JGSP **9** (2007) 83–95

JOURNAL OF
Geometry and Symmetry in Physics

GEOMETRY OF KALMAN FILTERS

MACIEJ P. WOJTKOWSKI

Communicated by Martin Schlichenmaier

Abstract. We present a geometric explanation of Kalman filters in terms of a symplectic linear space and a special quadratic form on it. It is an extension of the work of Bougerol with application of a different metric we have introduced earlier. The new results are contained in Theorem 1 and Theorem 4.

1. Kalman Filters and Two Contractions

Let us consider a linear dynamical system with noise. In the simplest case of discrete time we have a linear map $F : \mathbb{R}^k \to \mathbb{R}^k$ with additive noise

$$X_n = FX_{n-1} + G\varepsilon_n, \qquad n \geq 1$$

where X_n is the state of the system at time $n \geq 0$ and the noise $\{\varepsilon_n\}, n \geq 1$, is a sequence of independent normalized gaussian vectors in $\mathbb{R}^l$ and G is a $k \times l$ matrix, with $l \leq k$. The initial state X_0 has also a gaussian distribution with mean $\widehat{X}_0$ and covariance matrix P_0.

We take measurements on the system and the result at time n is

$$Y_n = HX_n + \eta_n, \qquad n \geq 1$$

where H is a $m \times k$ matrix and $\{\eta_n\}$ is the observational noise which is assumed again to be a sequence of independent normalized gaussian vectors. The dimension of the measurement vector (m) is in general smaller than the dimension of the state vector (k), e.g., $m = 1$.

The problem addressed by the Kalman filter is to estimate X_n given the values of $Y_1, ..., Y_n$. The ideal answer is the estimator

$$\widehat{X}_n = \mathbb{E}(X_n | \mathcal{F}_n)$$