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ON UNIVALENT HARMONIC FUNCTIONS

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ABSTRACT. Two classes of univalent harmonic functions on unit disc satisfying the conditions $\sum_{n=2}^{\infty}(n-\alpha)(|a_n|+|b_n|) \leq (1-\alpha)(1-|b_1|)$ and $\sum_{n=2}^{\infty} n(n-\alpha)(|a_n|+|b_n|) \leq (1-\alpha)(1-|b_1|)$ are given. That the ranges of the functions belonging to these two classes are starlike and convex, respectively. Sharp coefficient relations and distortion theorems are given for these functions. Furthermore results concerning the convolutions of functions satisfying above inequalities with univalent, harmonic and convex functions in the unit disk and with harmonic functions having positive real part.

Key words and phrases: Convex harmonic functions, Starlike harmonic functions, Extremal problems.

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1. INTRODUCTION

Let U denote the open unit disc and S_H denote the class of all complex valued, harmonic, orientation-preserving, univalent functions f in U normalized by $f(0) = f_z(0) - 1 = 0$. Each $f \in S_H$ can be expressed as $f = h + \overline{g}$ where h and g belong to the linear space $H(U)$ of all analytic functions on U.

Firstly, Clunie and Sheil-Small [3] studied S_H together with some geometric subclasses of S_H. They proved that although S_H is not compact, it is normal with respect to the topology of uniform convergence on compact subsets of U. Meanwhile the subclass S_H^0 of S_H consisting of the functions having the property $f_{\overline{z}}(0) = 0$ is compact.

In this article we concentrate on two specific subclasses of univalent harmonic mappings. These classes have corresponding meaning in the class of convex and starlike analytic functions of order α. The geometrical properties of the functions in these classes together with the neighborhoods in the meaning of Ruscheveyh and convolution products are considered.

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2. THE CLASSES HS(α) AND HC(α)

Let $U_r = \{z : |z| < r,\ 0 < r \leq 1\}$ and $U_1 = U$. Harmonic, complex-valued, orientation-preserving, univalent mappings f defined on U can be written as

$$f(z) = h(z) + \overline{g(z)}, \tag{2.1}$$

where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{2.2}$$

are analytic in U.

Denote by $HS(\alpha)$ the class of all functions of the form (2.1) that satisfy the condition

$$\sum_{n=2}^{\infty} (n-\alpha)(|a_n| + |b_n|) \leq (1-\alpha)(1-|b_1|) \tag{2.3}$$

and by $HC(\alpha)$ the subclass of $HS(\alpha)$ that consists of all functions subject to the condition

$$\sum_{n=2}^{\infty} n(n-\alpha)(|a_n| + |b_n|) \leq (1-\alpha)(1-|b_1|),$$

where $0 \leq \alpha < 1$ and $0 \leq |b_1| < 1$. The corresponding subclasses of $HS(\alpha)$ and $HC(\alpha)$ with $b_1 = 0$ will be denoted by $HS^0(\alpha)$ and $HC^0(\alpha)$, respectively. When $\alpha = 0$, these classes are denoted by HS and HC and have been studied by Y. Avcı and E. Zlotkiewicz [2]. If $|b_1| = 1$ and (2.3) is satisfied, then the mappings $z + \overline{b_1 z}$ are not univalent in U and of no interest.

If h, g, H, G are of the form (2.2) and if

$$f(z) = h(z) + \overline{g(z)} \quad \text{and} \quad F(z) = H(z) + \overline{G(z)}$$

then the convolution of f and F is defined to be the function

$$f * F(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \overline{\sum_{n=1}^{\infty} b_n B_n z^n}$$

while the integral convolution is defined by

$$f \diamond F(z) = z + \sum_{n=2}^{\infty} \frac{a_n A_n}{n} z^n + \overline{\sum_{n=1}^{\infty} \frac{b_n B_n}{n} z^n}.$$

The $\delta-$neighborhood of f is the set

$$N_\delta(f) = \left\{ F : \sum_{n=2}^{\infty} n(|a_n - A_n| + |b_n - B_n|) + |b_1 - B_1| \leq \delta \right\}$$

(see [1], [5]). In this case, let us define the generalized $\delta-$neighborhood of f to be the set

$$N(f) = \left\{ F : \sum_{n=2}^{\infty} (n-\alpha)(|a_n - A_n| + |b_n - B_n|) + (1-\alpha)|b_1 - B_1| \leq (1-\alpha)\delta \right\}.$$

3. MAIN RESULTS

First, let us give the interrelations between the classes $HS(\alpha)$ and HS, $HC(\alpha)$ and HC.

Theorem 3.1. *$HS(\alpha) \subset HS$ and $HC(\alpha) \subset HC$. Consequently $HS^0(\alpha) \subset HS^0$ and $HC^0(\alpha) \subset HC^0$. In particular if $0 \le \alpha_1 \le \alpha_2 < 1$ then $HS(\alpha_2) \subset HS(\alpha_1)$ and $HC(\alpha_2) \subset HC(\alpha_1)$.*

Proof. Since

$$\sum_{n=2}^{\infty} n(|a_n| + |b_n|) \le \sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}(|a_n| + |b_n|) \le 1 - |b_1| \tag{3.1}$$

and

$$\sum_{n=2}^{\infty} n^2(|a_n| + |b_n|) \le \sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}(|a_n| + |b_n|) \le 1 - |b_1|$$

we have the proof of theorem. □

Corollary 3.2. (i) *Each member of $HS^0(\alpha)$ maps U onto a domain starlike with respect to the origin.*

(ii) *Functions of the class $HC^0(\alpha)$ maps U_r onto convex domains.*

Theorem 3.3. *The class $HS(\alpha)$ consist of univalent sense preserving harmonic mappings.*

Proof. From [2, Theorem 1] for f in $HS(\alpha)$ and for $z_1, z_2 \in U$ with $z_1 \ne z_2$ we have

$$|h(z_1) - h(z_2)| \ge |z_1 - z_2| \left(1 - |z_2| \sum_{n=2}^{\infty} n|a_n|\right)$$

and

$$|g(z_1) - g(z_2)| \le |z_1 - z_2| \left(|b_1| + |z_2| \sum_{n=2}^{\infty} n|b_n|\right).$$

When we consider the relation (3.1), it follows that

$$\begin{aligned}
|f(z_1) - f(z_2)| &\ge |z_1 - z_2| \left(1 - |b_1| - |z_2| \sum_{n=2}^{\infty} n(|a_n| + |b_n|)\right) \\
&= |z_1 - z_2| \left(1 - |b_1| - |z_2| \sum_{n=2}^{\infty} (n-\alpha)(|a_n| + |b_n|) \right.\\
&\qquad \left. -\alpha|z_2| \sum_{n=2}^{\infty} (|a_n| + |b_n|)\right) \\
&\ge |z_1 - z_2|[1 - |b_1| - |z_2|(1-\alpha)(1-|b_1|) - \alpha|z_2|(1-|b_1|)] \\
&= |z_1 - z_2|(1-|b_1|)(1-|z_2|) > 0.
\end{aligned}$$

So f is univalent. Since

$$\begin{aligned}
J_f(z) &= |h'(z)|^2 - |g'(z)|^2 \\
&\ge (|h'(z)| + |g'(z)|) \left(1 - |z| \sum_{n=2}^{\infty} n|a_n| - |b_1| - |z| \sum_{n=2}^{\infty} n|b_n|\right) \\
&\ge (|h'(z)| + |g'(z)|)(1 - |b_1|)(1 - |z|) > 0
\end{aligned}$$

f is sense preserving. □

Remark 3.4. (i) The functions $f_n(z) = z + \frac{n-1}{n+1}\, e^{i\theta}\overline{z}$ are in $HS(\alpha)$ and the sequence converges uniformly to $z + e^{i\theta}\overline{z}$. Thus the class $HS(\alpha)$ is not compact.

(ii) If $f \in HS(\alpha)$, then for each r, $0 < r < 1$, $r^{-1}f(rz) \in HS(\alpha)$.

(iii) If $f \in HS(\alpha)$ and $f_0(z) = (f(z) - \overline{b_1 f(z)})/(1 - |b_1|^2)$ then $f_0 \in HS^0(\alpha)$, but $f(z) = f_0(z) + \overline{b_1 f_0(z)}$ may not be in $HS(\alpha)$.

(iv) If $f = h + \overline{g} \in HS^0(\alpha)$ then the function

$$F(z) = \int_0^1 \frac{f(tz)}{t}dt = \int_0^z \frac{h(u)}{u}du + \overline{\int_0^z \frac{g(u)}{u}du}$$

satisfies (2.3) with $b_1 = 0$, hence $F(z)$ is a convex harmonic mapping. Convexity of $F(z)$ however, does not imply starlikeness of $f(z)$ (or even univalence) in general situation.

Theorem 3.5. *Each function in the class $HS^0(\alpha)$ maps disks U_r, $r \leq \frac{1-\alpha}{2-\alpha}$ onto convex domains. The constant $\frac{1-\alpha}{2-\alpha}$ is best possible.*

Proof. If $f \in HS^0(\alpha)$ and r, $0 < r < 1$ be fixed, then $r^{-1}f(rz) \in HS^0(\alpha)$. We must find an upper bound for r such that

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}(|a_n| + |b_n|)r^{n-1} \leq 1.$$

Since $f \in HS^0$ we have

$$\sum_{n=2}^{\infty} \frac{n(n-\alpha)}{1-\alpha}(|a_n| + |b_n|)r^{n-1} = \sum_{n=2}^{\infty} n(|a_n| + |b_n|)\frac{(n-\alpha)r^{n-1}}{1-\alpha} \leq 1$$

provided $(n-\alpha)r^{n-1}/(1-\alpha) \leq 1$ which is true if $r \leq (1-\alpha)/(2-\alpha)$. □

Theorem 3.6. *If $f \in HS(\alpha)$, then*

$$|f(z)| \leq |z|(1 + |b_1|) + \frac{(1-\alpha^2)(1-|b_1|)}{2}|z|^2$$

and

$$|f(z)| \geq (1 - |b_1|)\left(|z| - (1-\alpha^2)\frac{|z|^2}{2}\right).$$

Equalities are attained by the functions

$$f_\theta(z) = z + |b_1|e^{i\theta}\overline{z} + \frac{1-|b_1|}{2}(1-\alpha^2)\overline{z}^2$$

for properly chosen real θ.

Proof. We have

$$|f(z)| \leq |z|(1 + |b_1|) + |z|^2 \sum_{n=2}^{\infty}(|a_n| + |b_n|).$$

Since

$$\begin{aligned}\sum_{n=2}^{\infty}(|a_n|+|b_n|) &\le \frac{1}{2}(1-\alpha)(1-|b_1|)+\frac{\alpha}{2}(|a_2|+|b_2|)\\ &\quad -\frac{1}{2}\sum_{n=3}^{\infty}(n-\alpha-2)(|a_n|+|b_n|)\\ &\le \frac{1}{2}(1-\alpha)(1-|b_1|)+\frac{\alpha}{2}(1-\alpha)(1-|b_1|)\\ &= \frac{1}{2}(1-\alpha^2)(1-|b_1|)\end{aligned}$$

it follows that

$$|f(z)| \le |z|(1+|b_1|)+\frac{(1-\alpha^2)(1-|b_1|)}{2}|z|^2$$

for z in U.

Similarly we get

$$\begin{aligned}|f(z)| &\ge |z|(1-|b_1|)-|z|^2\sum_{n=2}^{\infty}(|a_n|+|b_n|)\\ &\ge |z|(1-|b_1|)-\frac{(1-\alpha^2)(1-|b_1|)}{2}|z|^2\\ &= (1-|b_1|)\left(|z|-(1-\alpha^2)\frac{|z|^2}{2}\right).\end{aligned}$$

The classes $HS(\alpha)$ and $HC(\alpha)$ are uniformly bounded, hence they are normal. □

Remark 3.7. We can give a similar result for $HC(\alpha)$ to that given for $HS(\alpha)$. As the proof is similar we shall omit it.

Theorem 3.8. *If $f \in HC(\alpha)$, then*

$$|f(z)| \le |z|(1+|b_1|)+\frac{3-\alpha-2\alpha^2}{2\alpha}(1-|b_1|)|z|^2$$

and

$$|f(z)| \ge (1-|b_1|)\left(|z|-\frac{3-\alpha-2\alpha^2}{2\alpha}|z|^2\right).$$

Equalities are attained by the functions

$$f_\theta(z)=z+|b_1|e^{i\theta}\overline{z}+\frac{3-\alpha-2\alpha^2}{2\alpha}\overline{z}^2$$

for properly chosen real θ.

Theorem 3.9. *The extreme points of $HS^0(\alpha)$ are only the functions of the form: $z+a_nz^n$ or $z+\overline{b_mz^m}$ with*

$$|a_n|=\frac{1-\alpha}{n-\alpha},\qquad |b_m|=\frac{1-\alpha}{m-\alpha},\qquad 0\le\alpha<1.$$

Proof. Suppose that

$$f(z)=z+\sum_{n=2}^{\infty}(a_nz^n+\overline{b_nz^n})$$

is such that

$$\sum_{n=2}^{\infty}\frac{n-\alpha}{1-\alpha}(|a_n|+|b_n|)<1,\qquad a_k>0.$$

Then, if $\lambda > 0$ is small enough we can replace a_k by $a_k - \lambda$, $a_k + \lambda$ and we obtain two functions $f_1(z),\ f_2(z)$ that satisfy the same condition and for which one gets $f(z) = \frac{1}{2}[f_1(z) + f_2(z)]$. Hence f is not a possible extreme point of $HS^0(\alpha)$.

Let now $f \in HS^0(\alpha)$ be such that

$$\sum_{n=2}^{\infty} \frac{n-\alpha}{1-\alpha}(|a_n| + |b_n|) = 1, \qquad a_k \neq 0, b_l \neq 0 \tag{3.2}$$

If $\lambda > 0$ is small enough and if μ, ζ with $|\mu| = |\zeta| = 1$ are properly chosen complex numbers, then leaving all but a_k, b_l coefficients of $f(z)$ unchanged and replacing a_k, b_l by

$$a_k + \lambda \frac{1-\alpha}{k-\alpha}\mu, \qquad b_l - \lambda \frac{1-\alpha}{l-\alpha}\zeta$$

$$a_k - \lambda \frac{1-\alpha}{k-\alpha}\mu, \qquad b_l + \lambda \frac{1-\alpha}{l-\alpha}\zeta$$

we obtain functions $f_1(z),\ f_2(z)$ that satisfy (3.2) and such that $f(z) = \frac{1}{2}[f_1(z) + f_2(z)]$. In this case f can not be an extreme point. Thus for $|a_n| = (1-\alpha)/(n-\alpha)$, $|b_m| = (1-\alpha)/(m-\alpha)$, $f(z) = z + a_n z^n$ or $f(z) = z + \overline{b_m z^m}$ are extreme points of $HS^0(\alpha)$. □

Similarly we can obtain that the following result is true.

Theorem 3.10. *The extreme points of $HC^0(\alpha)$ are only the functions of the form: $z + a_n z^n$ or $z + \overline{b_m z^m}$ with*

$$|a_n| = \frac{1-\alpha}{n(n-\alpha)}, \qquad |b_m| = \frac{1-\alpha}{m(m-\alpha)}, \qquad 0 \leq \alpha < 1.$$

Let K_H^0 denote the class of harmonic univalent functions of the form (2.1) with $b_1 = 0$ that map U onto convex domains. It is known [3, Theorem 5.10] that the sharp inequalities

$$2|A_n| \leq n+1, \qquad 2|B_n| \leq n-1$$

are true.

Theorem 3.11. *Suppose that*

$$F(z) = z + \sum_{n=2}^{\infty}(A_n z^n + \overline{B_n z^n})$$

belongs to K_H^0. Then

(i) *If $f \in HC^0(\alpha)$ then $f * F$ is starlike univalent and $f \diamond F$ is convex.*

(ii) *If $f(z)$ satisfies the condition*

$$\sum_{n=2}^{\infty} n^2(n-\alpha)(|a_n| + |b_n|) \leq 1-\alpha$$

then $f * F$ is convex univalent.

Proof. We justify the case (i). Since

$$\begin{aligned} \sum_{n=2}^{\infty}(n-\alpha)(|a_n A_n| + |b_n B_n|) &= \sum_{n=2}^{\infty} n(n-\alpha)\left(|a_n|\left|\frac{A_n}{n}\right| + |b_n|\left|\frac{B_n}{n}\right|\right) \\ &\leq \sum_{n=2}^{\infty} n(n-\alpha)(|a_n| + |b_n|) \leq 1-\alpha \end{aligned}$$

it follows that $f * F$ is in $HS^0(\alpha)$. Namely $f * F$ is starlike univalent.

Furthermore, the transformation

$$\int_0^1 \frac{f * F(tz)}{t} dt = f \diamond F(z)$$

now shows that $f \diamond F \in HC^0(\alpha)$. □

Let S denote the class of analytic univalent functions of the form $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$. It is well known that the sharp inequality $|A_n| \le n$ is true.

Theorem 3.12. *If $f \in HC^0(\alpha)$ and $F \in S$ then for $|\varepsilon| \le 1$, $f * (F + \varepsilon \overline{F})$ is starlike univalent.*

Proof. Since

$$\sum_{n=2}^{\infty} (n-\alpha)(|a_n A_n| + |b_n B_n|) \le \sum_{n=2}^{\infty} n(n-\alpha)(|a_n| + |b_n|) \le 1-\alpha$$

it follows that $f * (F + \varepsilon \overline{F})$ is starlike univalent.

Let P_H^0 denote the class of functions F complex and harmonic in U, $f = h + \overline{g}$ such that $\operatorname{Re} f(z) > 0$, $z \in U$ and

$$H(z) = 1 + \sum_{n=1}^{\infty} A_n z^n, \qquad G(z) = \sum_{n=2}^{\infty} B_n z^n.$$

It is known [4, Theorem 3] that the sharp inequalities

$$|A_n| \le n+1, \qquad |B_n| \le n-1$$

are true. □

Theorem 3.13. *Suppose that*

$$F(z) = 1 + \sum_{n=1}^{\infty} (A_n z^n + \overline{B_n z^n})$$

belongs to P_H^0. Then

(i) *If $f \in HC^0(\alpha)$ then for $\frac{3}{2} \le |A_1| \le 2$, $\frac{1}{A_1} f * F$ is starlike univalent and $\frac{1}{A_1} f \diamond F$ is convex.*

(ii) *If $f(z)$ satisfies the condition*

$$\sum_{n=2}^{\infty} n^2 (n-\alpha)(|a_n| + |b_n|) \le 1-\alpha$$

then $\frac{1}{A_1} f * F$ is convex univalent.

Proof. We justify the case (ii). Since

$$\begin{aligned} \sum_{n=2}^{\infty} n(n-\alpha)\left(\left|\frac{a_n A_n}{A_1}\right| + \left|\frac{b_n B_n}{A_1}\right|\right) &\le \sum_{n=2}^{\infty} n^2 (n-\alpha) \left(|a_n| \frac{n+1}{|A_1| n} + |b_n| \frac{n-1}{|A_1| n} \right) \\ &\le \sum_{n=2}^{\infty} n^2 (n-\alpha)(|a_n| + |b_n|) \le 1-\alpha \end{aligned}$$

$\frac{1}{A_1} f * F$ is convex univalent. □

Remark 3.14. If $f \in HS^0(\alpha)$ and $F \in K_H^0$, then $f * F$ need not be univalent. For example, if $f(z) = z + \frac{1-\alpha}{n-\alpha} z^n$ and $F(z) = Re\left(\frac{z}{1-z}\right) + i\, Im\left(\frac{z}{(1-z)^2}\right)$ then $f * F(z) = z + \frac{(n+1)(1-\alpha)}{2(n-\alpha)} z^n$ is not univalent in U. But $f \diamond F$ is univalent and starlike.

Theorem 3.15. *Let*

$$f(z) = z + \overline{b_1 z} + \sum_{n=2}^{\infty} (a_n z^n + \overline{b_n z^n})$$

is a member of $HC(\alpha)$. *If* $\delta \leq \frac{1-\alpha}{2-\alpha}(1 - |b_1|)$, *then* $N(f) \subset HS(\alpha)$.

Proof. Let $f \in HC(\alpha)$ and $F(z) = z + \overline{B_1 z} + \sum_{n=2}^{\infty} (A_n z^n + \overline{B_n z^n})$ belong to $N(f)$. We have

$$\begin{aligned}
(1-\alpha)|B_1| &+ \sum_{n=2}^{\infty} (n-\alpha)(|A_n| + |B_n|) \\
&\leq (1-\alpha)|B_1 - b_1| + (1-\alpha)|b_1| \\
&\quad + \sum_{n=2}^{\infty} (n-\alpha)(|A_n - a_n| + |B_n - b_n|) + \sum_{n=2}^{\infty} (n-\alpha)(|a_n| + |b_n|) \\
&\leq (1-\alpha)\delta + (1-\alpha)|b_1| + \frac{1}{2-\alpha} \sum_{n=2}^{\infty} n(n-\alpha)(|a_n| + |b_n|) \\
&\leq (1-\alpha)\delta + (1-\alpha)|b_1| + \frac{1-\alpha}{2-\alpha}(1 - |b_1|) \\
&\leq 1 - \alpha.
\end{aligned}$$

Hence, for

$$\delta \leq \frac{1-\alpha}{2-\alpha}(1 - |b_1|)$$

$F(z) \in HS(\alpha)$. □

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