Journal of Inequalities in Pure and Applied Mathematics

ERDŐS-TURÁN TYPE INEQUALITIES

LAURENŢIU PANAITOPOL

University of Bucharest
Faculty of Mathematics
14 Academiei St.
RO–70109 Bucharest
Romania.
EMail: pan@al.math.unibuc.ro

volume 4, issue 1, article 23, 2003.

Received 10 January, 2003; accepted 19 February, 2003.

Communicated by: J. Sándor

Abstract
Contents
Home Page
Go Back
Close
Quit

ISSN (electronic): 1443-5756
004-03

Abstract

Denoting by $(r_n)_{n\geq 1}$ the increasing sequence of the numbers p^α with p prime and $\alpha \geq 2$ integer, we prove that $r_{n+1} - 2r_n + r_{n-1}$ is positive for infinitely many values of n and negative also for infinitely many values of n. We prove similar properties for $r_n^2 - r_{n-1}r_{n+1}$ and $\frac{1}{r_{n-1}} - \frac{2}{r_n} + \frac{1}{r_{n+1}}$ as well.

2000 Mathematics Subject Classification: 11A25, 11N05.
Key words: Powers of prime numbers, Inequalities, Erdős-Turán theorems.

Contents

1. Introduction

Let $(r_n)_{n\geq 0}$ be the increasing sequence of the powers of prime numbers (p^α with p prime and $\alpha \geq 2$ integer). Thus, we have $r_1 = 4$, $r_2 = 8$, $r_3 = 9$, $r_4 = 16$, etc. Properties of the sequence $(r_n)_{n\geq 1}$ were studied in [5] and [3].

Denote by p_n the n-th prime number. In [1], Erdős and Turán proved that $p_{n+1} - 2p_n + p_{n-1}$ is positive for infinitely many values of n and negative also for infinitely many values of n. Until now, no answer is known for the following question raised by Erdős and Turán: Do there exist infinitely many numbers n such that

$$p_{n+1} - 2p_n + p_{n-1} = 0?$$

Erdős and Turán also proved that each of the sequences $(p_n^2 - p_{n-1}p_{n+1})_{n\geq 2}$ and $\left(\frac{1}{p_{n-1}} - \frac{2}{p_n} + \frac{1}{p_{n+1}}\right)_{n\geq 2}$ has infinitely many positive terms and infinitely many negative ones.

Denoting by $(q_n)_{n\geq 1}$ the increasing sequence of the powers of prime numbers, the author proved in [4] that the value of $q_{n+1} - 2q_n + q_{n-1}$ changes its sign infinitely many times.

In the present paper, we raise similar problems for the sequence $(r_n)_{n\geq 1}$. We need a few preliminary properties, which will be proved in the next section.

2. On the Difference $r_{n+1} - r_n$

Property 2.1. *We have*

$$\limsup_{n\to\infty}(r_{n+1} - r_n) = \infty. \tag{2.1}$$

Proof. Let $m \geq 4$. We show that, among the numbers

$$m! + 2, m! + 3, \dots, m! + [\sqrt{m}],$$

there is no term of the sequence $(r_n)_{n\geq 1}$.

Assume that there exists an integer a such that $2 \leq a \leq [\sqrt{m}]$ and

$$m! + a = p^i \tag{2.2}$$

where p is prime and $i \geq 2$.

The relation (2.2) can also be written in the form

$$a\left(\frac{m!}{a} + 1\right) = p^i, \text{ whence } a = p^j \text{ with } 1 \leq j \leq i.$$

It follows that

$$\frac{m!}{p^j} + 1 = p^{i-j}, \text{ hence } \frac{m!}{p^j} \text{ is not divisible by } p.$$

If $e_p(n)$ is Legendre's function, we have $e_p(m) = j$, that is,

$$\sum_{s=1}^{\infty}\left[\frac{m}{p^s}\right] = j. \tag{2.3}$$

Since $a \leq \sqrt{m}$, it follows that $p^j \leq \sqrt{m}$, that is, $m \geq p^{2j}$, and then (2.3) implies that

$$\begin{aligned} j &\geq \left[\frac{p^{2j}}{p}\right] + \left[\frac{p^{2j}}{p^2}\right] + \cdots + \left[\frac{p^{2j}}{p^{2j}}\right] \\ &= p^{2j-1} + p^{2j-2} + \cdots + p + 1 \\ &\geq 2^{2j-1} + 2^{2j-2} + \cdots + 2 + 1 \\ &= 2^{2j} - 1. \end{aligned}$$

Since for $j \geq 1$ we have $2^{2j} - 1 > j$, we obtained a contradiction.

Since our assumption turned out to be false, it follows that for every $m \geq 4$ there exists $k = k(m)$ such that

$$r_k \leq m! + 1 \text{ and } r_{k+1} \geq m! + [\sqrt{m}] + 1,$$

whence $r_{k+1} - r_k \geq [\sqrt{m}]$, and finally

$$\limsup_{n \to \infty} (r_{n+1} - r_n) = \infty,$$

and the proof ends. □

We now denote $a_n = \frac{r_{n+1} - r_n}{n \log^2 n}$ and recall that, in [2], H. Meier proved that

$$\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\log n} < 0.248. \tag{2.4}$$

In connection with this result, we prove:

Property 2.2. *We have*

$$\liminf_{n\to\infty} a_n < 0.496. \tag{2.5}$$

Proof. We consider the indices m such that

$$\frac{p_{m+1}-p_m}{\log m} < 0.248.$$

Both the numbers p_m^2 and p_{m+1}^2 occur in the sequence $(r_n)_{n\geq 1}$, that is, $p_m^2 = r_k$ and $p_{m+1}^2 = r_h$, with $k = k(m)$, $h = h(m)$ and $h \geq k+1$. In [5], it was proved that, for $m \geq 1783$, we have

$$p_m^2 \geq r_m > m^2 \log^2 m. \tag{2.6}$$

Since $p_m \sim m \log m$, it follows that $r_k \sim k^2 \log^2 k$. But $r_k = p_m^2$, hence $k \log k \sim m \log m$. One can show without difficulty that $k(m) \sim m$. It then follows that

$$\frac{\sqrt{r_{k+1}}-\sqrt{r_k}}{\log k} < \frac{\sqrt{r_h}-\sqrt{r_k}}{\log k} = \frac{p_{m+1}-p_m}{\log k}.$$

Since $\log k \sim \log m$, we get

$$\liminf_{k\to\infty} \frac{\sqrt{r_{k+1}}-\sqrt{r_k}}{\log k} \leq \liminf_{m\to\infty} \frac{p_{m+1}-p_m}{\log m} < 0.248.$$

Since $\sqrt{r_k} \sim k \log k$ and $\sqrt{r_{k+1}} \sim (k+1)\log(k+1) \sim k \log k$, it follows that

$$\liminf_{k\to\infty} \frac{r_{k+1}-r_k}{k \log^2 k} < 0.496,$$

where $k = k(m)$. Consequently,

$$\liminf_{n \to \infty} \frac{r_{n+1} - r_n}{n \log^2 n} < 0.496.$$

□

3. Erdős-Turán Type Properties

For $k \geq 2$ we denote

$$R_k = r_{k+1} - 2r_k + r_{k-1},$$

and prove

Property 3.1. *There exist infinitely many values of n such that*

$$R_n > 0,$$

and also infinitely many ones such that

$$R_n < 0.$$

Proof. Denoting $S_m = \sum_{k=2}^{m} R_k$, we have $S_m = r_{m+1} - r_m - r_2 + 1$. By (2.1) we have $\limsup_{m\to\infty} S_m = \infty$, hence $R_n > 0$ for infinitely many values of n.

Denoting $\sigma_m = \sum_{k=2}^{m} kR_k$, we have

$$\sigma_m = m(r_{m+1} - r_m) - r_m - r_2 + 2r_1 = m^2 \log^2 m \left(a_m - \frac{r_m}{m^2 \log^2 m} \right).$$

Since $r_m \sim m^2 \log^2 m$, we get by (2.5) that $\liminf_{m\to\infty} \sigma_m = -\infty$, hence $R_n < 0$ for infinitely many values of n. □

For $k \geq 2$, denoting $\rho_k = \frac{1}{r_{k-1}} - \frac{2}{r_k} + \frac{1}{r_{k+1}}$, we have

Property 3.2. *There exist infinitely many values of n such that*

$$\rho_n > 0,$$

and also infinitely many ones such that

$$\rho_n < 0.$$

Proof. For $\alpha > 3$, denoting $S'_m(\alpha) = \sum_{k=2}^{m} k^\alpha \rho_k$, we get

$$S'_m(\alpha) = -\frac{m^\alpha(r_{m+1} - r_m)}{r_m r_{m+1}} - \frac{m^\alpha - (m-1)^\alpha}{r_m} + \sum_{k=2}^{m-1} \frac{k^\alpha - 2(k-1)^\alpha + (k-2)^\alpha}{r_k} + O(1).$$

We have

$$r_k \sim k^2 \log^2 k,$$
$$k^\alpha - (k-1)^\alpha \sim \alpha k^{\alpha-1},$$
$$k^\alpha - 2(k-1)^\alpha + (k-2)^\alpha \sim \alpha(\alpha-1)k^{\alpha-2},$$

whence

$$\frac{m^\alpha(r_{m+1} - r_m)}{r_m r_{m+1}} \sim \frac{m^{\alpha-3} a_m}{\log^2 m},$$
$$\frac{m^\alpha - (m-1)^\alpha}{r_m} \sim \frac{\alpha m^{\alpha-3}}{\log^2 m},$$
$$\frac{k^\alpha - 2(k-1)^\alpha + (k-2)^\alpha}{r_k} \sim \frac{\alpha(\alpha-1)k^{\alpha-4}}{\log^2 k}.$$

Since

$$\sum_{k=2}^{m-1} \frac{k^{\alpha-4}}{\log^2 m} \sim \frac{(\alpha-3)m^{\alpha-3}}{\log^2 m},$$

it follows that

$$S'_m(\alpha) \sim \frac{m^{\alpha-3}}{\log^2 m} \cdot \big(- a_m - \alpha + \alpha(\alpha-1)(\alpha-3)\big).$$

Then $\lim_{m\to\infty} S'_m(3.1) = -\infty$, and thus there exist infinitely many values of n such that $\rho_n < 0$.

On the other hand, we have by (2.5) that $\limsup_{m\to\infty} S'_m(4) = \infty$, which shows that there exist infinitely many values of n such that $\rho_n > 0$. □

A consequence of Properties 3.1 and 3.2 is the following.

Property 3.3. *There exist infinitely many values of n such that*

$$r_{n-1}r_{n+1} > r_n^2,$$

and also infinitely many ones such that

$$r_{n-1}r_{n+1} < r_n^2.$$

Proof. If $r_n > \frac{r_{n+1}+r_{n-1}}{2}$, then $r_n > \sqrt{r_{n-1}r_{n+1}}$. On the other hand, if $\frac{2}{r_n} > \frac{1}{r_{n-1}} + \frac{1}{r_{n+1}}$, then

$$r_n < 2 \Big/ \left(\frac{1}{r_{n-1}} + \frac{1}{r_{n+1}} \right) < \sqrt{r_{n-1}r_{n+1}},$$

and then the desired conclusion follows by Properties 3.1 and 3.2. □

Open problem. Do there exist infinitely many values of n such that

$$r_{n+1} - 2r_n + r_{n-1} = 0?$$

References

[1] P. ERDŐS AND P. TURÁN, On some new question on the distribution of prime numbers, *Bull. Amer. Math. Soc.*, **54**(4) (1948), 371–378.

[2] H. MEIER, Small difference between prime numbers, *Michigan Math. J.*, **35** (1988), 324–344.

[3] G. MINCU, An asymptotic expansion, (in press).

[4] L. PANAITOPOL, Some of the properties of the sequence of powers of prime numbers, *Rocky Mountain J. Math.*, **31**(4) (2001), 1407–1415.

[5] L. PANAITOPOL, The sequence of the powers of prime numbers revisited, *Math. Reports*, **5**(55)(1) (2003), (in press).