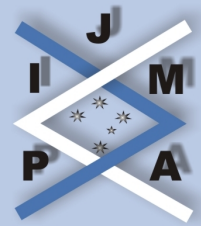


# A NOTE ON THE UPPER BOUNDS FOR THE DISPERSION



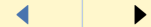
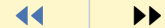
Upper Bounds for the  
Dispersion

Yu Miao and Guangyu Yang

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[Title Page](#)

[Contents](#)



Page 1 of 13

[Go Back](#)

[Full Screen](#)

[Close](#)

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*Abstract:* In this note we provide upper bounds for the standard deviation  $(\sigma(X))$ , for the quantity  $\sigma^2(X) + (x - \mathbb{E}(X))^2$  and for the  $L^p$  absolute deviation of a random variable. These improve and extend current results in the literature.

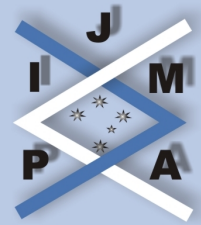
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# Contents

1	Introduction	3
2	Standard Deviation	4
3	$L^p$ Absolute Deviation	9



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vol. 8, iss. 3, art. 83, 2007

---

Title Page

Contents



Page 2 of 13

Go Back

Full Screen

Close

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Title Page

Contents

◀◀ ▶▶

◀ ▶

Page 3 of 13

Go Back

Full Screen

Close

## 1. Introduction

Let  $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}_+$  be the probability density function (p.d.f.) of the random variable  $X$  and  $\mathbb{E}(X)$  and  $\sigma(X)$  the mean and standard deviation respectively.

In [2], the authors gave upper bounds for the dispersion  $\sigma(X)$  and for  $\sigma^2(X) + (x - \mathbb{E}(X))^2$  in terms of the p.d.f. for a continuous random variable. Recently, Agbeko [1] also obtained some upper bounds for the same two quantities, namely,

$$(1.1) \quad \sigma(X) \leq \min\{\max\{|a|, |b|\}, (b - a)\},$$

and

$$(1.2) \quad \sqrt{\sigma^2(X) + (x - \mathbb{E}(X))^2} \leq 2 \min\{\max\{|a|, |b|\}, (b - a)\},$$

for all  $x \in [a, b]$ .

Both the work of Barnett et al. [2] and Agbeko [1] exclude the discrete case. In this present note, we remove this exclusion and improve the bounds of (1.1) and (1.2) in Section 2. In Section 3, we consider the  $L^p$  absolute deviation. The symbolism  $1_A$  denotes the indicator function on the set  $A$ .



## 2. Standard Deviation

Let  $\mathbb{P}$  be the distribution of the random variable  $X$ , then the expectation of  $X$  can be denoted by

$$\mathbb{E}(X) = \int_a^b x d\mathbb{P}(x)$$

and the dispersion or standard deviation of  $X$  by

$$\sigma(X) = \sqrt{\int_a^b x^2 d\mathbb{P}(x) - \left(\int_a^b x d\mathbb{P}(x)\right)^2}.$$

**Theorem 2.1.** *Let  $X$  be a random variable with  $a \leq X \leq b$ . Then we have*

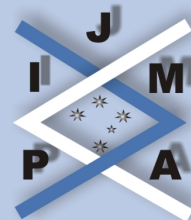
$$(2.1) \quad \sigma(X) \leq \min\{\max\{|a|, |b|\}, M(a, b, \mathbb{E}(X))(b - a)\},$$

where

$$M(a, b, \mathbb{E}(X)) = \frac{1}{\sqrt{2}} \sqrt{2 - \exp\left\{-\frac{2(b - \mathbb{E}(X))^2}{(b - a)^2}\right\} - \exp\left\{-\frac{2(\mathbb{E}(X) - a)^2}{(b - a)^2}\right\}}.$$

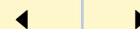
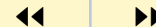
*Proof.* Since  $\sigma^2(X) = \mathbb{E}(X - \mathbb{E}(X))^2$ , by the Fubini's theorem, we have

$$(2.2) \quad \begin{aligned} \mathbb{E}(X - \mathbb{E}(X))^2 &= \mathbb{E} \int_0^{(X - \mathbb{E}(X))^2} ds \\ &= \mathbb{E} \int_0^\infty 1_{\{(X - \mathbb{E}(X))^2 \geq s\}} ds \\ &= \int_0^\infty \mathbb{P}((X - \mathbb{E}(X))^2 \geq s) ds \end{aligned}$$



Title Page

Contents



Page 5 of 13

Go Back

Full Screen

Close

$$\begin{aligned} &= \int_0^{(b-\mathbb{E}(X))^2} \mathbb{P}(X - \mathbb{E}(X) \geq \sqrt{s}) ds \\ &\quad + \int_0^{(\mathbb{E}(X)-a)^2} \mathbb{P}(-(X - \mathbb{E}(X)) \geq \sqrt{s}) ds. \end{aligned}$$

For any  $\lambda > 0$ , we have

$$\begin{aligned} \mathbb{E}e^{\lambda(X-\mathbb{E}(X))} &\leq e^{-\lambda\mathbb{E}(X)} \left( \frac{b - \mathbb{E}(X)}{b - a} e^{\lambda a} + \frac{\mathbb{E}(X) - a}{b - a} e^{\lambda b} \right) \\ &= e^{\lambda(a-\mathbb{E}(X))} \left( 1 - \frac{\mathbb{E}(X) - a}{b - a} + \frac{\mathbb{E}(X) - a}{b - a} e^{\lambda(b-a)} \right) \\ &\triangleq e^{L(\lambda)}, \end{aligned}$$

where

$$L(\lambda) = \lambda(a - \mathbb{E}(X)) + \log \left( 1 - \frac{\mathbb{E}(X) - a}{b - a} + \frac{\mathbb{E}(X) - a}{b - a} e^{\lambda(b-a)} \right).$$

The first two derivatives of  $L(\lambda)$  are

$$\begin{aligned} L'(\lambda) &= a - \mathbb{E}(X) + \frac{(\mathbb{E}(X) - a)e^{\lambda(b-a)}}{\left( 1 - \frac{\mathbb{E}(X)-a}{b-a} + \frac{\mathbb{E}(X)-a}{b-a} e^{\lambda(b-a)} \right)} \\ &= a - \mathbb{E}(X) + \frac{\mathbb{E}(X) - a}{\left( \left( 1 - \frac{\mathbb{E}(X)-a}{b-a} \right) e^{-\lambda(b-a)} + \frac{\mathbb{E}(X)-a}{b-a} \right)}, \\ L''(\lambda) &= \frac{(b - \mathbb{E}(X))(\mathbb{E}(X) - a)e^{-\lambda(b-a)}}{\left( \left( 1 - \frac{\mathbb{E}(X)-a}{b-a} \right) e^{-\lambda(b-a)} + \frac{\mathbb{E}(X)-a}{b-a} \right)^2}. \end{aligned}$$



Title Page

Contents



Page 6 of 13

Go Back

Full Screen

Close

Noting that

$$L''(\lambda) \leq \frac{(b-a)^2}{4},$$

then by Taylor's formula, we have

$$L(\lambda) \leq L(0) + L'(0)\lambda + \frac{(b-a)^2}{8}\lambda^2 = \frac{(b-a)^2}{8}\lambda^2.$$

Therefore from Markov's inequality, we have

$$\begin{aligned} (2.3) \quad \mathbb{P}(X - \mathbb{E}(X) \geq \sqrt{s}) &\leq \inf_{\lambda > 0} e^{-\sqrt{s}\lambda} \mathbb{E}e^{\lambda(X - \mathbb{E}(X))} \\ &\leq \inf_{\lambda > 0} \exp \left\{ -\sqrt{s}\lambda + \frac{(b-a)^2}{8}\lambda^2 \right\} \\ &= \exp \left\{ -\frac{2s}{(b-a)^2} \right\}. \end{aligned}$$

Similarly,

$$\mathbb{P}(-(X - \mathbb{E}(X)) \geq \sqrt{s}) \leq \exp \left\{ -\frac{2s}{(b-a)^2} \right\}.$$

From (2.2), it follows that

$$\begin{aligned} (2.4) \quad \mathbb{E}(X - \mathbb{E}(X))^2 &= \int_0^{(b-\mathbb{E}(X))^2} \mathbb{P}((X - \mathbb{E}(X)) \geq \sqrt{s}) ds \\ &\quad + \int_0^{(\mathbb{E}(X)-a)^2} \mathbb{P}(-(X - \mathbb{E}(X)) \geq \sqrt{s}) ds \end{aligned}$$

$$\begin{aligned} &\leq \int_0^{(b-\mathbb{E}(X))^2} \exp\left\{-\frac{2s}{(b-a)^2}\right\} ds + \int_0^{(\mathbb{E}(X)-a)^2} \exp\left\{-\frac{2s}{(b-a)^2}\right\} ds \\ &= \frac{(b-a)^2}{2} \left( 2 - \exp\left\{-\frac{2(b-\mathbb{E}(X))^2}{(b-a)^2}\right\} - \exp\left\{-\frac{2(\mathbb{E}(X)-a)^2}{(b-a)^2}\right\} \right). \end{aligned}$$

Furthermore,

$$\mathbb{E}(X - \mathbb{E}(X))^2 = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \leq \mathbb{E}(X^2) \leq \max\{|a|^2, |b|^2\},$$

which, by (2.4), implies the result.  $\square$

*Remark 1.* In fact, by,

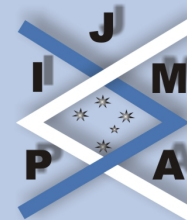
$$\begin{aligned} M(a, b, \mathbb{E}(X)) &= \frac{1}{\sqrt{2}} \sqrt{2 - \exp\left\{-\frac{2(b-\mathbb{E}(X))^2}{(b-a)^2}\right\} - \exp\left\{-\frac{2(\mathbb{E}(X)-a)^2}{(b-a)^2}\right\}} \\ &\leq \frac{1}{\sqrt{2}} \sqrt{2 - e^{-2}} < 1, \end{aligned}$$

$$\min\{\max\{|a|, |b|\}, M(a, b, \mathbb{E}(X))(b-a)\} \leq \min\{\max\{|a|, |b|\}, (b-a)\},$$

and is, therefore, a tighter bound than (1.1). If  $\mathbb{E}(X) = (b+a)/2$ , then

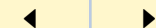
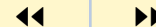
$$\begin{aligned} M(a, b, \mathbb{E}(X)) &= \frac{1}{\sqrt{2}} \sqrt{2 - \exp\left\{-\frac{2(b-\mathbb{E}(X))^2}{(b-a)^2}\right\} - \exp\left\{-\frac{2(\mathbb{E}(X)-a)^2}{(b-a)^2}\right\}} \\ &= \sqrt{1 - e^{-1/2}} < \frac{1}{\sqrt{2}}, \end{aligned}$$

which means that under some assumptions, this bound is better than the result of [2] for the case  $f \in L_1([a, b])$ .



Title Page

Contents



Page 7 of 13

Go Back

Full Screen

Close



Title Page

Contents



Page 8 of 13

Go Back

Full Screen

Close

**Corollary 2.2.** Let  $X$  be a random variable with  $a \leq X \leq b$ , then for any  $x \in [a, b]$ ,

$$(2.5) \quad \sqrt{\sigma^2(X) + (x - \mathbb{E}(X))^2} \leq \min\{2 \max\{|a|, |b|\}, N(a, b, \mathbb{E}(X))(b - a)\},$$

where

$$N^2(a, b, \mathbb{E}(X)) = 2 - \frac{1}{2} \left( \exp \left\{ -\frac{2(b - \mathbb{E}(X))^2}{(b - a)^2} \right\} + \exp \left\{ -\frac{2(\mathbb{E}(X) - a)^2}{(b - a)^2} \right\} \right).$$

*Proof.* It is clear that  $(x - \mathbb{E}(X))^2 \leq (b - a)^2$  and from the proof of Theorem 2.1, we know that

$$\begin{aligned} & \sigma^2(X) + (x - \mathbb{E}(X))^2 \\ & \leq \left( 2 - \frac{1}{2} \left( \exp \left\{ -\frac{2(b - \mathbb{E}(X))^2}{(b - a)^2} \right\} + \exp \left\{ -\frac{2(\mathbb{E}(X) - a)^2}{(b - a)^2} \right\} \right) \right) (b - a)^2. \end{aligned}$$

Furthermore,

$$\sigma^2(X) + (x - \mathbb{E}(X))^2 = \mathbb{E}(X - x)^2 \leq \max\{(x - y)^2, x, y \in [a, b]\}.$$

The remainder of the proof follows as Theorem 1.2 in Agbeko [1], giving

$$\max\{|x - y|, x, y \in [a, b]\} \leq 2 \min\{\max\{|a|, |b|\}, (b - a)\}.$$

□





[Title Page](#)

[Contents](#)

[◀](#) [▶](#)

[◀](#) [▶](#)

Page 9 of 13

[Go Back](#)

[Full Screen](#)

[Close](#)

journal of **inequalities**  
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### 3. $L^p$ Absolute Deviation

In fact by the method in Section 2, we could extend the case of  $L^p$  absolute deviation, i.e., the following quantity,

$$\sigma_p(X) = \mathbb{E} (|X - \mathbb{E}(X)|^p)^{1/p}.$$

We have the following

**Theorem 3.1.** *Let  $X$  be a random variable and  $p \geq 1$ . Assume that  $\mathbb{E}|X|^p < \infty$ , then its  $L^p$  absolute deviation has the following estimation for  $p > 2$ ,*

$$(\sigma_p(X))^p \leq \min\{|b - a|^p, M_1(a, b, p, X)\},$$

where

$$(3.1) \quad M_1(a, b, p, X) = \frac{(b - a)^2}{2} \left( 1 - \exp \left\{ -\frac{2(|b - \mathbb{E}(X)|^p \wedge 1)}{(b - a)^2} \right\} \right) \\ + (|b - \mathbb{E}(X)|^p - |b - \mathbb{E}(X)|^p \wedge 1) \\ \times \exp \left\{ -\frac{2(|b - \mathbb{E}(X)|^p \wedge 1)^{2/p}}{(b - a)^2} \right\} \\ + \frac{(b - a)^2}{2} \left( 1 - \exp \left\{ -\frac{2(|a - \mathbb{E}(X)|^p \wedge 1)}{(b - a)^2} \right\} \right) \\ + (|a - \mathbb{E}(X)|^p - |a - \mathbb{E}(X)|^p \wedge 1) \\ \times \exp \left\{ -\frac{2(|a - \mathbb{E}(X)|^p \wedge 1)^{2/p}}{(b - a)^2} \right\}.$$

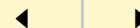
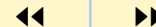
If  $1 \leq p < 2$ , then we have

$$(\sigma_p(X))^p \leq \min\{|b - a|^p, M_2(a, b, p, X)\},$$



Title Page

Contents



Page 10 of 13

Go Back

Full Screen

Close

where

$$(3.2) \quad M_2(a, b, p, X) \\ = |b - \mathbb{E}(X)|^p \wedge 1 + \frac{(b-a)^2}{2} \left( \exp \left\{ -\frac{2|b - \mathbb{E}(X)|^p \wedge 1}{(b-a)^2} \right\} - \exp \left\{ -\frac{2|b - \mathbb{E}(X)|^p}{(b-a)^2} \right\} \right) \\ + |a - \mathbb{E}(X)|^p \wedge 1 + \frac{(b-a)^2}{2} \left( \exp \left\{ -\frac{2|a - \mathbb{E}(X)|^p \wedge 1}{(b-a)^2} \right\} - \exp \left\{ -\frac{2|a - \mathbb{E}(X)|^p}{(b-a)^2} \right\} \right).$$

*Proof.* On one hand, by the Fubini's theorem, we have

$$(3.3) \quad \mathbb{E}|X - \mathbb{E}(X)|^p = \mathbb{E} \int_0^{|X - \mathbb{E}(X)|^p} ds \\ = \mathbb{E} \int_0^\infty 1_{\{|X - \mathbb{E}(X)|^p \geq s\}} ds \\ = \int_0^\infty \mathbb{P}(|X - \mathbb{E}(X)|^p \geq s) ds \\ = \int_0^{|b - \mathbb{E}(X)|^p} \mathbb{P}((X - \mathbb{E}(X)) \geq s^{1/p}) ds \\ + \int_0^{|\mathbb{E}(X) - a|^p} \mathbb{P}(-(X - \mathbb{E}(X)) \geq s^{1/p}) ds.$$

From the estimation (2.3), we have

$$\mathbb{E}|X - \mathbb{E}(X)|^p = \int_0^{|b - \mathbb{E}(X)|^p} \mathbb{P}(X - \mathbb{E}(X) \geq s^{1/p}) ds \\ + \int_0^{|\mathbb{E}(X) - a|^p} \mathbb{P}(-(X - \mathbb{E}(X)) \geq s^{1/p}) ds$$



Title Page

Contents



Page 11 of 13

Go Back

Full Screen

Close

$$\leq \int_0^{|b-\mathbb{E}(X)|^p} \exp\left\{-\frac{2s^{2/p}}{(b-a)^2}\right\} ds + \int_0^{|\mathbb{E}(X)-a|^p} \exp\left\{-\frac{2s^{2/p}}{(b-a)^2}\right\} ds.$$

(Case:  $p > 2$ .) By simple calculating, we have

$$\begin{aligned} & \int_0^{|b-\mathbb{E}(X)|^p} \exp\left\{-\frac{2s^{2/p}}{(b-a)^2}\right\} ds \\ & \leq \int_0^{|b-\mathbb{E}(X)|^p \wedge 1} \exp\left\{-\frac{2s}{(b-a)^2}\right\} ds + \int_{|b-\mathbb{E}(X)|^p \wedge 1}^{|b-\mathbb{E}(X)|^p} \exp\left\{-\frac{2s^{2/p}}{(b-a)^2}\right\} ds \\ & \leq \frac{(b-a)^2}{2} \left(1 - \exp\left\{-\frac{2(|b-\mathbb{E}(X)|^p \wedge 1)}{(b-a)^2}\right\}\right) \\ & \quad + (|b-\mathbb{E}(X)|^p - |b-\mathbb{E}(X)|^p \wedge 1) \exp\left\{-\frac{2(|b-\mathbb{E}(X)|^p \wedge 1)^{2/p}}{(b-a)^2}\right\} \end{aligned}$$

and with the same reason

$$\begin{aligned} & \int_0^{|\mathbb{E}(X)-a|^p} \exp\left\{-\frac{2s^{2/p}}{(b-a)^2}\right\} ds \leq \frac{(b-a)^2}{2} \left(1 - \exp\left\{-\frac{2(|a-\mathbb{E}(X)|^p \wedge 1)}{(b-a)^2}\right\}\right) \\ & \quad + (|a-\mathbb{E}(X)|^p - |a-\mathbb{E}(X)|^p \wedge 1) \exp\left\{-\frac{2(|a-\mathbb{E}(X)|^p \wedge 1)^{2/p}}{(b-a)^2}\right\}. \end{aligned}$$

(Case  $1 \leq p < 2$ ) In this case, we have

$$\begin{aligned} & \int_0^{|b-\mathbb{E}(X)|^p} \exp\left\{-\frac{2s^{2/p}}{(b-a)^2}\right\} ds \\ & \leq \int_0^{|b-\mathbb{E}(X)|^p \wedge 1} \exp\left\{-\frac{2s^{2/p}}{(b-a)^2}\right\} ds + \int_{|b-\mathbb{E}(X)|^p \wedge 1}^{|b-\mathbb{E}(X)|^p} \exp\left\{-\frac{2s}{(b-a)^2}\right\} ds \end{aligned}$$



Title Page

Contents



Page 12 of 13

Go Back

Full Screen

Close

$$\leq |b - \mathbb{E}(X)|^p \wedge 1 + \frac{(b-a)^2}{2} \left( \exp \left\{ -\frac{2|b - \mathbb{E}(X)|^p \wedge 1}{(b-a)^2} \right\} - \exp \left\{ -\frac{2|b - \mathbb{E}(X)|^p}{(b-a)^2} \right\} \right)$$

and with the same reason

$$\int_0^{|\mathbb{E}(X)-a|^p} \exp \left\{ -\frac{2s^{2/p}}{(b-a)^2} \right\} ds \leq |a - \mathbb{E}(X)|^p \wedge 1 + \frac{(b-a)^2}{2} \left( \exp \left\{ -\frac{2|a - \mathbb{E}(X)|^p \wedge 1}{(b-a)^2} \right\} - \exp \left\{ -\frac{2|a - \mathbb{E}(X)|^p}{(b-a)^2} \right\} \right).$$

On the other hand, for any  $p \geq 1$ , it is clear that

$$\mathbb{E}|X - \mathbb{E}(X)|^p \leq (b-a)^p.$$

From the above discussion, the desired results are obtained.  $\square$

*Remark 2.*  $M_1(a, b, p, X)$  and  $M_2(a, b, p, X)$  are sometimes better than  $(b-a)^p$ , e.g., if  $p > 2$ , taking  $\mathbb{E}(X) = (a+b)/2$  and letting  $1/2 < (b-a)/2 < 1$ , then

$$\begin{aligned} M_1(a, b, p, X) &= (b-a)^2 (1 - \exp \{-2^{1-p}(b-a)^{p-2}\}) \\ &< (b-a)^p (1 - \exp \{-2^{1-p}(b-a)^{p-2}\}) \\ &< (b-a)^p. \end{aligned}$$

Further, if  $1 \leq p < 2$ , taking  $\mathbb{E}(X) = (a+b)/2$  and letting  $(b-a)/2 < 1$ , then

$$M_2(a, b, p, X) = 2 \left( \frac{(b-a)^p}{2^p} \right) = 2^{1-p}(b-a)^p \leq (b-a)^p.$$

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---

Upper Bounds for the  
Dispersion

Yu Miao and Guangyu Yang

vol. 8, iss. 3, art. 83, 2007

---

Title Page

Contents



Page 13 of 13

Go Back

Full Screen

Close

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