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INEQUALITIES FOR INSCRIBED SIMPLEX AND APPLICATIONS

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Abstract

In this paper, we study the problem of geometric inequalities for the inscribed simplex of an n-dimensional simplex. An inequality for the inscribed simplex of a simplex is established. Applying it we get a generalization of n-dimensional Euler inequality and an inequality for the pedal simplex of a simplex.

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Contents

1. Main Results

Let σ_n be an n-dimensional simplex in the n-dimensional Euclidean space E^n, V denote the volume of σ_n, R and r the circumradius and inradius of σ_n, respectively. Let $A_0, A_1, \ldots, A_n$ be the vertices of σ_n, $a_{ij} = |A_iA_j|$ $(0 \le i < j \le n)$, F_i denote the area of the ith face $f_i = A_0 \cdots A_{i-1}A_{i+1} \cdots A_n$ ($(n-1)$-dimensional simplex) of σ_n, points O and G be the circumcenter and barycenter of σ_n, respectively. For $i = 0, 1, \ldots, n$, let A_i' be an arbitrary interior point of the ith face f_i of σ_n. The n-dimensional simplex $\sigma_n' = A_0'A_1' \cdots A_n'$ is called the inscribed simplex of the simplex σ_n. Let $a_{ij}' = |A_i'A_j'|$ $(0 \le i < j \le n)$, R' denote the circumradius of σ_n', P be an arbitrary interior point of σ_n, P_i be the orthogonal projection of the point P on the ith face f_i of σ_n. The n-dimensional simplex $\sigma_n'' = P_0P_1 \cdots P_n$ is called the pedal simplex of the point P with respect to the simplex σ_n [1] – [2], let V'' denote the volume of σ_n'', R'' and r'' denote the circumradius and inradius of σ_n'', respectively. We note that the pedal simplex σ_n'' is an inscribed simplex of the simplex σ_n. Our main results are following theorems.

Theorem 1.1. *Let σ_n' be an inscribed simplex of the simplex σ_n, then we have*

$$(R')^2 \left(R^2 - \overline{OG}^2\right)^{n-1} \ge n^{2(n-1)} r^{2n}, \tag{1.1}$$

with equality if the simplex σ_n is regular and σ_n' is the tangent point simplex of σ_n.

Let T_i be the tangent point where the inscribed sphere of the simplex σ_n touches the ith face f_i of σ_n. The simplex $\overline{\sigma}_n = T_0T_1 \cdots T_n$ is called the tangent point simplex of σ_n [3]. If we take $A_i' \equiv T_i$ $(i = 0, 1, \ldots, n)$ in Theorem 1.1, then σ'_n and $\overline{\sigma}_n$ are the same and $R' = r$, we get a generalization of the n-dimensional Euler inequality [4] as follows.

Corollary 1.2. *For an n-dimensional simplex σ_n, we have*

$$R^2 \geq n^2 r^2 + \overline{OG}^2, \tag{1.2}$$

with equality if the simplex σ_n is regular.

Inequality (1.2) improves the n-dimensional Euler inequality [5] as follows.

$$R \geq nr. \tag{1.3}$$

Theorem 1.3. *Let P be an interior point of the simplex σ_n, and σ'_n the pedal simplex of the point P with respect to σ_n, then*

$$R'' R^{n-1} \geq n^{2n-1} \left(r''\right)^n, \tag{1.4}$$

with equality if the simplex σ_n is regular and σ''_n is the tangent point simplex of σ_n.

2. Some Lemma and Proofs of Theorems

To prove the theorems stated above, we need some lemmas as follows.

Lemma 2.1. *Let σ_n' be an inscribed simplex of the n-dimensional simplex σ_n, then we have*

$$\left(\sum_{0\le i<j\le n}\left(a_{ij}'\right)^2\right)\left(\sum_{i=0}^{n}F_i^2\right)\ge n^2(n+1)V^2, \tag{2.1}$$

with equality if the simplex σ_n is regular and σ_n' is the tangent point simplex of σ_n.

Proof. Let B be an interior point of the simplex σ_n, and $(\lambda_0,\lambda_1,\dots,\lambda_n)$ the barycentric coordinates of the point B with respect to coordinate simplex σ_n. Here $\lambda_i = V_iV^{-1}\,(i=0,1,\dots,n)$, V_i is the volume of the simplex $\sigma_n(i) = BA_0\cdots A_{i-1}A_{i+1}\cdots A_n$ and $\sum_{i=0}^{n}\lambda_i = 1$. Let Q be an arbitrary point in E^n, then

$$\overrightarrow{QB}=\sum_{i=0}^{n}\lambda_i\overrightarrow{QA_i}.$$

From this we have

$$\sum_{i=0}^{n}\lambda_i\overrightarrow{BA_i}=\sum_{i=0}^{n}\lambda_i\left(\overrightarrow{QA_i}-\overrightarrow{QB}\right)=\overrightarrow{0},$$

$$\sum_{i=0}^{n}\lambda_i\left(\overrightarrow{QA_i}\right)^2=\sum_{i=0}^{n}\lambda_i\left(\overrightarrow{QB}+\overrightarrow{BA_i}\right)^2 \tag{2.2}$$
$$=\sum_{i=0}^{n}\lambda_i\overrightarrow{QB}^2+2\overrightarrow{QB}\cdot\sum_{i=0}^{n}\lambda_i\overrightarrow{BA_i}+\sum_{i=0}^{n}\lambda_i\left(\overrightarrow{BA_i}\right)^2$$
$$=\overrightarrow{QB}^2+\sum_{i=0}^{n}\lambda_i\left(\overrightarrow{BA_i}\right)^2.$$

For $j=0,1,\dots,n$, taking $Q\equiv A_j$ in (2.2) we get

$$\sum_{i=0}^{n}\lambda_i\lambda_j\left(\overrightarrow{A_iA_j}\right)^2 \tag{2.3}$$
$$=\lambda_j\left(\overrightarrow{BA_j}\right)^2+\lambda_j\sum_{i=0}^{n}\lambda_i\left(\overrightarrow{BA_i}\right)^2\quad(j=0,1,\dots,n).$$

Adding up these equalities in (2.3) and noting that $\sum_{j=0}^{n}\lambda_j=1$, we get

$$\sum_{0\le i<j\le n}\lambda_i\lambda_j\left(\overrightarrow{A_iA_j}\right)^2=\sum_{i=0}^{n}\lambda_i\left(\overrightarrow{BA_i}\right)^2. \tag{2.4}$$

For any real numbers $x_i>0\ (i=0,1,\dots,n)$ and an inscribed simplex $\sigma_n'=A_0'A_1'\cdots A_n'$ of σ_n, we take an interior point B' of σ_n' such that $(\lambda_0',\lambda_1',\dots,\lambda_n')$ is the barycentric coordinates of the point B' with respect to coordinate simplex

σ'_n, here $\lambda'_i = x_i \Big/ \sum_{j=0}^{n} x_j \ \ (i = 0, 1, \ldots, n)$. Using equality (2.4) we have

$$\sum_{0 \le i < j \le n} \lambda'_i \lambda'_j \left(a'_{ij}\right)^2 = \sum_{i=0}^{n} \lambda'_i \left(\overrightarrow{B'A'_i}\right)^2,$$

i.e.

$$\sum_{0 \le i < j \le n} x_i x_j \left(a'_{ij}\right)^2 = \left(\sum_{i=0}^{n} x_i\right) \left(\sum_{i=0}^{n} x_i \left(\overrightarrow{B'A'_i}\right)^2\right). \tag{2.5}$$

Since B' is an interior point of σ'_n and σ'_n is an inscribed simplex of σ_n, so B' is an interior point of σ_n. Let the point Q_i be the orthogonal projection of the point B' on the ith face f_i of σ_n, then

$$\sum_{i=0}^{n} x_i \left(\overrightarrow{B'A'_i}\right)^2 \ge \sum_{i=0}^{n} x_i \left(\overrightarrow{B'Q_i}\right)^2. \tag{2.6}$$

Equality in (2.6) holds if and only if $Q_i \equiv A'_i \ (i = 0, 1, \ldots, n)$. In addition, we have

$$\sum_{i=0}^{n} \left|\overrightarrow{B'Q_i}\right| F_i = nV. \tag{2.7}$$

By the Cauchy's inequality and (2.7) we have

$$\left(\sum_{i=0}^{n} x_i \overrightarrow{B'Q_i}^2\right) \left(\sum_{i=0}^{n} x_i^{-1} F_i^2\right) \ge \left(\sum_{i=0}^{n} \left|\overrightarrow{B'Q_i}\right| \cdot F_i\right)^2 = (nV)^2. \tag{2.8}$$

Using (2.5), (2.6) and (2.8), we get

$$\left(\sum_{0\le i<j\le n} x_i x_j \left(a'_{ij}\right)^2\right)\left(\sum_{i=0}^{n} x^{-1}F_i^2\right) \ge n^2\left(\sum_{i=0}^{n} x_i\right)V^2. \tag{2.9}$$

Taking $x_0 = x_1 = \cdots = x_n = 1$ in (2.9), we get inequality (2.1). It is easy to prove that equality in (2.1) holds if the simplex σ_n is regular and σ'_n is the tangent point simplex of σ_n. □

Lemma 2.2 ([1, 6]). *For the n-dimensional simplex σ_n, we have*

$$\sum_{i=0}^{n} F_i^2 \le \left[n^{n-4}(n!)^2(n+1)^{n-2}\right]^{-1}\left(\sum_{0\le i<j\le n} a_{ij}^2\right), \tag{2.10}$$

with equality if the simplex σ_n is regular.

Lemma 2.3 ([2]). *Let P be an interior point of the simplex σ, σ''_n the pedal simplex of the point P with respect to σ_n, then*

$$V \ge n^n V'', \tag{2.11}$$

with equality if the simplex σ_n is regular.

Lemma 2.4 ([1]). *For the n-dimensional simplex σ_n, we have*

$$V \ge \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!} r^n, \tag{2.12}$$

with equality if the simplex σ_n is regular.

Lemma 2.5 ([4]). *For the n-dimensional simplex σ_n, we have*

$$\sum_{0\le i<j\le n} a_{ij}^2 = (n+1)^2 \left(R^2 - \overline{OG}^2\right). \tag{2.13}$$

Here O and G are the circumcenter and barycenter of the simplex σ_n, respectively.

Proof of Theorem 1.1. Using inequalities (2.1) and (2.10), we get

$$\left(\sum_{0\le i<j\le n} (a'_{ij})^2\right)\left(\sum_{0\le i<j\le n} a_{ij}^2\right)^{n-1} \ge n^{n-2}(n!)^2(n+1)^{n-1}V^2. \tag{2.14}$$

By Lemma 2.5 we have

$$\sum_{0\le i<j\le n} (a'_{ij})^2 \le (n+1)^2 (R')^2. \tag{2.15}$$

From (2.13), (2.14) and (2.15) we get

$$(R')^2 \left(R^2 - \overline{OG}^2\right)^{n-1} \ge \frac{n^{n-1}(n!)^2}{(n+1)^{n+1}} V^2. \tag{2.16}$$

Using inequalities (2.16) and (2.12), we get inequality (1.1). It is easy to prove that equality in (1.1) holds if the simplex σ_n is regular and σ'_n is the tangent point simplex of σ_n. □

Proof of Theorem 1.3. Since the pedal simplex σ_n'' is an inscribed simplex of the simplex σ_n, thus inequality (2.16) holds for the pedal simplex σ_n'', i.e.

$$(R'')^2\left(R^2-\overline{OG}^2\right)^{n-1}\geq\frac{n^{n-2}(n!)^2}{(n+1)^{n+1}}V^2. \tag{2.17}$$

Using inequalities (2.17) and (2.11), we get

$$(R'')^2 R^{2(n-1)}\geq(R'')^2\left(R^2-\overline{OG}^2\right)^{n-1}\geq\frac{n^{3n-2}(n!)^2}{(n+1)^{n+1}}(V'')^2 \tag{2.18}$$

By Lemma 2.4 we have

$$V''\geq\frac{n^{n/2}(n+1)^{(n+1)/2}}{n!}(r'')^n. \tag{2.19}$$

From (2.18) and (2.19) we obtain inequality (1.4). It is easy to prove that equality in (1.4) holds if the simplex σ_n is regular and σ_n'' is the tangent point simplex of σ_n. □

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