

# Journal of Inequalities in Pure and Applied Mathematics

## INEQUALITIES FOR INSCRIBED SIMPLEX AND APPLICATIONS

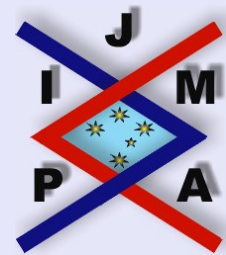
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©2000 Victoria University  
ISSN (electronic): 1443-5756  
019-05



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volume 7, issue 5, article 165,  
2006.

*Received 18 January, 2005;*  
*accepted 06 June, 2005.*

*Communicated by: W.S. Cheung*

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## Abstract

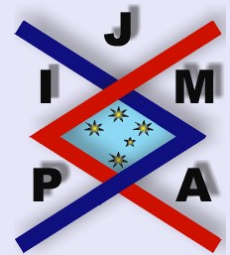
In this paper, we study the problem of geometric inequalities for the inscribed simplex of an  $n$ -dimensional simplex. An inequality for the inscribed simplex of a simplex is established. Applying it we get a generalization of  $n$ -dimensional Euler inequality and an inequality for the pedal simplex of a simplex.

*2000 Mathematics Subject Classification:* 51K16, 52A40.

*Key words:* Simplex, Inscribed simplex, Inradius, Circumradius, Inequality.

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# 1. Main Results

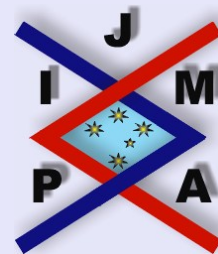
Let  $\sigma_n$  be an  $n$ -dimensional simplex in the  $n$ -dimensional Euclidean space  $E^n$ ,  $V$  denote the volume of  $\sigma_n$ ,  $R$  and  $r$  the circumradius and inradius of  $\sigma_n$ , respectively. Let  $A_0, A_1, \dots, A_n$  be the vertices of  $\sigma_n$ ,  $a_{ij} = |A_i A_j|$  ( $0 \leq i < j \leq n$ ),  $F_i$  denote the area of the  $i$ th face  $f_i = A_0 \cdots A_{i-1} A_{i+1} \cdots A_n$  ( $(n-1)$ -dimensional simplex) of  $\sigma_n$ , points  $O$  and  $G$  be the circumcenter and barycenter of  $\sigma_n$ , respectively. For  $i = 0, 1, \dots, n$ , let  $A'_i$  be an arbitrary interior point of the  $i$ th face  $f_i$  of  $\sigma_n$ . The  $n$ -dimensional simplex  $\sigma'_n = A'_0 A'_1 \cdots A'_n$  is called the inscribed simplex of the simplex  $\sigma_n$ . Let  $a'_{ij} = |A'_i A'_j|$  ( $0 \leq i < j \leq n$ ),  $R'$  denote the circumradius of  $\sigma'_n$ ,  $P$  be an arbitrary interior point of  $\sigma_n$ ,  $P_i$  be the orthogonal projection of the point  $P$  on the  $i$ th face  $f_i$  of  $\sigma_n$ . The  $n$ -dimensional simplex  $\sigma''_n = P_0 P_1 \cdots P_n$  is called the pedal simplex of the point  $P$  with respect to the simplex  $\sigma_n$  [1] – [2], let  $V''$  denote the volume of  $\sigma''_n$ ,  $R''$  and  $r''$  denote the circumradius and inradius of  $\sigma''_n$ , respectively. We note that the pedal simplex  $\sigma''_n$  is an inscribed simplex of the simplex  $\sigma_n$ . Our main results are following theorems.

**Theorem 1.1.** *Let  $\sigma'_n$  be an inscribed simplex of the simplex  $\sigma_n$ , then we have*

$$(1.1) \quad (R')^2 (R^2 - \overline{OG}^2)^{n-1} \geq n^{2(n-1)} r^{2n},$$

*with equality if the simplex  $\sigma_n$  is regular and  $\sigma'_n$  is the tangent point simplex of  $\sigma_n$ .*

Let  $T_i$  be the tangent point where the inscribed sphere of the simplex  $\sigma_n$  touches the  $i$ th face  $f_i$  of  $\sigma_n$ . The simplex  $\bar{\sigma}_n = T_0 T_1 \cdots T_n$  is called the tangent point simplex of  $\sigma_n$  [3]. If we take  $A'_i \equiv T_i$  ( $i = 0, 1, \dots, n$ ) in Theorem 1.1,



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then  $\sigma'_n$  and  $\bar{\sigma}_n$  are the same and  $R' = r$ , we get a generalization of the  $n$ -dimensional Euler inequality [4] as follows.

**Corollary 1.2.** *For an  $n$ -dimensional simplex  $\sigma_n$ , we have*

$$(1.2) \quad R^2 \geq n^2 r^2 + \overline{OG}^2,$$

*with equality if the simplex  $\sigma_n$  is regular.*

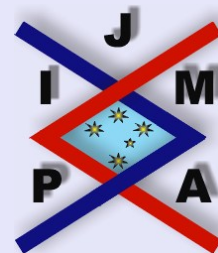
Inequality (1.2) improves the  $n$ -dimensional Euler inequality [5] as follows.

$$(1.3) \quad R \geq nr.$$

**Theorem 1.3.** *Let  $P$  be an interior point of the simplex  $\sigma_n$ , and  $\sigma'_n$  the pedal simplex of the point  $P$  with respect to  $\sigma_n$ , then*

$$(1.4) \quad R'' R^{n-1} \geq n^{2n-1} (r'')^n,$$

*with equality if the simplex  $\sigma_n$  is regular and  $\sigma''_n$  is the tangent point simplex of  $\sigma_n$ .*



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## 2. Some Lemma and Proofs of Theorems

To prove the theorems stated above, we need some lemmas as follows.

**Lemma 2.1.** *Let  $\sigma'_n$  be an inscribed simplex of the  $n$ -dimensional simplex  $\sigma_n$ , then we have*

$$(2.1) \quad \left( \sum_{0 \leq i < j \leq n} (a'_{ij})^2 \right) \left( \sum_{i=0}^n F_i^2 \right) \geq n^2(n+1)V^2,$$

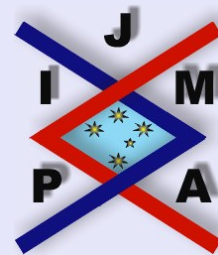
with equality if the simplex  $\sigma_n$  is regular and  $\sigma'_n$  is the tangent point simplex of  $\sigma_n$ .

*Proof.* Let  $B$  be an interior point of the simplex  $\sigma_n$ , and  $(\lambda_0, \lambda_1, \dots, \lambda_n)$  the barycentric coordinates of the point  $B$  with respect to coordinate simplex  $\sigma_n$ . Here  $\lambda_i = V_i V^{-1}$  ( $i = 0, 1, \dots, n$ ),  $V_i$  is the volume of the simplex  $\sigma_n(i) = BA_0 \cdots A_{i-1} A_{i+1} \cdots A_n$  and  $\sum_{i=0}^n \lambda_i = 1$ . Let  $Q$  be an arbitrary point in  $E^n$ , then

$$\overrightarrow{QB} = \sum_{i=0}^n \lambda_i \overrightarrow{QA_i}.$$

From this we have

$$\sum_{i=0}^n \lambda_i \overrightarrow{BA_i} = \sum_{i=0}^n \lambda_i (\overrightarrow{QA_i} - \overrightarrow{QB}) = \vec{0},$$



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$$\begin{aligned}
 (2.2) \quad \sum_{i=0}^n \lambda_i \left( \overrightarrow{QA_i} \right)^2 &= \sum_{i=0}^n \lambda_i \left( \overrightarrow{QB} + \overrightarrow{BA_i} \right)^2 \\
 &= \sum_{i=0}^n \lambda_i \overrightarrow{QB}^2 + 2\overrightarrow{QB} \cdot \sum_{i=0}^n \lambda_i \overrightarrow{BA_i} + \sum_{i=0}^n \lambda_i \left( \overrightarrow{BA_i} \right)^2 \\
 &= \overrightarrow{QB}^2 + \sum_{i=0}^n \lambda_i \left( \overrightarrow{BA_i} \right)^2.
 \end{aligned}$$

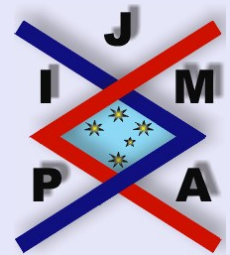
For  $j = 0, 1, \dots, n$ , taking  $Q \equiv A_j$  in (2.2) we get

$$\begin{aligned}
 (2.3) \quad \sum_{i=0}^n \lambda_i \lambda_j \left( \overrightarrow{A_i A_j} \right)^2 \\
 = \lambda_j \left( \overrightarrow{BA_j} \right)^2 + \lambda_j \sum_{i=0}^n \lambda_i \left( \overrightarrow{BA_i} \right)^2 \quad (j = 0, 1, \dots, n).
 \end{aligned}$$

Adding up these equalities in (2.3) and noting that  $\sum_{j=0}^n \lambda_j = 1$ , we get

$$(2.4) \quad \sum_{0 \leq i < j \leq n} \lambda_i \lambda_j \left( \overrightarrow{A_i A_j} \right)^2 = \sum_{i=0}^n \lambda_i \left( \overrightarrow{BA_i} \right)^2.$$

For any real numbers  $x_i > 0$  ( $i = 0, 1, \dots, n$ ) and an inscribed simplex  $\sigma'_n = A'_0 A'_1 \cdots A'_n$  of  $\sigma_n$ , we take an interior point  $B'$  of  $\sigma'_n$  such that  $(\lambda'_0, \lambda'_1, \dots, \lambda'_n)$  is the barycentric coordinates of the point  $B'$  with respect to coordinate simplex



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$\sigma'_n$ , here  $\lambda'_i = x_i / \sum_{j=0}^n x_j$  ( $i = 0, 1, \dots, n$ ). Using equality (2.4) we have

$$\sum_{0 \leq i < j \leq n} \lambda'_i \lambda'_j (a'_{ij})^2 = \sum_{i=0}^n \lambda'_i \left( \overrightarrow{B'A'_i} \right)^2,$$

i.e.

$$(2.5) \quad \sum_{0 \leq i < j \leq n} x_i x_j (a'_{ij})^2 = \left( \sum_{i=0}^n x_i \right) \left( \sum_{i=0}^n x_i \left( \overrightarrow{B'A'_i} \right)^2 \right).$$

Since  $B'$  is an interior point of  $\sigma'_n$  and  $\sigma'_n$  is an inscribed simplex of  $\sigma_n$ , so  $B'$  is an interior point of  $\sigma_n$ . Let the point  $Q_i$  be the orthogonal projection of the point  $B'$  on the  $i$ th face  $f_i$  of  $\sigma_n$ , then

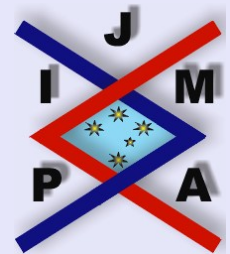
$$(2.6) \quad \sum_{i=0}^n x_i \left( \overrightarrow{B'A'_i} \right)^2 \geq \sum_{i=0}^n x_i \left( \overrightarrow{B'Q_i} \right)^2.$$

Equality in (2.6) holds if and only if  $Q_i \equiv A'_i$  ( $i = 0, 1, \dots, n$ ). In addition, we have

$$(2.7) \quad \sum_{i=0}^n \left| \overrightarrow{B'Q_i} \right| F_i = nV.$$

By the Cauchy's inequality and (2.7) we have

$$(2.8) \quad \left( \sum_{i=0}^n x_i \overrightarrow{B'Q_i}^2 \right) \left( \sum_{i=0}^n x_i^{-1} F_i^2 \right) \geq \left( \sum_{i=0}^n \left| \overrightarrow{B'Q_i} \right| \cdot F_i \right)^2 = (nV)^2.$$



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Using (2.5), (2.6) and (2.8), we get

$$(2.9) \quad \left( \sum_{0 \leq i < j \leq n} x_i x_j (a'_{ij})^2 \right) \left( \sum_{i=0}^n x^{-1} F_i^2 \right) \geq n^2 \left( \sum_{i=0}^n x_i \right) V^2.$$

Taking  $x_0 = x_1 = \dots = x_n = 1$  in (2.9), we get inequality (2.1). It is easy to prove that equality in (2.1) holds if the simplex  $\sigma_n$  is regular and  $\sigma'_n$  is the tangent point simplex of  $\sigma_n$ .  $\square$

**Lemma 2.2 ([1, 6]).** *For the  $n$ -dimensional simplex  $\sigma_n$ , we have*

$$(2.10) \quad \sum_{i=0}^n F_i^2 \leq [n^{n-4}(n!)^2(n+1)^{n-2}]^{-1} \left( \sum_{0 \leq i < j \leq n} a_{ij}^2 \right),$$

*with equality if the simplex  $\sigma_n$  is regular.*

**Lemma 2.3 ([2]).** *Let  $P$  be an interior point of the simplex  $\sigma$ ,  $\sigma''_n$  the pedal simplex of the point  $P$  with respect to  $\sigma_n$ , then*

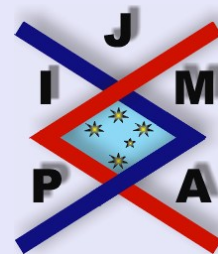
$$(2.11) \quad V \geq n^n V'',$$

*with equality if the simplex  $\sigma_n$  is regular.*

**Lemma 2.4 ([1]).** *For the  $n$ -dimensional simplex  $\sigma_n$ , we have*

$$(2.12) \quad V \geq \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!} r^n,$$

*with equality if the simplex  $\sigma_n$  is regular.*



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**Lemma 2.5 ([4]).** For the  $n$ -dimensional simplex  $\sigma_n$ , we have

$$(2.13) \quad \sum_{0 \leq i < j \leq n} a_{ij}^2 = (n+1)^2 (R^2 - \overline{OG}^2).$$

Here  $O$  and  $G$  are the circumcenter and barycenter of the simplex  $\sigma_n$ , respectively.

*Proof of Theorem 1.1.* Using inequalities (2.1) and (2.10), we get

$$(2.14) \quad \left( \sum_{0 \leq i < j \leq n} (a'_{ij})^2 \right) \left( \sum_{0 \leq i < j \leq n} a_{ij}^2 \right)^{n-1} \geq n^{n-2} (n!)^2 (n+1)^{n-1} V^2.$$

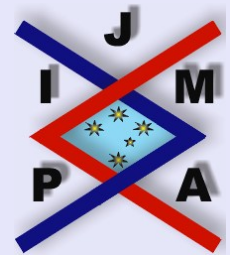
By Lemma 2.5 we have

$$(2.15) \quad \sum_{0 \leq i < j \leq n} (a'_{ij})^2 \leq (n+1)^2 (R')^2.$$

From (2.13), (2.14) and (2.15) we get

$$(2.16) \quad (R')^2 (R^2 - \overline{OG}^2)^{n-1} \geq \frac{n^{n-1} (n!)^2}{(n+1)^{n+1}} V^2.$$

Using inequalities (2.16) and (2.12), we get inequality (1.1). It is easy to prove that equality in (1.1) holds if the simplex  $\sigma_n$  is regular and  $\sigma'_n$  is the tangent point simplex of  $\sigma_n$ .  $\square$



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*Proof of Theorem 1.3.* Since the pedal simplex  $\sigma_n''$  is an inscribed simplex of the simplex  $\sigma_n$ , thus inequality (2.16) holds for the pedal simplex  $\sigma_n''$ , i.e.

$$(2.17) \quad (R'')^2 \left( R^2 - \overline{OG}^2 \right)^{n-1} \geq \frac{n^{n-2} (n!)^2}{(n+1)^{n+1}} V^2.$$

Using inequalities (2.17) and (2.11), we get

$$(2.18) \quad (R'')^2 R^{2(n-1)} \geq (R'')^2 \left( R^2 - \overline{OG}^2 \right)^{n-1} \geq \frac{n^{3n-2} (n!)^2}{(n+1)^{n+1}} (V'')^2$$

By Lemma 2.4 we have

$$(2.19) \quad V'' \geq \frac{n^{n/2} (n+1)^{(n+1)/2}}{n!} (r'')^n.$$

From (2.18) and (2.19) we obtain inequality (1.4). It is easy to prove that equality in (1.4) holds if the simplex  $\sigma_n$  is regular and  $\sigma_n''$  is the tangent point simplex of  $\sigma_n$ . □



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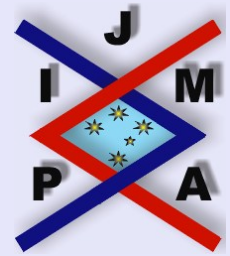
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