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INEQUALITIES FOR CONVEX SETS

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ABSTRACT. This paper collects together known inequalities relating the area, perimeter, width, diameter, inradius and circumradius of planar convex sets. Also, a technique for finding new inequalities is stated and illustrated.

Key words and phrases: planar convex set, inequality, area, perimeter, diameter, width, inradius, circumradius.

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1. INTRODUCTION

Let K be a convex set in the plane. Associated with K are a number of well-known functionals: the area $A = A(K)$, the perimeter $p = p(K)$, the diameter $d = d(K)$, the width $w = w(K)$, the inradius $r = r(K)$ and the circumradius $R = R(K)$. For many years we have been interested in inequalities involving these functionals. As the literature is extensive and spans more than 80 years, it will be helpful to summarize the known inequalities. This is done in sections 2 and 3. Then in section 4 we explore a technique for suggesting new inequalities, and give a number of conjectured new inequalities, mostly obtained by this method.

2. INEQUALITIES INVOLVING TWO FUNCTIONALS

Table 2.1 lists the known inequalities involving two functionals. The extremal sets referred to in the table are described below the table. These demonstrate that the inequalities are best possible. All the proofs of the results in this table can be found in the indicated sections of Yaglom and Boltyanskiĭ's book [17]. The (d, R) and (w, r) results are respectively known as Jung's Theorem and Blaschke's Theorem. Where a dagger (†) appears in the reference column, the result is trivial.

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Parameters	Inequality	Extremal Set	Reference
A, d	$4A \le \pi d^2$	$\bigcirc$	p. 239, ex. 610a
A, p	$4\pi A \le p^2$	$\bigcirc$	p. 207, ex. 5.8
A, r	$\pi r^2 \le A$	$\bigcirc$	†
A, R	$A \le \pi R^2$	$\bigcirc$	†
A, w	$w^2 \le \sqrt{3}A$	$\triangle_E$	p. 221, ex. 6.4
d, p	$2d < p \le \pi d$	$\vert, W$	†; p. 257, ex. 7.17a
d, r	$2r \le d$	$\bigcirc$	†
d, R	$\sqrt{3}R \le d \le 2R$	$\triangle_E, \bigcirc$	p. 213, ex. 6.1; †
d, w	$w \le d$	W	†
p, r	$2\pi r \le p$	$\bigcirc$	†
p, R	$4R < p \le 2\pi R$	$\vert, W$	†
p, w	$\pi w \le p$	W	p. 258, ex. 7.18a
r, R	$r \le R$	$\bigcirc$	†
r, w	$2r \le w \le 3r$	$\bigcirc, \triangle_E$	†; p. 215, ex. 6.2
R, w	$w \le 2R$	$\bigcirc$	†

Table 2.1: Inequalities involving two functionals.

The extremal sets

$\vert$	line segment
$\bigcirc$	circle
$\triangle_E$	equilateral triangle
W	sets of constant width

3. INEQUALITIES INVOLVING THREE FUNCTIONALS

Table 3.1 lists the known inequalities involving three functionals. Extra information, signified in the Notes column, appears after the table. The extremal sets referred to in the table are also listed after the table. These demonstrate that the inequalities are best possible. Some of the results in the table are established in [17]. These are indicated by a number in the References column, and details appear after the table. No inequalities appear to be known relating (p, r, R) or (p, R, w).

Param.	Condition	Inequality	Note	Ext. Set	Ref.
1. A, d, p		$8\phi A \le p\,(p - 2d\cos\phi)$	1a	()	[8][1]
		$d\,(p - 2d) \le 4A \le pd$		$\vert, \bigcirc$	[5]
	$2d \le p \le 3d$	$4A \ge (p - 2d)\sqrt{4pd - p^2}$		$\triangle_I$	[8, 9] [2]
	$3d \le p \le \pi d$	$4A \ge \sqrt{3}d\,(p - 2d)$	1b	$\triangle_E$	[9]
2. A, d, r		$A < 2dr$		$\Vert$	[6]
3. A, d, R		$(2R - d)\,A \le \pi\left(3\sqrt{3} - 5\right)R^3$	3a		[15]

4. A, d, w	$2w \leq \sqrt{3}d$ $\sqrt{3} < 2w < 2d$	$A < wd$ $2A \leq w\sqrt{d^2 - w^2}$ $+2d^2 \arcsin(w/d)$ $2A \geq wd$ $2A > \pi w^2 - \sqrt{3}d^2$ $+6w^2(\tan\delta - \delta)$ $2A \geq 3dw - \sqrt{3}d^2$ $2A \geq \pi w^2 - \sqrt{3}d^2$	4a	$\|$ ⦶ $\triangle$ $\triangle_Y$ $\triangle_E$ $\triangle_R$	[8] [8][3] [8][4] [16] [16] [16]
5. A, p, r		$A \leq r(p - \pi r)$ $2A \geq pr$		$\bigcirc$ $\triangle$	[2] [3]
6. A, p, R		$A \leq R(p - \pi R)$ $A < 2R(p - 2R)$ $A > R(p - 4R)$	6a	$\bigcirc$ $\vert$	[2] [6] [4]
7. A, p, w	$2\sqrt{3}w \leq p$ $\pi w < p < 2\sqrt{3}w$ $\pi w = p$	$2A \leq w\left(p - \frac{1}{2}\pi w\right)$ $A \geq A_1^*$ $2A \geq w\left(p - \sqrt{3}w\sec^2\gamma\right)$ $2A \geq \left(\pi - \sqrt{3}\right)w^2$ $6A \geq 4\sqrt{3}w^2 - pw$ $4A \geq pw - \frac{2}{\sqrt{3}}w^2$ $6A \geq pw$	7a 7b	○≡○ $\triangle_I$ $\triangle_Y$ $\triangle_R$ $\triangle_E$ $\triangle_E$ $\triangle_E$	[8][5] [18][6] [18] [10, 11][7] [16] [7] [16]
8. A, r, R		$A < 4rR$ $A > 2rR$		$\|$ $\vert$	[6] [6]
9. A, r, w		$4(w - 2r)A < w^3$ $\sqrt{3}(w - 2r)A \leq w^2 r$ $(w - 2r)A \leq \sqrt{3}wr^2$ $\leq 3\sqrt{3}r^3$		$\sqcup$ $\triangle_E$ $\triangle_E$	[14] [14] [14]
10. A, R, w		$A < 4Rw$ $A > \sqrt{3}Rw$		$\|$ $\triangle_E$	[6] [6]
11. d, p, r		$p < 2d + 4r$		$\|$	[6]
12. d, p, R		$(2R - d)p$ $\leq \left(2\sqrt{3} - 3\right)\pi R^2$		W	[15]
13. d, p, w		$p \leq 2\sqrt{d^2 - w^2}$ $+ 2d\arcsin(w/d)$ $p \geq 2\sqrt{d^2 - w^2}$ $+2w\arcsin(w/d)$		⦶ ◁○▷	[8][8] [8][9]
14. d, r, R		$(2R - d)r \leq \left(3\sqrt{3} - 5\right)R^2$		$\triangle_R$	[15]
15. d, r, w		$\sqrt{3}(w - 2r)d \leq 2wr \leq 6r^2$ $2(w - 2r)d \leq w^2$		$\triangle_E$ $\sqcup$	[13, 14] [14]
16. d, R, w		$(2R - d)w$ $\leq \sqrt{3}\left(2 - \sqrt{3}\right)R^2$ $3(2R - d)$ $\leq 2\left(2 - \sqrt{3}\right)w$		$\triangle_R$ $\triangle_E$	[15] [1]
17. p, r, R					

18. p, r, W		$\sqrt{3}\,(w-2r)\,p \le 2w^2$ $(w-2r)\,p \le 2\sqrt{3}wr$ $\le 6\sqrt{3}r^2$		$\triangle_E$ $\triangle_E$	[14] [14]
19. p, R, w					
20. r, R, w		$4\,(w-2r)\,R \le w^2$ $3\,(w-2r)\,R \le 2wr$ $\le 6r^2$		$\sqcup$ $\triangle_E$	[14] [14]

Table 3.1: Inequalities involving three functionals

The extremal sets

- $|$ line segment
- () lens: the intersection of two congruent circular disks
- ○≡○ convex hull of two congruent circles
- $\triangle$ triangle
- $\triangle_I$ isoceles triangle
- $\triangle_Y$ equilateral Yamanouti triarc, bounded by three circular arcs of radius w whose centres lie at the vertices of an equilateral triangle of side length d, and by the six tangents drawn from the vertices of this triangle to these arcs
- W sets of constant width
- ◯ circle
- ◁◯▷ convex hull of a disk and two symmetrically placed points
- (‖) intersection of a disk and a symmetrically placed strip
- $\triangle_E$ equilateral triangle
- $\triangle_R$ Reuleaux triangle
- $\sqcup$ half strip, occuring as the limit of an isoceles triangle with increasinng altitude on a given base
- $\|$ infinite strip bounded by two parallel lines

Notes

1. A, p, d Note 1a: $2\phi d = p \sin \phi$.
   Note 1b: This is not best possible unless $p = 3d$.

3. A, d, R Note 3a: This bound is not best possible. See Conjecture 4.7 below.

4. A, d, w Note 4a: Here $\delta = \arccos{(w/d)}$.

6. A, p, r Note 6a: This bound is not best possible.

7. A, d, w Note 7a: Here A_1^* is the middle root of the equation

$$128px^3 - 16w\left(5p^2 + w^2\right)x^2 + 16w^2p^3x - w^3p^4 = 0.$$

Note 7b: $6w\,(\tan\gamma - \gamma) = p - \pi w$.

References

The precise references to the proofs in Yaglom and Boltyanskiĭ [17] are:

[1]p. 240, ex. 6.11(a) [2]p. 229, ex. 6.8(a) [3]p. 240, ex. 6.10(b)
[4]p. 227, ex. 6.7 [5]p. 241, ex. 6.11(b) [6] p. 231, ex. 6.8(b)
[7]p. 260, ex. 7.20 [8]p. 257, ex. 7.17(b) [9] p. 258, ex. 7.18(b).

4. OBTAINING NEW INEQUALITIES

Osserman [12] takes the classical isoperimetric inequality

$$p^2 \geq 4\pi A \tag{4.1}$$

and from it derives the new inequality

$$p^2 - 4\pi A \geq \pi^2(R-r)^2. \tag{4.2}$$

We can think of this in the following way. The function $f(K) = \pi^2(R-r)^2$ takes the value 0 for the extremal set (the circle) of (4.1). Hence for the circle, (4.2) and (4.1) are identical. Osserman shows that in fact (4.2) is satisfied for all other sets K.

Let us see if we can adapt this method to find other new inequalities. The method will become clear by an example. From Table 2.1 we have Jung's Theorem: $d \geq \sqrt{3}R$, with equality when and only when the set K is an equilateral triangle. For an equilateral triangle, we know that $2A = dw$ or $d = \frac{2A}{w}$. Substituting into Jung's inequality *suggests* the inequality $2A \geq \sqrt{3}Rw$, which is in fact true and was discovered by Henk (see [6]). So the technique is as follows:

- Consider a known inequality and its extremal set.
- Take an equality relating functionals of the extremal set which includes a functional in the chosen inequality.
- Substitute for this functional in the chosen inequality to obtain a new inequality.

Of course the new inequality may be incorrect; it is then quickly discarded. Again, the new inequality may be trivial, in the sense that it is a combination of known simpler inequalities. For example, the technique gives

$$(w-2r)p \leq 2A, \text{ with extremal set } \triangle_E;$$

this occurs as a combination of $(w-2r)p \leq 2w^2/\sqrt{3}$ (Table 3.1) and $w^2 \leq \sqrt{3}A$ (Table 2.1). Similarly,

$$4\pi r^2 \leq pd, \text{ with extremal set } \bigcirc,$$

occurs as a combination of $2\pi r \leq p$ and $2r \leq d$ (Table 2.1). However, the following more interesting conjectures have been obtained by this method.

Conjecture 4.1. $2w^2 \leq \sqrt{3}pr$ *with extremal set* $\triangle_E$.

Conjecture 4.2. $\sqrt{3}wR \leq pr$, $\triangle_E$.

Conjecture 4.3. $(p-2d)w \leq 2A$, $\triangle_E$.

Conjecture 4.4. $wp \leq 9dr$, $\triangle$.

Conjecture 4.5. $3(w-2r)(p-2d) \leq 2A$, $\triangle_E$.

Conjecture 4.6. $dw \leq pr$, $\triangle$.

Finally we have the older [15]

Conjecture 4.7. $2(2R-d)A \leq 3(2-\sqrt{3})(\pi - \sqrt{3})R^3$, $\triangle_R$.

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