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ON ERROR BOUNDS FOR GAUSS–LEGENDRE AND LOBATTO QUADRATURE RULES

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ABSTRACT. The error bounds for Gauss–Legendre and Lobatto quadratures are proved for four times differentiable functions (instead of six times differentiable functions as in the classical results). Auxiliarily we establish some inequalities for 3–convex functions.

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1. INTRODUCTION

The classical error bounds for the Gauss–Legendre quadrature rule (with three knots) and for the Lobatto quadrature rule (with four knots) hold for six times differentiable functions. In this paper we obtain error bounds for these rules for four times differentiable functions. To prove our main results we establish some inequalities for so–called 3–convex functions. In [7] using the same technique the error bounds for Midpoint, Trapezoidal, Simpson and Radau quadrature rules were reproved. We prove our results for functions defined on $[-1,1]$ and next we translate them to the interval $[a,b]$.

Now we would like to recall the notions and results needed in this paper (cf. also the Introduction to [7]).

1.1. **Convex functions of higher orders.** Hopf's thesis [2] is probably the first work devoted to higher–order convexity. This concept was also studied among others by Popoviciu [4]. Let

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$I \subset \mathbb{R}$ be an interval and let $n \in \mathbb{N}$. Recall that the function $f : I \to \mathbb{R}$ is called *n–convex* if

$$D(x_0, x_1, \dots, x_{n+1}; f) := \begin{vmatrix} 1 & 1 & \dots & 1 \\ x_0 & x_1 & \dots & x_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_0^n & x_1^n & \dots & x_{n+1}^n \\ f(x_0) & f(x_1) & \dots & f(x_{n+1}) \end{vmatrix} \geq 0 \tag{1.1}$$

for any $x_0, x_1, \dots, x_{n+1} \in I$ such that $x_0 < x_1 < \dots < x_{n+1}$. Obviously 1–convex functions are convex in the classical sense. More information on the definition and properties of convex functions of higher orders can be found in [2, 3, 4, 6].

The following theorem (cf. [2, 3, 4]) characterizes n–convexity of $(n+1)$–times differentiable functions.

Theorem A. *Assume that $f : (a, b) \to \mathbb{R}$ is an $(n + 1)$–times differentiable function. Then f is n–convex if and only if $f^{(n+1)}(x) \geq 0$, $x \in (a, b)$.*

The next result holds for the interval $[a, b]$.

Theorem B. [7, Theorem 1.3] *Assume that $f : [a, b] \to \mathbb{R}$ is $(n + 1)$–times differentiable on (a, b) and continuous on $[a, b]$. If $f^{(n+1)}(x) \geq 0$, $x \in (a, b)$, then f is n–convex.*

1.2. **Quadrature Rules.** For a function $f : [-1, 1] \to \mathbb{R}$ we define some operators connected with the quadrature rules:

$$\mathcal{G}_2(f) := \frac{1}{2}\left(f\left(-\frac{\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)\right),$$

$$\mathcal{G}_3(f) := \frac{5}{18} f\left(-\frac{\sqrt{15}}{5}\right) + \frac{4}{9} f(0) + \frac{5}{18} f\left(\frac{\sqrt{15}}{5}\right),$$

$$\mathcal{L}(f) := \frac{1}{12} f(-1) + \frac{5}{12} f\left(-\frac{\sqrt{5}}{5}\right) + \frac{5}{12} f\left(\frac{\sqrt{5}}{5}\right) + \frac{1}{12} f(1),$$

$$\mathcal{S}(f) := \tfrac{1}{6}\left(f(-1) + 4f(0) + f(1)\right),$$

$$\mathcal{I}(f) := \frac{1}{2}\int_{-1}^{1} f(x)dx.$$

The operators $\mathcal{G}_2$ and $\mathcal{G}_3$ are connected with Gauss–Legendre quadrature rules. The operators $\mathcal{L}$ and $\mathcal{S}$ concern Lobatto and Simpson's quadrature rules, respectively. The operator $\mathcal{I}$ stands for the integral mean value. Obviously all these operators are linear.

Next we recall the well known quadrature rules (cf. e.g. [5], [8], [9], [10]).

Gauss–Legendre quadratures. *If $f \in \mathcal{C}^4\big([-1, 1]\big)$ then*

$$\mathcal{I}(f) = \mathcal{G}_2(f) + \frac{f^{(4)}(\xi)}{270} \quad \textit{for some } \xi \in (-1, 1). \tag{1.2}$$

If $f \in \mathcal{C}^6\big([-1, 1]\big)$ then

$$\mathcal{I}(f) = \mathcal{G}_3(f) + \frac{f^{(6)}(\xi)}{31500} \quad \textit{for some } \xi \in (-1, 1). \tag{1.3}$$

Lobatto quadrature. *If $f \in \mathcal{C}^6\big([-1, 1]\big)$ then*

$$\mathcal{I}(f) = \mathcal{L}(f) - \frac{f^{(6)}(\xi)}{23625} \quad \textit{for some } \xi \in (-1, 1). \tag{1.4}$$

Simpson's Rule. *If* $f \in \mathcal{C}^4\big([-1,1]\big)$ *then*

$$\mathcal{I}(f) = \mathcal{S}(f) - \frac{f^{(4)}(\xi)}{180} \quad \textit{for some } \xi \in (-1,1). \tag{1.5}$$

2. INEQUALITIES FOR 3–CONVEX FUNCTIONS

Let $V(x_1,\dots,x_n)$ be the Vandermonde determinant of the terms involved.

Lemma 2.1. *If* $f:[-1,1]\to\mathbb{R}$ *is 3–convex, then the inequality*

$$v^2\big(f(-u)+f(u)\big) \le u^2\big(f(-v)+f(v)\big) + 2(v^2-u^2)f(0)$$

holds for any $0<u<v\le 1$.

Proof. Let $0<u<v\le 1$. Since f is 3–convex and $-1\le -v<-u<0<u<v\le 1$, then by (1.1)

$$0 \le D(-v,-u,0,u,v;f) = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -v & -u & 0 & u & v \\ v^2 & u^2 & 0 & u^2 & v^2 \\ -v^3 & -u^3 & 0 & u^3 & v^3 \\ f(-v) & f(-u) & f(0) & f(u) & f(v) \end{vmatrix}.$$

Expanding this determinant by the last row we obtain

$$\begin{aligned} V(-u,0,u,v)f(-v) - V(-v,0,u,v)f(-u) + V(-v,-u,u,v)f(0) \\ - V(-v,-u,0,v)f(u) + V(-v,-u,0,u)f(v) \ge 0. \end{aligned}$$

Computing the Vandermonde determinants

$$\begin{aligned} V(-u,0,u,v) &= V(-v,-u,0,u) = 2u^3v(v^2-u^2), \\ V(-v,0,u,v) &= V(-v,-u,0,v) = 2uv^3(v^2-u^2), \\ V(-v,-u,u,v) &= 4uv(v^2-u^2)^2 \end{aligned}$$

and rearranging the above inequality we obtain

$$2uv^3(v^2-u^2)\big(f(-u)+f(u)\big) \le 2u^3v(v^2-u^2)\big(f(-v)+f(v)\big) + 4uv(v^2-u^2)^2 f(0),$$

from which, by $2uv(v^2-u^2)>0$, the lemma follows. □

Proposition 2.2. *If* $f:[-1,1]\to\mathbb{R}$ *is 3–convex, then* $\mathcal{G}_2(f)\le\mathcal{G}_3(f)\le\mathcal{S}(f)$ *and* $\mathcal{L}(f)\le\mathcal{S}(f)$.

Proof. Setting in Lemma 2.1 $u=\frac{\sqrt5}{5}$, $v=1$ we obtain

$$f\left(-\tfrac{\sqrt5}{5}\right) + f\left(\tfrac{\sqrt5}{5}\right) \le \frac{1}{5}\big(f(-1)+f(1)\big) + \frac{8}{5}f(0).$$

Then

$$5\left(f\left(-\tfrac{\sqrt5}{5}\right) + f\left(\tfrac{\sqrt5}{5}\right)\right) \le f(-1)+f(1)+8f(0),$$

whence

$$f(-1)+f(1)+5\left(f\left(-\tfrac{\sqrt5}{5}\right) + f\left(\tfrac{\sqrt5}{5}\right)\right) \le 2\left(f(-1)+f(1)\right)+8f(0).$$

Dividing both sides of this inequality by 12 we get $\mathcal{L}(f)\le\mathcal{S}(f)$. The proofs of the inequalities $\mathcal{G}_2(f)\le\mathcal{G}_3(f)$ and $\mathcal{G}_3(f)\le\mathcal{S}(f)$ are similar. □

3. Error Bounds for Quadrature Rules

In this section we assume that $f \in \mathcal{C}^4\big([-1,1]\big)$. Then

$$M_4(f) := \sup_{-1\le x\le 1}\left|f^{(4)}(x)\right| < \infty.$$

The classical error bound for the Gauss–Legendre quadrature rule $\mathcal{G}_3(f)$ holds for the six times differentiable function f. This is also the case for the Lobatto quadrature formula $\mathcal{L}(f)$. We prove the error bounds for these quadratures for less regular functions, i.e. for four times differentiable functions. We start with the result for 3–convex functions.

Theorem 3.1. *If $f \in \mathcal{C}^4\big([-1,1]\big)$ is 3–convex then* $|\mathcal{G}_3(f)-\mathcal{I}(f)| \le \frac{M_4(f)}{180}$.

Proof. On account of Theorem A, $f^{(4)} \ge 0$ on $(-1,1)$. Therefore we conclude from (1.2) that (for some $\xi \in (-1,1)$)

$$\mathcal{G}_2(f)-\mathcal{I}(f) = -\frac{f^{(4)}(\xi)}{270} \ge -\frac{f^{(4)}(\xi)}{180} \ge -\frac{M_4(f)}{180}. \tag{3.1}$$

By Proposition 2.2

$$\mathcal{G}_2(f) \le \mathcal{G}_3(f) \le \mathcal{S}(f). \tag{3.2}$$

Next, by (1.5) there exists an $\eta \in (-1,1)$ such that

$$\mathcal{S}(f)-\mathcal{I}(f) = \frac{f^{(4)}(\eta)}{180} \le \frac{M_4(f)}{180}. \tag{3.3}$$

By (3.1), (3.2) and (3.3) we obtain

$$-\frac{M_4(f)}{180} \le \mathcal{G}_2(f)-\mathcal{I}(f) \le \mathcal{G}_3(f)-\mathcal{I}(f) \le \mathcal{S}(f)-\mathcal{I}(f) \le \frac{M_4(f)}{180},$$

from which the result follows. □

To prove the next two results we need to make some observations.

Remark 3.2. For $f \in \mathcal{C}^4\big([-1,1]\big)$ we consider the function $g(x) = \frac{M_4(f)}{24}x^4$. Then

$$\left|f^{(4)}(x)\right| \le M_4(f) = g^{(4)}(x), \quad -1 \le x \le 1. \tag{3.4}$$

Hence $(g-f)^{(4)} \ge 0$ and $(g+f)^{(4)} \ge 0$. Thus Theorem B implies that the functions $g-f$ and $g+f$ are 3–convex. Moreover, using (3.4) we obtain

$$(g-f)^{(4)}(x) = g^{(4)}(x) - f^{(4)}(x) = M_4(f) - f^{(4)}(x) \le 2M_4(f)$$

and

$$(g+f)^{(4)}(x) = M_4(f) + f^{(4)}(x) \le 2M_4(f).$$

Then

$$M_4(g-f) \le 2M_4(f) \quad \text{and} \quad M_4(g+f) \le 2M_4(f). \tag{3.5}$$

By (1.3) and (1.4) we have also $\mathcal{G}_3(g) = \mathcal{L}(g) = \mathcal{I}(g)$.

Corollary 3.3. *If $f \in \mathcal{C}^4\big([-1,1]\big)$ then* $|\mathcal{G}_3(f)-\mathcal{I}(f)| \le \frac{M_4(f)}{90}$.

Proof. By Remark 3.2 the function $g+f$ is 3–convex and $\mathcal{G}_3(g) = \mathcal{I}(g)$, where $g(x) = \frac{M_4(f)}{24}x^4$. Theorem 3.1 and the linearity of the operators $\mathcal{G}_3$ and $\mathcal{I}$ now imply

$$\begin{aligned}|\mathcal{G}_3(f)-\mathcal{I}(f)| &= |\mathcal{G}_3(g)+\mathcal{G}_3(f)-\mathcal{I}(g)-\mathcal{I}(f)| \\ &= |\mathcal{G}_3(g+f)-\mathcal{I}(g+f)| \le \frac{M_4(g+f)}{180}.\end{aligned}$$

This inequality together with (3.5) concludes the proof. □

Before we prove the error bound for the Lobatto quadrature rule we make the following simple observation.

Remark 3.4. By Proposition 2.2 and (1.5) we obtain that for a 3–convex function $f \in \mathcal{C}^4\big([-1,1]\big)$ there exists a $\xi \in (-1,1)$ such that $\mathcal{L}(f) \le \mathcal{S}(f) = \mathcal{I}(f) + \frac{f^{(4)}(\xi)}{180}$. This gives

$$\mathcal{L}(f) - \mathcal{I}(f) \le \frac{M_4(f)}{180}. \tag{3.6}$$

Theorem 3.5. *If $f \in \mathcal{C}^4\big([-1,1]\big)$ then $\big|\mathcal{L}(f) - \mathcal{I}(f)\big| \le \frac{M_4(f)}{90}$.*

Proof. By Remark 3.2 the functions $g - f$ and $g + f$ are 3–convex, where $g(x) = \frac{M_4(f)}{24}x^4$. Then by (3.6)

$$\mathcal{L}(g-f) - \mathcal{I}(g-f) \le \frac{M_4(g-f)}{180} \quad \text{and} \quad \mathcal{L}(g+f) - \mathcal{I}(g+f) \le \frac{M_4(g+f)}{180}.$$

Because of $\mathcal{L}(g) = \mathcal{I}(g)$ and by linearity of the operators $\mathcal{L}$ and $\mathcal{I}$ we have

$$-\big(\mathcal{L}(f) - \mathcal{I}(f)\big) \le \frac{M_4(g-f)}{180} \quad \text{and} \quad \mathcal{L}(f) - \mathcal{I}(f) \le \frac{M_4(g+f)}{180}.$$

These inequalities together with (3.5) conclude the proof. □

4. ERROR BOUNDS FOR QUADRATURE RULES ON $[a,b]$

In the next section we translate the quadrature rules and error bounds obtained in Theorem 3.1, Corollary 3.3 and Theorem 3.5 to the interval $[a,b]$. To do this task we use the following change of variables: for $t \in [-1,1]$ let

$$x = \frac{1-t}{2}a + \frac{1+t}{2}b. \tag{4.1}$$

Then $x \in [a,b]$. For a function $f : [a,b] \to \mathbb{R}$ we define $F : [-1,1] \to \mathbb{R}$ by

$$F(t) := f(x). \tag{4.2}$$

Remark 4.1. If f is n–convex on $[a,b]$ then the function F given by (4.2) is n–convex on $[-1,1]$ (cf. Popoviciu [4], Chapter II, §1, point 12).

By the substitution (4.1) we obtain

$$\mathcal{I}(F) = \frac{1}{b-a}\int_a^b f(x)dx. \tag{4.3}$$

Let $f \in \mathcal{C}^4\big([a,b]\big)$. Using (4.1) and considering the function F defined by (4.2) we have $F^{(4)}(t) = \left(\frac{b-a}{2}\right)^4 f^{(4)}(x)$ for $x \in [a,b]$, $t \in [-1,1]$. It is easy to see that $F \in \mathcal{C}^4\big([-1,1]\big)$. Let

$$M_4(F) := \sup_{-1\le t\le 1}\left|F^{(4)}(t)\right| \quad \text{and} \quad M_4(f) := \sup_{a\le x\le b}\left|f^{(4)}(x)\right|.$$

Then $M_4(F) = \left(\frac{b-a}{2}\right)^4 M_4(f)$. If moreover f is 3–convex then by Remark 4.1 F is also 3–convex.

Corollary 4.2. *If* $f \in \mathcal{C}^4\big([a,b]\big)$ *then*

$$\left|\frac{5}{18}f\left(\frac{5+\sqrt{15}}{10}a+\frac{5-\sqrt{15}}{10}b\right)+\frac{4}{9}f\left(\frac{a+b}{2}\right)\right.$$
$$\left.+\frac{5}{18}f\left(\frac{5-\sqrt{15}}{10}a+\frac{5+\sqrt{15}}{10}b\right)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^4M_4(f)}{1440}. \tag{4.4}$$

If moreover f *is 3–convex then the right hand side of* (4.4) *can be replaced by* $\frac{(b-a)^4M_4(f)}{2880}$.

Proof. By (4.1) and (4.2) we get

$$f\left(\frac{5+\sqrt{15}}{10}a+\frac{5-\sqrt{15}}{10}b\right)=F\left(-\frac{\sqrt{15}}{5}\right), \qquad f\left(\frac{a+b}{2}\right)=F(0)$$

and

$$f\left(\frac{5-\sqrt{15}}{10}a+\frac{5+\sqrt{15}}{10}b\right)=F\left(\frac{\sqrt{15}}{5}\right).$$

Since $F \in \mathcal{C}^4\big([-1,1]\big)$ then using Corollary 3.3 and (4.3) we obtain

$$|\mathcal{G}_3(F)-\mathcal{I}(F)| \le \frac{M_4(F)}{90} = \left(\frac{b-a}{2}\right)^4 \cdot \frac{M_4(f)}{90},$$

which proves the desired inequality (4.4). For a 3–convex function f we argue similarly using Theorem 3.1. ☐

Using Theorem 3.5 we obtain by the same reasoning

Corollary 4.3. *If* $f \in \mathcal{C}^4\big([a,b]\big)$ *then*

$$\left|\frac{1}{12}f(a)+\frac{5}{12}f\left(\frac{5+\sqrt{5}}{10}a+\frac{5-\sqrt{5}}{10}b\right)\right.$$
$$\left.+\frac{5}{12}f\left(\frac{5-\sqrt{5}}{10}a+\frac{5+\sqrt{5}}{10}b\right)+\frac{1}{12}f(b)-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^4M_4(f)}{1440}. \tag{4.5}$$

Remark 4.4. For six times differentiable functions inequalities similar to (4.4) and (4.5) can be obtained using Bessenyei and Pales' results [1, Corollary 5] and the method of convex functions of higher orders presented in this paper (cf. also [7]). However, our results are obtained for less regular functions.

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