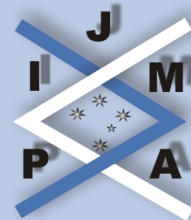
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2. Notations and Preliminaries

Throughout this paper, we assume $q \in]0, 1[$, and refer to the general reference [6] for the definitions, notations and properties of the q -shifted factorials and the q -hypergeometric functions.

We write $\mathbb{R}_q = \{\pm q^n : n \in \mathbb{Z}\}$, $\mathbb{R}_{q,+} = \{q^n : n \in \mathbb{Z}\}$,

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad x \in \mathbb{C} \quad \text{and} \quad [n]_q! = \frac{(q; q)_n}{(1 - q)^n}, \quad n \in \mathbb{N}.$$

The q^2 -analogue differential operator is (see [11, 12])

$$(2.1) \quad \partial_q(f)(z) = \begin{cases} \frac{f(q^{-1}z) + f(-q^{-1}z) - f(qz) + f(-qz) - 2f(-z)}{2(1-q)z} & \text{if } z \neq 0 \\ \lim_{x \rightarrow 0} \partial_q(f)(x) & \text{(in } \mathbb{R}_q) \quad \text{if } z = 0. \end{cases}$$

We remark that if f is differentiable at z , then $\lim_{q \rightarrow 1} \partial_q(f)(z) = f'(z)$.

A repeated application of the q^2 -analogue differential operator is denoted by:

$$\partial_q^0 f = f, \quad \partial_q^{n+1} f = \partial_q(\partial_q^n f).$$

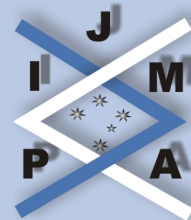
The following lemma lists some useful computational properties of ∂_q .

Lemma 2.1.

1. For all functions f on \mathbb{R}_q ,

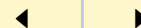
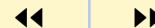
$$\partial_q f(z) = \frac{f_e(q^{-1}z) - f_e(z)}{(1 - q)z} + \frac{f_o(z) - f_o(qz)}{(1 - q)z},$$

where, f_e and f_o are, respectively, the even and the odd parts of f .



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2. For two functions f and g on \mathbb{R}_q , we have

- if f is even and g is odd,

$$\begin{aligned}\partial_q(fg)(z) &= q\partial_q(f)(qz)g(z) + f(qz)\partial_q(g)(z) \\ &= \partial_q(g)(z)f(z) + qg(qz)\partial_q(f)(qz); \end{aligned}$$

- if f and g are even,

$$\partial_q(fg)(z) = \partial_q(f)(z)g(q^{-1}z) + f(z)\partial_q(g)(z).$$

The operator ∂_q induces a q -analogue of the classical exponential function (see [11, 12])

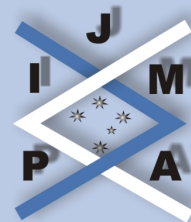
$$(2.2) \quad e(z; q^2) = \sum_{n=0}^{\infty} a_n \frac{z^n}{[n]_q!}, \quad \text{with} \quad a_{2n} = a_{2n+1} = q^{n(n+1)}.$$

The q -Jackson integrals are defined by (see [8])

$$\begin{aligned}\int_0^a f(x) d_q x &= (1-q)a \sum_{n=0}^{\infty} q^n f(aq^n), \\ \int_a^b f(x) d_q x &= \int_0^b f(x) d_q x - \int_0^a f(x) d_q x, \\ \int_0^{\infty} f(x) d_q x &= (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n), \end{aligned}$$

and

$$\int_{-\infty}^{\infty} f(x) d_q x = (1-q) \sum_{n=-\infty}^{\infty} q^n f(q^n) + (1-q) \sum_{n=-\infty}^{\infty} q^n f(-q^n),$$



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provided the sums converge absolutely.

The q -Gamma function is given by (see [8])

$$\Gamma_q(x) = \frac{(q; q)_\infty}{(q^x; q)_\infty} (1 - q)^{1-x}, \quad x \neq 0, -1, -2, \dots$$

- $\mathcal{S}_q(\mathbb{R}_q)$ the space of functions f defined on \mathbb{R}_q satisfying

$$\forall n, m \in \mathbb{N}, \quad P_{n,m,q}(f) = \sup_{x \in \mathbb{R}_q} |x^m \partial_q^n f(x)| < +\infty$$

and

$$\lim_{x \rightarrow 0} \partial_q^n f(x) \quad (\text{in } \mathbb{R}_q) \quad \text{exists;}$$

- $L_q^\infty(\mathbb{R}_q) = \left\{ f : \|f\|_{\infty,q} = \sup_{x \in \mathbb{R}_q} |f(x)| < \infty \right\};$
- $L_{\alpha,q}^p(\mathbb{R}_q) = \left\{ f : \|f\|_{p,\alpha,q} = \left(\int_{-\infty}^{\infty} |f(x)|^p |x|^{2\alpha+1} d_q x \right)^{\frac{1}{p}} < \infty \right\};$
- $L_{\alpha,q}^p([-a, a]) = \left\{ f : \|f\|_{p,\alpha,q} = \left(\int_{-a}^a |f(x)|^p |x|^{2\alpha+1} d_q x \right)^{\frac{1}{p}} < \infty \right\}.$

For the particular case $p = 2$, we denote by $\langle \cdot; \cdot \rangle$ the inner product of the Hilbert space $L_{\alpha,q}^2(\mathbb{R}_q)$.



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3. The q -Dunkl Operator and the q -Dunkl Transform

In this section, we collect some basic properties of the q -Dunkl operator and the q -Dunkl transform introduced in [2] which will be useful in the sequel.

For $\alpha \geq -\frac{1}{2}$, the q -Dunkl operator is defined by

$$\Lambda_{\alpha,q}(f)(x) = \partial_q [H_{\alpha,q}(f)](x) + [2\alpha + 1]_q \frac{f(x) - f(-x)}{2x},$$

where

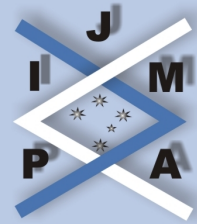
$$H_{\alpha,q} : f = f_e + f_o \longmapsto f_e + q^{2\alpha+1} f_o.$$

It satisfies the following relations:

- For $\alpha = -\frac{1}{2}$, $\Lambda_{\alpha,q} = \partial_q$.
- $\Lambda_{\alpha,q}$ lives $\mathcal{S}_q(\mathbb{R}_q)$ invariant.
- If f is odd then $\Lambda_{\alpha,q}(f)(x) = q^{2\alpha+1} \partial_q f(x) + [2\alpha + 1]_q \frac{f(x)}{x}$ and if f is even then $\Lambda_{\alpha,q}(f)(x) = \partial_q f(x)$.
- For all $a \in \mathbb{C}$, $\Lambda_{\alpha,q}[f(ax)] = a \Lambda_{\alpha,q}(f)(ax)$.
- For all f and g such that $\int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1} d_q x$ exists, we have

$$\begin{aligned} (3.1) \quad \int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(f)(x)g(x)|x|^{2\alpha+1} d_q x \\ = - \int_{-\infty}^{+\infty} \Lambda_{\alpha,q}(g)(x)f(x)|x|^{2\alpha+1} d_q x. \end{aligned}$$

It was shown in [2] that for each $\lambda \in \mathbb{C}$, the function



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$$(3.2) \quad \psi_{\lambda}^{\alpha, q} : x \mapsto j_{\alpha}(\lambda x; q^2) + \frac{i\lambda x}{[2\alpha + 2]_q} j_{\alpha+1}(\lambda x; q^2)$$

is the unique solution of the q -differential-difference equation:

$$\begin{cases} \Lambda_{\alpha, q}(f) = i\lambda f \\ f(0) = 1, \end{cases}$$

where $j_{\alpha}(\cdot; q^2)$ is the normalized third Jackson's q -Bessel function given by

$$(3.3) \quad j_{\alpha}(x; q^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q^2; q^2)_n (q^{2(\alpha+1)}; q^2)_n} ((1-q)x)^{2n}.$$

The function $\psi_{\lambda}^{\alpha, q}(x)$, has a unique extension to $\mathbb{C} \times \mathbb{C}$ and verifies the following properties.

- $\psi_{a\lambda}^{\alpha, q}(x) = \psi_{\lambda}^{\alpha, q}(ax) = \psi_{ax}^{\alpha, q}(\lambda), \quad \forall a, x, \lambda \in \mathbb{C}.$
- For all $x, \lambda \in \mathbb{R}_q,$

$$(3.4) \quad |\psi_{\lambda}^{\alpha, q}(x)| \leq \frac{4}{(q; q)_{\infty}}.$$

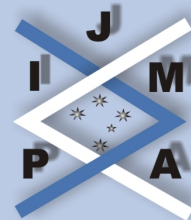
The q -Dunkl transform $F_D^{\alpha, q}$ is defined on $L_{\alpha, q}^1(\mathbb{R}_q)$ (see [2]) by

$$F_D^{\alpha, q}(f)(\lambda) = \frac{c_{\alpha, q}}{2} \int_{-\infty}^{+\infty} f(x) \psi_{-\lambda}^{\alpha, q}(x) |x|^{2\alpha+1} d_q x,$$

where

$$c_{\alpha, q} = \frac{(1+q)^{-\alpha}}{\Gamma_{q^2}(\alpha+1)}.$$

It satisfies the following properties:



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- For $\alpha = -\frac{1}{2}$, $F_D^{\alpha,q}$ is the q^2 -analogue Fourier transform $\widehat{f}(\cdot; q^2)$ given by (see [12, 11])

$$\widehat{f}(\lambda; q^2) = \frac{(1+q)^{1/2}}{2\Gamma_{q^2}(\frac{1}{2})} \int_{-\infty}^{+\infty} f(x)e(-i\lambda x; q^2)d_q x.$$

- On the even functions space, $F_D^{\alpha,q}$ coincides with the q -Bessel transform given by (see [2])

$$\mathcal{F}_{\alpha,q}(f)(\lambda) = c_{\alpha,q} \int_0^{+\infty} f(x)j_{\alpha}(\lambda x; q^2)x^{2\alpha+1}d_q x.$$

- For all $f \in L^1_{\alpha,q}(\mathbb{R}_q)$, we have:

$$(3.5) \quad \|F_D^{\alpha,q}(f)\|_{\infty,q} \leq \frac{2c_{\alpha,q}}{(q; q)_{\infty}} \|f\|_{1,\alpha,q}.$$

- For all $f \in L^1_{\alpha,q}(\mathbb{R}_q)$, such that $xf \in L^1_{\alpha,q}(\mathbb{R}_q)$,

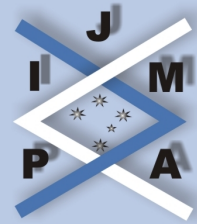
$$(3.6) \quad F_D^{\alpha,q}(\Lambda_{\alpha,q}f)(\lambda) = i\lambda F_D^{\alpha,q}(f)(\lambda)$$

and

$$(3.7) \quad \Lambda_{\alpha,q}(F_D^{\alpha,q}(f)) = -iF_D^{\alpha,q}(xf).$$

- The q -Dunkl transform $F_D^{\alpha,q}$ is an isomorphism from $L^2_{\alpha,q}(\mathbb{R}_q)$ (resp. $\mathcal{S}_q(\mathbb{R}_q)$) onto itself and satisfies the following Plancherel formula:

$$(3.8) \quad \|F_D^{\alpha,q}(f)\|_{2,\alpha,q} = \|f\|_{2,\alpha,q}, \quad f \in L^2_{\alpha,q}(\mathbb{R}_q).$$



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4. q -Analogue of the Heisenberg Inequality

In this section, we provide a Heisenberg uncertainty principle for the q -Dunkl transform. For this purpose, inspired by the approach given in [10], we follow the steps of [1], using the operator $\Lambda_{\alpha,q}$ instead of the operator ∂_q , and consider the operators

$$L_{\alpha,q}(f)(x) = f_e(x) + q^{2\alpha+2}f_o(qx) \quad \text{and} \quad Qf(x) = xf(x),$$

and the q -commutator:

$$[D_{\alpha,q}, Q]_q = D_{\alpha,q}Q - qQD_{\alpha,q},$$

where

$$D_{\alpha,q} = L_{\alpha,q}\Lambda_{\alpha,q}.$$

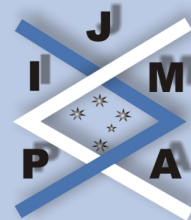
The following theorem gives a Heisenberg uncertainty principle for the q -Dunkl transform $F_D^{\alpha,q}$.

Theorem 4.1. For $f \in \mathcal{S}_q(\mathbb{R}_q)$, we have

$$(4.1) \quad \frac{q^{2\alpha+1}}{1+q+q^{\alpha-1}+q^\alpha} \left| q\|f\|_{2,\alpha,q}^2 + \left(1 - q - \frac{[2\alpha+1]_q}{q^{2\alpha}}\right) \|f_o\|_{2,\alpha,q}^2 \right| \\ \leq \|xf\|_{2,\alpha,q} \|xF_D^{\alpha,q}(f)(x)\|_{2,\alpha,q}.$$

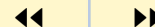
Proof. By Lemma 2.1 and simple calculus, we obtain

$$[D_{\alpha,q}, Q]_q f = q^{2\alpha+2}f_e + q^{2\alpha+1} \left(1 - \frac{[2\alpha+1]_q}{q^{2\alpha}}\right) f_o.$$



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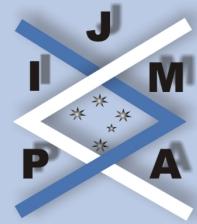
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Then, using the Cauchy-Schwarz inequality and the properties of the q -Dunkl operator, one can write

$$\begin{aligned}
 & \left| q^{2\alpha+2} \|f_e\|_{2,\alpha,q}^2 + q^{2\alpha+1} \left(1 - \frac{[2\alpha+1]_q}{q^{2\alpha}} \right) \|f_o\|_{2,\alpha,q}^2 \right| \\
 &= |\langle [D_{\alpha,q}, Q]_q f; f \rangle| \\
 &= |\langle D_{\alpha,q} Q f - q Q D_{\alpha,q} f; f \rangle| \leq |\langle D_{\alpha,q} Q f; f \rangle| + q |\langle Q D_{\alpha,q} f; f \rangle| \\
 &= |\langle D_{\alpha,q} (x f_e + x f_o); f \rangle| + q |\langle D_{\alpha,q} f; x f \rangle| \\
 &= |\langle \Lambda_{\alpha,q} (x f_e) + q^{2\alpha+2} \Lambda_{\alpha,q} (x f_o)(qx); f \rangle| \\
 &\quad + q |\langle q^{2\alpha+2} \Lambda_{\alpha,q} (f_e)(qx) + \Lambda_{\alpha,q} (f_o)(x); x f \rangle| \\
 &\leq |\langle \Lambda_{\alpha,q} (x f_e); f \rangle| + q^{2\alpha+2} |\langle \Lambda_{\alpha,q} (x f_o)(qx); f \rangle| \\
 &\quad + q^{2\alpha+3} |\langle \Lambda_{\alpha,q} (f_e)(qx); x f \rangle| + q |\langle \Lambda_{\alpha,q} (f_o)(x); x f \rangle| \\
 &= |\langle \Lambda_{\alpha,q} (x f_e); f \rangle| + q^{2\alpha+1} |\langle \Lambda_{\alpha,q} (x f_o(q\cdot)); f \rangle| \\
 &\quad + q^{2\alpha+2} |\langle \Lambda_{\alpha,q} (f_e(q\cdot)); x f \rangle| + q |\langle \Lambda_{\alpha,q} (f_o)(x); x f \rangle| \\
 &\leq \|x f_e\|_{2,\alpha,q} \|x F_D^{\alpha,q}(f)\|_{2,\alpha,q} + q^{\alpha-1} \|x f_o\|_{2,\alpha,q} \|x F_D^{\alpha,q}(f)\|_{2,\alpha,q} \\
 &\quad + q^\alpha \|x f_e\|_{2,\alpha,q} \|x F_D^{\alpha,q}(f)\|_{2,\alpha,q} + q \|x f_o\|_{2,\alpha,q} \|x F_D^{\alpha,q}(f)\|_{2,\alpha,q} \\
 &\leq (1 + q + q^{\alpha-1} + q^\alpha) \|x f\|_{2,\alpha,q} \|x F_D^{\alpha,q}(f)\|_{2,\alpha,q},
 \end{aligned}$$

which achieves the proof. \square

As a consequence, we obtain a Heisenberg-Weyl uncertainty principle for the q^2 -analogue Fourier transform (by taking $\alpha = -1/2$) and the q -Bessel transform (in the even case).



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Corollary 4.2.

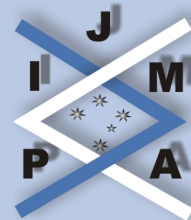
1. For $f \in \mathcal{S}_q(\mathbb{R}_q)$, we have

$$(4.2) \quad \frac{q}{1+q+q^{-3/2}+q^{-1/2}} \|f\|_{2,q}^2 \leq \|xf\|_{2,q} \|\lambda \widehat{f}(\lambda; q^2)\|_{2,q}.$$

2. For an even function $f \in \mathcal{S}_q(\mathbb{R}_q)$, we have

$$(4.3) \quad \frac{q^{2\alpha+2}}{1+q+q^{\alpha-1}+q^\alpha} \|f\|_{2,\alpha,q}^2 \leq \|xf\|_{2,\alpha,q} \|\lambda \mathcal{F}_{\alpha,q}(f)(\lambda)\|_{2,\alpha,q}.$$

We remark that when q tends to 1^- , (4.2) tends at least formally to the classical Heisenberg uncertainty principle given by (1.1).



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5. Local Uncertainty Principles

In this section, we will state, for the q -Dunkl transform, two local uncertainty principles leading to a new Heisenberg-Weyl type inequality.

Notations: For $E \subset \mathbb{R}_q$ and f defined on \mathbb{R}_q , we write

$$\int_E f(t) d_q t = \int_{-\infty}^{\infty} f(t) \chi_E(t) d_q t \quad \text{and} \quad |E|_{\alpha} = \int_E |t|^{2\alpha+1} d_q t,$$

where χ_E is the characteristic function of E .

Theorem 5.1. *If $0 < a < \alpha + 1$, then for all bounded subsets E of \mathbb{R}_q and all $f \in L_{\alpha,q}^2(\mathbb{R}_q)$, we have*

$$(5.1) \quad \int_E |F_D^{\alpha,q}(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d_q \lambda \leq K_{a,\alpha} |E|^{\frac{a}{\alpha+1}} \|x^a f\|_{2,\alpha,q}^2,$$

where

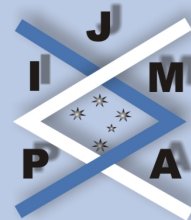
$$K_{a,\alpha} = \left(\frac{2\tilde{c}_{\alpha,q}}{\sqrt{[2(\alpha+1-a)]_q}} \left(\frac{\alpha+1-a}{a} \right) \right)^{\frac{2\alpha}{\alpha+1}} \left(\frac{\alpha+1}{\alpha+1-a} \right)^2$$

and $\tilde{c}_{\alpha,q} = \frac{2c_{\alpha,q}}{(q;q)_{\infty}}$.

Proof. For $r > 0$, let $\chi_r = \chi_{[-r,r]}$ the characteristic function of $[-r, r]$ and $\tilde{\chi}_r = 1 - \chi_r$.

Then for $r > 0$, we have, since $f \cdot \chi_r \in L_q^1(\mathbb{R}_q)$,

$$\left(\int_E |F_D^{\alpha,q}(f)(\lambda; q^2)|^2 |\lambda|^{2\alpha+1} d_q \lambda \right)^{1/2}$$



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$$\begin{aligned}
 &= \|F_D^{\alpha,q}(f) \cdot \chi_E\|_{2,\alpha,q} \\
 &\leq \|F_D^{\alpha,q}(f \cdot \chi_r)\chi_E\|_{2,\alpha,q} + \|F_D^{\alpha,q}(f \cdot \tilde{\chi}_r)\chi_E\|_{2,\alpha,q} \\
 &\leq |E|_{\alpha}^{1/2} \|F_D^{\alpha,q}(f \cdot \chi_r)\|_{\infty,q} + \|F_D^{\alpha,q}(f \cdot \tilde{\chi}_r)\|_{2,\alpha,q}.
 \end{aligned}$$

Now, on the one hand, we have by the relation (3.5) and the Cauchy-Schwartz inequality,

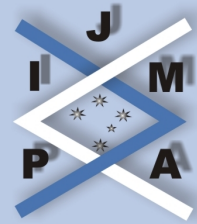
$$\begin{aligned}
 \|F_D^{\alpha,q}(f \cdot \chi_r)\|_{\infty,q} &\leq \tilde{c}_{\alpha,q} \|f \cdot \chi_r\|_{1,\alpha,q} \\
 &= \tilde{c}_{\alpha,q} \|x^{-a} \chi_r \cdot x^a f\|_{1,\alpha,q} \\
 &\leq \tilde{c}_{\alpha,q} \|x^{-a} \chi_r\|_{2,\alpha,q} \|x^a f\|_{2,\alpha,q} \\
 &\leq \frac{2\tilde{c}_{\alpha,q}}{\sqrt{[2(\alpha+1-a)]_q}} r^{(\alpha+1)-a} \|x^a f\|_{2,\alpha,q}.
 \end{aligned}$$

On the other hand, since $f \in L_{\alpha,q}^2(\mathbb{R}_q)$, we have $f \cdot \tilde{\chi}_r \in L_{\alpha,q}^2(\mathbb{R}_q)$ and by the Plancherel formula, we obtain

$$\begin{aligned}
 \|F_D^{\alpha,q}(f \cdot \tilde{\chi}_r)\|_{2,\alpha,q} &= \|f \cdot \tilde{\chi}_r\|_{2,\alpha,q} \\
 &= \|x^{-a} \tilde{\chi}_r \cdot x^a f\|_{2,\alpha,q} \\
 &\leq \|x^{-a} \tilde{\chi}_r\|_{\infty,q} \|x^a f\|_{2,\alpha,q} \\
 &\leq r^{-a} \|x^a f\|_{2,\alpha,q}.
 \end{aligned}$$

So,

$$\begin{aligned}
 &\left(\int_E |F_D^{\alpha,q}(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d_q \lambda \right)^{\frac{1}{2}} \\
 &\leq \left(\frac{2\tilde{c}_{\alpha,q}}{\sqrt{[2(\alpha+1-a)]_q}} |E|_{\alpha}^{\frac{1}{2}} r^{\alpha+1-a} + r^{-a} \right) \|x^a f\|_{2,\alpha,q}.
 \end{aligned}$$



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The desired result is obtained by minimizing the right hand side of the previous inequality over $r > 0$. \square

Corollary 5.2. For $\alpha \geq -\frac{1}{2}$, $0 < a < \alpha + 1$ and $b > 0$, we have for all $f \in L^2_{\alpha,q}(\mathbb{R}_q)$,

$$(5.2) \quad \|f\|_{2,\alpha,q}^{(a+b)} \leq K_{a,b,\alpha} \|x^a f\|_{2,\alpha,q}^b \|\lambda^b F_D^{\alpha,q}(f)\|_{2,\alpha,q}^a,$$

with

$$K_{a,b,\alpha} = \left[\left(\frac{b}{a}\right)^{\frac{a}{a+b}} + \left(\frac{a}{b}\right)^{\frac{b}{a+b}} \right]^{\frac{a+b}{2}} (2K_{a,\alpha})^{\frac{b}{2}} \frac{q^{-(2\alpha+1)(a+b)}}{([2\alpha + 2]_q)^{\frac{ab}{2(\alpha+1)}}}$$

where $K_{a,\alpha}$ is the constant given in Theorem 5.1.

Proof. For $r > 0$, we put $E_r =]-r, r[\cap \mathbb{R}_q$ and \tilde{E}_r the supplementary of E_r in \mathbb{R}_q . We have E_r is a bounded subset of \mathbb{R}_q and $|E_r|_{\alpha} \leq 2 \frac{r^{2\alpha+2}}{[2\alpha+2]_q}$. Then the Plancherel formula and the previous theorem lead to

$$\begin{aligned} \|f\|_{2,\alpha,q}^2 &= \|F_D^{\alpha,q}(f)\|_{2,\alpha,q}^2 \\ &= \int_{E_r} |F_D^{\alpha,q}(f)|^2(\lambda) |\lambda|^{2\alpha+1} d_q \lambda + \int_{\tilde{E}_r} |F_D^{\alpha,q}(f)|^2(\lambda) |\lambda|^{2\alpha+1} d_q \lambda \\ &\leq 2K_{a,\alpha} |E_r|_{\alpha}^{\frac{a}{\alpha+1}} \|x^a f\|_{2,\alpha,q}^2 + r^{-2b} \|\lambda^b F_D^{\alpha,q}(f)\|_{2,\alpha,q}^2 \\ &\leq 2 \frac{K_{a,\alpha}}{[2\alpha + 2]_q^{\frac{a}{\alpha+1}}} r^{2a} \|x^a f\|_{2,\alpha,q}^2 + r^{-2b} \|\lambda^b F_D^{\alpha,q}(f)\|_{2,\alpha,q}^2. \end{aligned}$$

The desired result follows by minimizing the right expressions over $r > 0$. \square



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Theorem 5.3. For $\alpha \geq -\frac{1}{2}$ and $a > \alpha + 1$, there exists a constant $K'_{a,\alpha,q}$ such that for all bounded subsets E of \mathbb{R}_q and all f in $L^2_{\alpha,q}(\mathbb{R}_q)$, we have

$$(5.3) \quad \int_E |F_D^{\alpha,q}(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d_q \lambda \leq K'_{a,\alpha,q} |E|_\alpha \|f\|_{2,\alpha,q}^{2(1-\frac{\alpha+1}{a})} \|x^a f\|_{2,\alpha,q}^{2\frac{\alpha+1}{a}}.$$

The proof of this result needs the following lemmas.

Lemma 5.4. Suppose $a > \alpha + 1$, then for all $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ such that $x^a f \in L^2_{\alpha,q}(\mathbb{R}_q)$,

$$(5.4) \quad \|f\|_{1,\alpha,q}^2 \leq K_2 [\|f\|_{2,\alpha,q}^2 + \|x^a f\|_{2,\alpha,q}^2],$$

where

$$K_2 = 2(1-q) \frac{(q^{2a}, q^{2a}, -q^{2\alpha+2}, -q^{2(a-\alpha-1)}; q^{2a})_\infty}{(q^{2\alpha+2}, q^{2(a-\alpha-1)}, -q^{2a}, -1; q^{2a})_\infty}.$$

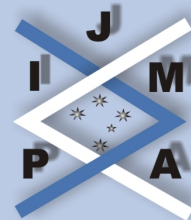
Proof. From ([4, Example 1]) and Hölder's inequality, we have

$$\begin{aligned} \|f\|_{1,\alpha,q}^2 &= \left[\int_{-\infty}^{+\infty} (1 + |x|^{2a})^{\frac{1}{2}} |f(x)| (1 + |x|^{2a})^{-\frac{1}{2}} |x|^{2\alpha+1} d_q x \right]^2 \\ &\leq K_2 [\|f\|_{2,\alpha,q}^2 + \|x^a f\|_{2,\alpha,q}^2], \end{aligned}$$

where

$$\begin{aligned} K_2 &= 2 \int_0^{+\infty} \frac{x^{2\alpha+1}}{1 + x^{2a}} d_q x \\ &= 2(1-q) \frac{(q^{2a}, q^{2a}, -q^{2\alpha+2}, -q^{2(a-\alpha-1)}; q^{2a})_\infty}{(q^{2\alpha+2}, q^{2(a-\alpha-1)}, -q^{2a}, -1; q^{2a})_\infty}. \end{aligned}$$

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Lemma 5.5. Suppose $a > \alpha + 1$, then for all $f \in L^2_{\alpha,q}(\mathbb{R}_q)$ such that $x^a f \in L^2_{\alpha,q}(\mathbb{R}_q)$, we have

$$(5.5) \quad \|f\|_{1,\alpha,q} \leq K_3 \|f\|_{2,\alpha,q}^{(1-\frac{\alpha+1}{a})} \|x^a f\|_{2,\alpha,q}^{\frac{\alpha+1}{a}},$$

where

$$K_3 = K_3(a, \alpha, q) = \left[\left(q^{2(\alpha+1)} \left(\frac{a}{\alpha+1} - 1 \right) \right)^{\frac{\alpha+1}{a}} q^{-2(\alpha+1)} \left(1 + \frac{\alpha+1}{a-\alpha-1} \right) K_2 \right]^{\frac{1}{2}}.$$

Proof. For $s \in \mathbb{R}_q$, define the function f_s by $f_s(x) = f(sx)$, $x \in \mathbb{R}_q$.

We have

$$\|f_s\|_{1,\alpha,q} = s^{-2(\alpha+1)} \|f\|_{1,\alpha,q}, \quad \|x^a f_s\|_{2,\alpha,q}^2 = s^{-2(\alpha+a+1)} \|x^a f\|_{2,\alpha,q}^2.$$

Replacement of f by f_s in Lemma 5.4 gives:

$$\|f\|_{1,\alpha,q}^2 \leq K_2 \left[s^{2(\alpha+1)} \|f\|_{2,\alpha,q}^2 + s^{2(\alpha-a+1)} \|x^a f\|_{2,\alpha,q}^2 \right].$$

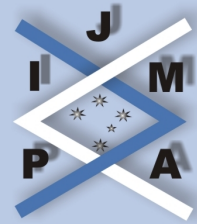
Now, for all $r > 0$, put $\alpha(r) = \frac{\text{Log}(r)}{\text{Log}(q)} - E\left(\frac{\text{Log}(r)}{\text{Log}(q)}\right)$. We have $s = \frac{r}{q^{\alpha(r)}} \in \mathbb{R}_q$ and $r \leq s < \frac{r}{q}$. Then, for all $r > 0$,

$$\|f\|_{1,\alpha,q}^2 \leq K_2 \left[\left(\frac{r}{q} \right)^{2(\alpha+1)} \|f\|_{2,\alpha,q}^2 + r^{2(\alpha-a+1)} \|x^a f\|_{2,\alpha,q}^2 \right].$$

The right hand side of this inequality is minimized by choosing

$$r = \left(\frac{a}{\alpha+1} - 1 \right)^{\frac{1}{2a}} q^{\frac{\alpha+1}{a}} \|f\|_{2,\alpha,q}^{-\frac{1}{a}} \|x^a f\|_{2,\alpha,q}^{\frac{1}{a}}.$$

When this is done we obtain the result. □



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Proof of Theorem 5.3. Let E be a bounded subset of \mathbb{R}_q . When the right hand side of the inequality is finite, Lemma 5.4 implies that $f \in L_q^1(\mathbb{R}_q)$, so, $F_D^{\alpha,q}(f)$ is defined and bounded on \mathbb{R}_q . Using Lemma 5.5, the relation (3.5) and the fact that

$$\int_E |F_D^{\alpha,q}(f)(\lambda)|^2 |\lambda|^{2\alpha+1} d_q \lambda \leq |E|_\alpha \|F_D^{\alpha,q}(f)\|_{\infty,q}^2,$$

we obtain the result with

$$\begin{aligned} K'_{a,\alpha,q} &= \frac{4(1+q)^{-2\alpha}}{\Gamma_{q^2}^2(\alpha+1)(q;q)_\infty^2} K_3^2 \\ &= \frac{8(1-q)(1+q)^{-2\alpha}}{\Gamma_{q^2}^2(\alpha+1)(q;q)_\infty^2} \left(q^{2(\alpha+1)} \left(\frac{a}{\alpha+1} - 1 \right) \right)^{\frac{\alpha+1}{a}} q^{-2(\alpha+1)} \left(1 + \frac{\alpha+1}{a-\alpha-1} \right) \\ &\quad \times \frac{(q^{2a}, q^{2a}, -q^{2\alpha+2}, -q^{2(a-\alpha-1)}; q^{2a})_\infty}{(q^{2\alpha+2}, q^{2(a-\alpha-1)}, -q^{2a}, -1; q^{2a})_\infty}. \end{aligned}$$

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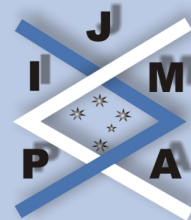
Corollary 5.6. For $\alpha \geq -\frac{1}{2}$, $a > \alpha + 1$ and $b > 0$, we have for all $f \in L_{\alpha,q}^2(\mathbb{R}_q)$,

$$(5.6) \quad \|f\|_{2,\alpha,q}^{(a+b)} \leq K'_{a,b,\alpha} \|x^a f\|_{2,\alpha,q}^b \|\lambda^b F_D^{\alpha,q}(f)\|_{2,\alpha,q}^a,$$

with

$$K'_{a,b,\alpha} = \left(\frac{K'_{a,\alpha,q}}{[2\alpha+2]_q} \right)^{\frac{ab}{2\alpha+2}} \left(q^{-(4\alpha+2)} \left[\left(\frac{b}{\alpha+1} \right)^{\frac{\alpha+1}{\alpha+b+1}} + \left(\frac{b}{\alpha+1} \right)^{-\frac{b}{\alpha+b+1}} \right] \right)^{\frac{a(\alpha+b+1)}{2(\alpha+1)}},$$

where $K'_{a,\alpha,q}$ is the constant given in the previous theorem.



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Proof. The same techniques as in Corollary 5.2 give the result. \square

The following result gives a new Heisenberg-Weyl type inequality for the q -Dunkl transform.

Theorem 5.7. For $\alpha \geq -\frac{1}{2}$, $\alpha \neq 0$, we have for all $f \in L_{\alpha,q}^2(\mathbb{R}_q)$,

$$(5.7) \quad \|f\|_{2,\alpha,q}^2 \leq K_\alpha \|xf\|_{2,\alpha,q} \|\lambda F_D^{\alpha,q}(f)\|_{2,\alpha,q},$$

with

$$K_\alpha = \begin{cases} K_{1,1,\alpha} & \text{if } \alpha > 0 \\ K'_{1,1,\alpha} & \text{if } \alpha < 0. \end{cases}$$

Proof. The result follows from Corollaries 5.2 and 5.6, by taking $a = b = 1$. \square

Remark 1. Note that Theorem 4.1 and Theorem 5.7 are both Heisenberg-Weyl type inequalities for the q -Dunkl transform. However, the constants in the two theorems are different, the first one seems to be more optimal. Moreover, Theorem 4.1 is true for every $\alpha > -\frac{1}{2}$ and uses both f and f_0 , in contrast to Theorem 5.7, which is true only for $\alpha \neq 0$ and uses only f .

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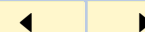
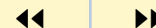
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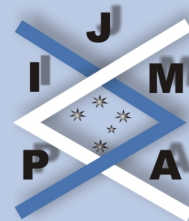
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