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ON AN OPEN PROBLEM OF F. QI

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ABSTRACT. In the paper *Several integral inequalities* published in J. Inequal. Pure Appl. Math. **1** (2000), no. 2, Art. 19 (http://jipam.vu.edu.au/v1n2.html.001_00.html) and RGMIA Res. Rep. Coll. **2**(7) (1999), Art. 9, 1039–1042 (http://rgmia.vu.edu.au/v2n7.html) by F. Qi, an open problem was posed. In this article we give the solution and further generalizations of this problem. Reverse inequalities to the posed one are considered and, finally, the derived results are extended to weighted integral inequalities.

Key words and phrases: Integral inequality, Integral Hölder inequality, Reverse Hölder inequality, Nehari inequality, Barnes-Godunova-Levin inequality, Weighted integral inequality, Pečarić inequality.

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1. INTRODUCTION AND PREPARATION

The following problem was posed by F. Qi in his paper [7]:

Under what condition does the inequality

$$\int_a^b [f(x)]^t dx \geq \left(\int_a^b f(x)dx\right)^{t-1} \tag{1.1}$$

hold for $t > 1$?

In the above-mentioned paper F. Qi considered integral inequalities with nonnegative differentiable integrands and derived his results by the use of analytic methods. Furthermore, in his joint paper with K.-W. Yu, F. Qi gave an affirmative answer to the quoted problem using the integral version of Jensen's inequality and a lemma of convexity. Namely, they showed that (1.1) holds true for all $f \in C[a,b]$ such that $\int_a^b f(x)dx \geq (b-a)^{t-1}$ for given $t > 1$, see [9]. N. Towghi [8] solved F. Qi's problem by imposing sufficient conditions upon the integrand f being differentable at $x = a$.

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In this paper, our approach in deriving (1.1) will be somewhat different. We avoid the assumptions of differentiability used in [7, 8] and the convexity criteria used in [9]. Instead, by using the classical integral inequalities due to Hölder, Nehari, Barnes and their generalizations by Godunova and Levin, we will show that it is enough to consider suitable bounded integrands for the validity of a few inequalities which contain some results of [7] and [9] as corollaries.

In this article we consider a finite integration domain, i.e. in $\int_a^b f$ we take $-\infty < a < b < \infty$ (possible extensions to an infinite integration domain are left to the interested reader). We denote $\int_a^b f d\mu$ as the weighted (Stieltjes type) integral where μ is nonnegative, monotonic nondecreasing, left continuous, finite total variation, Stieltjes measure. In other words μ can be taken to be a probability distribution function.

Finally, we are interested in describing the class of functions for which the reverse variants of (1.1) hold true. This will be done using reversed Hölder inequalities obtained by Nehari, Barnes and Godunova-Levin. See [1], [2], [3], and [5].

Now, we give the Nehari inequality and the Barnes-Godunova-Levin inequality.

Lemma 1.1 ([3], Nehari Inequality)**.** *Let f, g be nonnegative concave functions on $[a, b]$. Then, for $p, q > 0$ such that $p^{-1} + q^{-1} = 1$, we have*

$$\left(\int_a^b f^p\right)^{1/p} \left(\int_a^b g^q\right)^{1/q} \le N(p,q) \int_a^b fg, \tag{1.2}$$

where

$$N(p,q) = \frac{6}{(1+p)^{1/p}(1+q)^{1/q}}. \tag{1.3}$$

Lemma 1.2 ([2, 3, 4], Barnes-Godunova-Levin Inequality)**.** *Let f, g be nonnegative concave functions on $[a, b]$. When $p, q > 1$ it holds true that*

$$\left(\int_a^b f^p\right)^{1/p} \left(\int_a^b g^q\right)^{1/q} \le B(p,q) \int_a^b fg, \tag{1.4}$$

where

$$B(p,q) = \frac{6(b-a)^{1/p+1/q-1}}{(1+p)^{1/p}(1+q)^{1/q}}. \tag{1.5}$$

For further details about inequalities (1.2) and (1.4), please refer to §4.3 of the monograph [4].

In order to obtain weighted variants of our principal results we need the weighted Hölder inequality and the weighted inverse Hölder inequality. The last one was considered (to the best of my knowledge) only by Pečarić in [4, 6]. Here we give his result in a form suitable for us.

Lemma 1.3 ([4, 6], Pečarić Inequality)**.** *Let f, g be nonnegative, concave functions and fg be μ-integrable on $[a, b]$, where μ is a probability distribution function. Let f be nondecreasing and g nonincreasing. Then for $p, q \ge 1$ we have*

$$\left(\int_a^b f^p d\mu\right)^{1/p} \left(\int_a^b g^q d\mu\right)^{1/q} \le P(p,q;\mu) \int_a^b fg d\mu, \tag{1.6}$$

where

$$P(p,q;\mu) = \frac{\left(\int_a^b (x-a)^p d\mu\right)^{1/p} \left(\int_a^b (b-x)^q d\mu\right)^{1/q}}{\int_a^b (b-x)(x-a) d\mu}. \tag{1.7}$$

2. THE DIRECT INEQUALITY

In this section we will give an inequality which contains, as a special case, an answer to F. Qi's problem (1.1), see Corollary 2.1.1.

Theorem 2.1. *Suppose that* $\beta > 0,\ \max\{\beta, 1\} < \alpha$ *and let* f^α *be integrable on* $[a, b]$. *For* $f(x) \geq (b-a)^{\frac{\beta-1}{\alpha-\beta}}$, *we have*

$$\int_a^b f^\alpha \geq \left(\int_a^b f\right)^\beta. \tag{2.1}$$

Proof. As $f(x) > 0$ on the integration domain, by the Hölder inequality we get

$$\int_a^b f \leq (b-a)^{1-1/\alpha} \left(\int_a^b f^\alpha\right)^{1/\alpha},$$

i.e.

$$\int_a^b f^\alpha \geq (b-a)^{1-\alpha} \left(\int_a^b f\right)^{\alpha-\beta} \left(\int_a^b f\right)^\beta \geq \underbrace{(b-a)^{1-\beta}(\min_{[a,b]} f)^{\alpha-\beta}}_{K_1} \left(\int_a^b f\right)^\beta.$$

Now, by $f(x) \geq (b-a)^{\frac{\beta-1}{\alpha-\beta}}$ we deduce that $K_1 \geq 1$. Hence, the inequality (2.1) is proved. $\square$

Corollary 2.2. *For all* $f(x) \geq (b-a)^{t-2},\ f^t$ *integrable, the inequality* (1.1) *holds for* $t > 1$.

Proof. Set $\alpha = t = \beta + 1$ into (2.1). $\square$

3. THE REVERSE INEQUALITY

We now derive a reverse variant of F. Qi's inequality (1.1).

Theorem 3.1. *Let* f *be nonnegative, concave and integrable on* $[a, b]$, $\beta > 0$ *and* $\max\{\beta, 1\} < \alpha$. *Suppose*

$$f(x) \leq \left(\frac{(1+\alpha)(2\alpha-1)^{\alpha-1}}{6^\alpha(\alpha-1)^{\alpha-1}(b-a)^{1-\beta}}\right)^{\frac{1}{\alpha-\beta}}, \quad x \in [a, b]. \tag{3.1}$$

Then it follows that

$$\int_a^b f^\alpha \leq \left(\int_a^b f\right)^\beta. \tag{3.2}$$

Proof. Put $g \equiv 1,\ p \equiv \alpha$ into the Nehari inequality (1.2). Then we clearly obtain

$$\begin{aligned}\int_a^b f^\alpha &\leq \frac{N^\alpha(\alpha, \frac{\alpha}{\alpha-1})}{(b-a)^{\alpha-1}} \left(\int_a^b f\right)^{\alpha-\beta} \left(\int_a^b f\right)^\beta \\ &\leq \underbrace{N^\alpha(\alpha, \tfrac{\alpha}{\alpha-1})(b-a)^{1-\beta}(\max_{[a,b]} f)^{\alpha-\beta}}_{K_2} \left(\int_a^b f\right)^\beta.\end{aligned}$$

Here $N(\cdot,\cdot)$ is the Nehari constant given by (1.3). Now, by (3.1) we conclude that $K_2 \leq 1$. Thus, (3.2) is proved. $\square$

In the next theorem q is a scaling parameter. Thus, we obtain an inequality scaled by three parameters.

Theorem 3.2. *Let f be nonnegative concave and integrable on $[a,b]$. If*

$$f(x) \le \left(\frac{(1+\alpha)(1+q)^{\frac{\alpha}{q}}}{6^\alpha (b-a)^{1-\beta}} \right)^{\frac{1}{\alpha-\beta}}, \quad x \in [a,b], \tag{3.3}$$

then (3.2) *holds for all $\alpha > 0$ and $\beta > 0$ such that $\max\{\beta, 1\} < \alpha$ and $q > 1$.*

Proof. Substituting $g \equiv 1$, $p \equiv \alpha$ in the Barnes-Godunova-Levin inequality (1.4) and repeating the procedure of previous theorem, we obtain

$$\int_a^b f^\alpha \le \underbrace{B^\alpha(\alpha, q)(b-a)^{\alpha-\beta-\frac{\alpha}{q}} (\max_{[a,b]} f)^{\alpha-\beta}}_{K_3} \left(\int_a^b f \right)^\beta .$$

Here, $B(\cdot,\cdot)$ is the Barnes-Godunova-Levin constant described by (1.5). Using the assumed upper bound (3.3), we have that $K_3 \le 1$ and the proof is complete. □

4. WEIGHTED VARIANTS OF INEQUALITIES

In this section we are interested in the weighted variants of inequalities similar to (2.1) and (3.2). We follow the approach used in the previous sections. Here the main mathematical tools will be the weighted integral Hölder inequality and the reversed weighted Hölder inequality (1.6), derived by J. Pečarić, see Lemma 1.3.

Theorem 4.1. *Let $\beta > 0,\ \max\{\beta, 1\} < \alpha$ and let f^α be μ - integrable on $[a,b]$. If*

$$f(x) \ge (\mu(b) - \mu(a))^{\frac{\beta-1}{\alpha-\beta}},$$

then

$$\int_a^b f^\alpha d\mu \ge \left(\int_a^b f d\mu \right)^\beta . \tag{4.1}$$

The proof is a identical to the proof of Theorem 2.1, therefore it is omitted.

Theorem 4.2. *Let f be nonnegative, concave nondecreasing (nonincreasing) and μ - integrable on $[a,b]$. If $\beta > 0$, $0 < \max\{\beta, 1\} < \alpha$, $q \ge 1$ and*

$$f(x) \le \frac{(\mu(b)-\mu(a))^{\frac{\alpha}{q(\alpha-\beta)}-1}}{(P(\alpha,q;\mu))^{\alpha/(\alpha-\beta)}}, \quad x \in [a,b], \tag{4.2}$$

where $P(\alpha,q;\mu)$ is given by (1.7)*, the following inequality holds*

$$\int_a^b f^\alpha d\mu \le \left(\int_a^b f d\mu \right)^\beta . \tag{4.3}$$

Proof. Substituting $p \equiv \alpha, g \equiv 1$ into the inequality (1.6) we get

$$\begin{aligned} \int_a^b f^\alpha d\mu &\le \frac{P^\alpha(\alpha,q;\mu)}{(\mu(b)-\mu(a))^{\alpha/q}} \left(\int_a^b f d\mu \right)^{\alpha-\beta} \left(\int_a^b f d\mu \right)^\beta \\ &\le \underbrace{\frac{P^\alpha(\alpha,q;\mu)(\max_{[a,b]} f)^{\alpha-\beta}}{(\mu(b)-\mu(a))^{\alpha/q+\beta-\alpha}}}_{K_4} \left(\int_a^b f d\mu \right)^\beta . \end{aligned}$$

Now, taking into account the upper bound (4.2) of f on $[a,b]$ and the exact value of $P(\cdot,\cdot;\mu)$, we clearly conclude that $K_4 \le 1$, i.e. (4.3) is proved. □

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