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## ERDŐS-TURÁN TYPE INEQUALITIES

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## Abstract

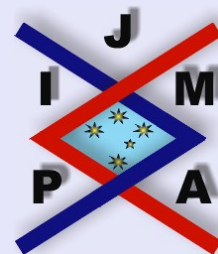
Denoting by  $(r_n)_{n \geq 1}$  the increasing sequence of the numbers  $p^\alpha$  with  $p$  prime and  $\alpha \geq 2$  integer, we prove that  $r_{n+1} - 2r_n + r_{n-1}$  is positive for infinitely many values of  $n$  and negative also for infinitely many values of  $n$ . We prove similar properties for  $r_n^2 - r_{n-1}r_{n+1}$  and  $\frac{1}{r_{n-1}} - \frac{2}{r_n} + \frac{1}{r_{n+1}}$  as well.

*2000 Mathematics Subject Classification:* 11A25, 11N05.

*Key words:* Powers of prime numbers, Inequalities, Erdős-Turán theorems.

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# 1. Introduction

Let  $(r_n)_{n \geq 0}$  be the increasing sequence of the powers of prime numbers ( $p^\alpha$  with  $p$  prime and  $\alpha \geq 2$  integer). Thus, we have  $r_1 = 4$ ,  $r_2 = 8$ ,  $r_3 = 9$ ,  $r_4 = 16$ , etc. Properties of the sequence  $(r_n)_{n \geq 1}$  were studied in [5] and [3].

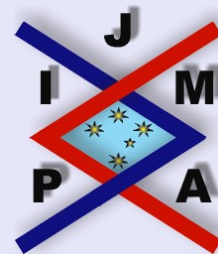
Denote by  $p_n$  the  $n$ -th prime number. In [1], Erdős and Turán proved that  $p_{n+1} - 2p_n + p_{n-1}$  is positive for infinitely many values of  $n$  and negative also for infinitely many values of  $n$ . Until now, no answer is known for the following question raised by Erdős and Turán: Do there exist infinitely many numbers  $n$  such that

$$p_{n+1} - 2p_n + p_{n-1} = 0?$$

Erdős and Turán also proved that each of the sequences  $(p_n^2 - p_{n-1}p_{n+1})_{n \geq 2}$  and  $\left(\frac{1}{p_{n-1}} - \frac{2}{p_n} + \frac{1}{p_{n+1}}\right)_{n \geq 2}$  has infinitely many positive terms and infinitely many negative ones.

Denoting by  $(q_n)_{n \geq 1}$  the increasing sequence of the powers of prime numbers, the author proved in [4] that the value of  $q_{n+1} - 2q_n + q_{n-1}$  changes its sign infinitely many times.

In the present paper, we raise similar problems for the sequence  $(r_n)_{n \geq 1}$ . We need a few preliminary properties, which will be proved in the next section.



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## 2. On the Difference $r_{n+1} - r_n$

**Property 2.1.** *We have*

$$(2.1) \quad \limsup_{n \rightarrow \infty} (r_{n+1} - r_n) = \infty.$$

*Proof.* Let  $m \geq 4$ . We show that, among the numbers

$$m! + 2, m! + 3, \dots, m! + [\sqrt{m}],$$

there is no term of the sequence  $(r_n)_{n \geq 1}$ .

Assume that there exists an integer  $a$  such that  $2 \leq a \leq [\sqrt{m}]$  and

$$(2.2) \quad m! + a = p^i$$

where  $p$  is prime and  $i \geq 2$ .

The relation (2.2) can also be written in the form

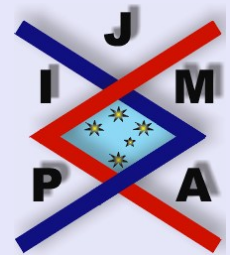
$$a \left( \frac{m!}{a} + 1 \right) = p^i, \text{ whence } a = p^j \text{ with } 1 \leq j \leq i.$$

It follows that

$$\frac{m!}{p^j} + 1 = p^{i-j}, \text{ hence } \frac{m!}{p^j} \text{ is not divisible by } p.$$

If  $e_p(n)$  is Legendre's function, we have  $e_p(m) = j$ , that is,

$$(2.3) \quad \sum_{s=1}^{\infty} \left[ \frac{m}{p^s} \right] = j.$$



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Since  $a \leq \sqrt{m}$ , it follows that  $p^j \leq \sqrt{m}$ , that is,  $m \geq p^{2j}$ , and then (2.3) implies that

$$\begin{aligned} j &\geq \left\lfloor \frac{p^{2j}}{p} \right\rfloor + \left\lfloor \frac{p^{2j}}{p^2} \right\rfloor + \cdots + \left\lfloor \frac{p^{2j}}{p^{2j}} \right\rfloor \\ &= p^{2j-1} + p^{2j-2} + \cdots + p + 1 \\ &\geq 2^{2j-1} + 2^{2j-2} + \cdots + 2 + 1 \\ &= 2^{2j} - 1. \end{aligned}$$

Since for  $j \geq 1$  we have  $2^{2j} - 1 > j$ , we obtained a contradiction.

Since our assumption turned out to be false, it follows that for every  $m \geq 4$  there exists  $k = k(m)$  such that

$$r_k \leq m! + 1 \text{ and } r_{k+1} \geq m! + \lfloor \sqrt{m} \rfloor + 1,$$

whence  $r_{k+1} - r_k \geq \lfloor \sqrt{m} \rfloor$ , and finally

$$\limsup_{n \rightarrow \infty} (r_{n+1} - r_n) = \infty,$$

and the proof ends. □

We now denote  $a_n = \frac{r_{n+1} - r_n}{n \log^2 n}$  and recall that, in [2], H. Meier proved that

$$(2.4) \quad \liminf_{n \rightarrow \infty} \frac{p_{n+1} - p_n}{\log n} < 0.248.$$

In connection with this result, we prove:



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**Property 2.2.** We have

$$(2.5) \quad \liminf_{n \rightarrow \infty} a_n < 0.496.$$

*Proof.* We consider the indices  $m$  such that

$$\frac{p_{m+1} - p_m}{\log m} < 0.248.$$

Both the numbers  $p_m^2$  and  $p_{m+1}^2$  occur in the sequence  $(r_n)_{n \geq 1}$ , that is,  $p_m^2 = r_k$  and  $p_{m+1}^2 = r_h$ , with  $k = k(m)$ ,  $h = h(m)$  and  $h \geq k + 1$ . In [5], it was proved that, for  $m \geq 1783$ , we have

$$(2.6) \quad p_m^2 \geq r_m > m^2 \log^2 m.$$

Since  $p_m \sim m \log m$ , it follows that  $r_k \sim k^2 \log^2 k$ . But  $r_k = p_m^2$ , hence  $k \log k \sim m \log m$ . One can show without difficulty that  $k(m) \sim m$ . It then follows that

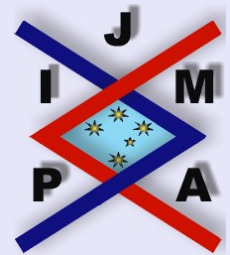
$$\frac{\sqrt{r_{k+1}} - \sqrt{r_k}}{\log k} < \frac{\sqrt{r_h} - \sqrt{r_k}}{\log k} = \frac{p_{m+1} - p_m}{\log k}.$$

Since  $\log k \sim \log m$ , we get

$$\liminf_{k \rightarrow \infty} \frac{\sqrt{r_{k+1}} - \sqrt{r_k}}{\log k} \leq \liminf_{m \rightarrow \infty} \frac{p_{m+1} - p_m}{\log m} < 0.248.$$

Since  $\sqrt{r_k} \sim k \log k$  and  $\sqrt{r_{k+1}} \sim (k+1) \log(k+1) \sim k \log k$ , it follows that

$$\liminf_{k \rightarrow \infty} \frac{r_{k+1} - r_k}{k \log^2 k} < 0.496,$$



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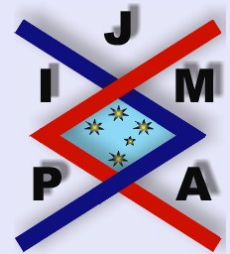
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where  $k = k(m)$ . Consequently,

$$\liminf_{n \rightarrow \infty} \frac{r_{n+1} - r_n}{n \log^2 n} < 0.496.$$

□



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### 3. Erdős-Turán Type Properties

For  $k \geq 2$  we denote

$$R_k = r_{k+1} - 2r_k + r_{k-1},$$

and prove

**Property 3.1.** *There exist infinitely many values of  $n$  such that*

$$R_n > 0,$$

*and also infinitely many ones such that*

$$R_n < 0.$$

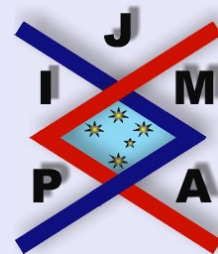
*Proof.* Denoting  $S_m = \sum_{k=2}^m R_k$ , we have  $S_m = r_{m+1} - r_m - r_2 + 1$ . By (2.1) we have  $\limsup_{m \rightarrow \infty} S_m = \infty$ , hence  $R_n > 0$  for infinitely many values of  $n$ .

Denoting  $\sigma_m = \sum_{k=2}^m kR_k$ , we have

$$\sigma_m = m(r_{m+1} - r_m) - r_m - r_2 + 2r_1 = m^2 \log^2 m \left( a_m - \frac{r_m}{m^2 \log^2 m} \right).$$

Since  $r_m \sim m^2 \log^2 m$ , we get by (2.5) that  $\liminf_{m \rightarrow \infty} \sigma_m = -\infty$ , hence  $R_n < 0$  for infinitely many values of  $n$ .  $\square$

For  $k \geq 2$ , denoting  $\rho_k = \frac{1}{r_{k-1}} - \frac{2}{r_k} + \frac{1}{r_{k+1}}$ , we have



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**Property 3.2.** *There exist infinitely many values of  $n$  such that*

$$\rho_n > 0,$$

*and also infinitely many ones such that*

$$\rho_n < 0.$$

*Proof.* For  $\alpha > 3$ , denoting  $S'_m(\alpha) = \sum_{k=2}^m k^\alpha \rho_k$ , we get

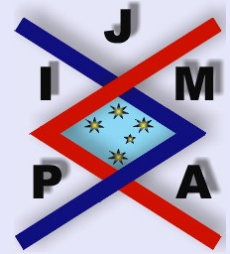
$$S'_m(\alpha) = -\frac{m^\alpha(r_{m+1} - r_m)}{r_m r_{m+1}} - \frac{m^\alpha - (m-1)^\alpha}{r_m} + \sum_{k=2}^{m-1} \frac{k^\alpha - 2(k-1)^\alpha + (k-2)^\alpha}{r_k} + O(1).$$

We have

$$\begin{aligned} r_k &\sim k^2 \log^2 k, \\ k^\alpha - (k-1)^\alpha &\sim \alpha k^{\alpha-1}, \\ k^\alpha - 2(k-1)^\alpha + (k-2)^\alpha &\sim \alpha(\alpha-1)k^{\alpha-2}, \end{aligned}$$

whence

$$\begin{aligned} \frac{m^\alpha(r_{m+1} - r_m)}{r_m r_{m+1}} &\sim \frac{m^{\alpha-3} a_m}{\log^2 m}, \\ \frac{m^\alpha - (m-1)^\alpha}{r_m} &\sim \frac{\alpha m^{\alpha-3}}{\log^2 m}, \\ \frac{k^\alpha - 2(k-1)^\alpha + (k-2)^\alpha}{r_k} &\sim \frac{\alpha(\alpha-1)k^{\alpha-4}}{\log^2 k}. \end{aligned}$$



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Since

$$\sum_{k=2}^{m-1} \frac{k^{\alpha-4}}{\log^2 m} \sim \frac{(\alpha-3)m^{\alpha-3}}{\log^2 m},$$

it follows that

$$S'_m(\alpha) \sim \frac{m^{\alpha-3}}{\log^2 m} \cdot (-a_m - \alpha + \alpha(\alpha-1)(\alpha-3)).$$

Then  $\lim_{m \rightarrow \infty} S'_m(3.1) = -\infty$ , and thus there exist infinitely many values of  $n$  such that  $\rho_n < 0$ .

On the other hand, we have by (2.5) that  $\limsup_{m \rightarrow \infty} S'_m(4) = \infty$ , which shows that there exist infinitely many values of  $n$  such that  $\rho_n > 0$ .  $\square$

A consequence of Properties 3.1 and 3.2 is the following.

**Property 3.3.** *There exist infinitely many values of  $n$  such that*

$$r_{n-1}r_{n+1} > r_n^2,$$

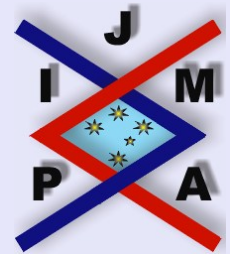
*and also infinitely many ones such that*

$$r_{n-1}r_{n+1} < r_n^2.$$

*Proof.* If  $r_n > \frac{r_{n+1}+r_{n-1}}{2}$ , then  $r_n > \sqrt{r_{n-1}r_{n+1}}$ . On the other hand, if  $\frac{2}{r_n} > \frac{1}{r_{n-1}} + \frac{1}{r_{n+1}}$ , then

$$r_n < 2 \left/ \left( \frac{1}{r_{n-1}} + \frac{1}{r_{n+1}} \right) \right. < \sqrt{r_{n-1}r_{n+1}},$$

and then the desired conclusion follows by Properties 3.1 and 3.2.  $\square$



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**Open problem.** Do there exist infinitely many values of  $n$  such that

$$r_{n+1} - 2r_n + r_{n-1} = 0?$$



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