JGSP **50** (2018) 97–100

JOURNAL OF
Geometry and Symmetry in Physics
ISSN 1312-5192

# BOOK REVIEW

*The Restricted Three-Body Problem and Holomorphic Curves*, by Urs Frauenfelder and Otto van Koert, Birkhäuser, Cham 2018, xi+374pp, ISBN 978-3-319-72277-1.

This book is an updated compilation of the recent and not so recent developments on the so called *restricted three-body problem*, which is a special case of the general three-body problem consisting of taking the limit of null mass of the third body that is called the *satellite*. In this way, the two primary bodies (for instance Earth and the Moon) are moving each one around the other in a constant plane since the satellite does not alter their motion. If the satellite also moves in the same plane, the problem is called the *planar restricted three-body problem*. If the satellite is allowed to move in another plane, it is called the *spatial restricted three-body problem*. An additional and usual simplification is to consider that the two primary masses are rotating in circular orbits around their center of mass. The problem is then called the *circular restricted three-body problem*. In this case, by taking a rotating frame placed at the center of mass, the two primaries remain at fixed positions on a line through the origin of coordinates. According to the authors, Jacobi showed that the Hamiltonian of the circular restricted three-body problem in this rotating frame is autonomous (time-independent) and therefore a constant of the motion, which no longer happens for the elliptic restricted three-body problem. These approximations are usually applied to study the motions of satellites around the Earth-Moon system, but also around the Sun-Jupiter system or even around the Pluto-Charon or Jupiter-Europa systems. The tools the authors handle to study this problem with the above mentioned approximations are symplectic geometry and holomorphic curves. In the introduction the authors sustain a fervent and detailed defence of these tools as the best way to attack the restricted three-body problem. At the same time, the reader then becomes aware that the book will not be easy to read. The book has been published in the collection “Pathways in Mathematics” by Springer, which explains:

> Each “Pathways in Mathematics” book offers a roadmap to a currently well developing mathematical research field and is a first-hand information and inspiration for further study, aimed both at students and researchers. It is

> written in an educational style, i.e., in a way that is accessible for advanced undergraduate and graduate students. It also serves as an introduction to and survey of the field for researchers who want to be quickly informed about the state of the art. The point of departure is typically a bachelor/masters level background, from which the reader is expeditiously guided to the frontiers. This is achieved by focusing on ideas and concepts underlying the development of the subject while keeping technicalities to a minimum.

Contrary to the intention of the publisher, this book is not aimed at students but only at researchers on this topic. Moreover, it is not an introduction to the field of the restricted three-body problem but a systematic treatise and deep exposition of the state of this art. In this way, technicalities are raised to a maximum for a problem that is very elemental from a conceptual point of view. Its index already indicates the complexity of this book, whose chapters are: introduction, symplectic geometry and Hamiltonian mechanics, symmetries, regularization of two-body collisions, the restricted three-body problem, contact geometry and the restricted three-body problem, periodic orbits in Hamiltonian systems, periodic orbits in the restricted three-body problem, global surfaces of section, the Maslov index, spectral flow, convexity, finite energy planes, Sierfring's intersection theory for fast finite energy planes, the moduli space of fast finite energy planes, compactness, construction of global surfaces of section, and numerics and dynamics via global surfaces of section. The book looks rather a treatise of symplectic geometry than a book on the restricted three-body problem. Those interested in symplectic geometry should be recommended to read the reference [6]. In spite of the good willingness of Frauenfelder and van Koert, their book could already be obsolete. They omit two very important references, the first one by Broucke and Lass [3], and the second one by Hestenes [5, pp 398-418]. The first paper outlined, for the first time, the general equations of the motion of the three-body problem given with relative coordinates and velocities in 1973. The latter author reminded us of that paper in his *New Foundations for Classical Mechanics*, an excellent book aimed at students. Perhaps, without Hestenes' reminder, the treatment of the three-body problem by Broucke and Lass would have fallen into oblivion. Frauenfelder and van Koert seem not to be aware of both important references. In our paper [4] devoted to the three-body problem and recently published in this journal, we provide the kinetic energy as a quadratic form of the relative velocities which, together with the potential energy depending on the relative coordinates, allows us to solve easily the Lagrange equations of motion [4, Equation (33)], the same equations outlined by Broucke and Lass [3, Equation (6)]. In this way, we have applied them to the system Sun-Earth-Moon with spectacular results which supersede Hill's lunar theory and can be summarized into the following items:

1) The undetermined Lagrange multiplier vector is developed in a series expansion of the quotient of the distances from Earth to the Moon and from the Sun to Earth, which is about $2.5 \cdot 10^{-3}$.

2) In the zero-order approach, Earth moves around the Sun on an elliptic orbit in the ecliptic, while the Moon moves around Earth on another elliptic orbit in a plane with constant orientation independent of and different from the ecliptic.

3) The first-order approximation accounts for perturbations to the Moon's Keplerian motion and fully explains almost all of them, such as retrogradation of the nodes, oscillation of the inclination of the orbit and changes in the length of the draconic month as well as the variations of the elliptic orbit.

4) The perturbations to all the parameters of the lunar orbit are always described by an addition of three sinusoidal functions whose amplitudes and frequencies are directly related with the mean orbital parameters of the Moon.

5) The Moon's elliptic orbit taken as input of the first-order approximation does not yield the advance of the perigee. Most likely, the perturbed orbit causes it due to its asymmetry, but this point has not been checked yet.

Our lunar theory is an analytic theory that can be easily explained to any undergraduate student and makes no use of the advanced mathematical tools Frauenfelder and van Koert's book deals with. These authors should try to apply the Langrangian [4, Equation (32)] of the three-body problem to the system Earth-Moon-satellite, although one must also remember the problematic approximations assumed in the restricted three-body problem. For instance, the supposition that the system Earth-Moon-satellite is an inertial frame is a rough approximation, since the gravity of the Sun is very present and, strictly speaking, it is a system of four bodies. Anyway, if one wishes to suppose that the center of mass behaves like an inertial frame, this is also a rough approximation because the direction of its motion changes by about $1^\circ$ per day, and the travel of a satellite going from Earth to the Moon can last several days. On the other hand, the distance from Earth to the Moon is not constant but continuously changing, and not according to a Keplerian ellipse but in a more intricate way. Therefore, the approximation of circular orbits is also a rough approximation. And finally, the inclination of the orbital plane of the Moon is continuously changing, so that the assumption of a constant plane of motion of Earth and the Moon is another rough approximation. Therefore, the practical application of the book is necessarily very limited owing to the errors in the computation of the orbit of the satellite caused by the above approximations.

Those who are interested in the restricted three-body problem and do not want to struggle with an abstruse mathematical formalism can read Broucke's report [1] of 1968. It is very complete and much more understandable than Frauenfelder and van Koert's book. The stability of the orbits was also studied in his later paper [2]. In fact, Broucke's job was the computation of the orbits of satellites at the Jet Propulsion Laboratory, where Hestenes later worked and learnt about him. In 1973, Broucke became the executive editor of *Celestial Mechanics*.

## Acknowledgments

We thank very much Professor Charles-Michel Marle for sharing with us the final version of his book [6] before being published.

## References

[1] Broucke R., *Periodic Orbits in the Restricted Three-Body Problem With Earth-Moon Masses*, NASA Technical Report 32-1168, JP Lab., Pasadena 1968, https://ntrs.nasa.gov/archive/nasa /casi.ntrs.nasa.gov/19680013800.pdf

[2] Broucke R., *Stability of Periodic Orbits in the Elliptic, Restricted Three-Body Problem*, AIAA J. **7** (1969) 1003-1009.

[3] Broucke R. and Lass H., *A Note on Relative Motion in the General Three-Body Problem*, Cel. Mech. **8** (1973) 5-10.

[4] González R., *The Solution to the Three-Body Problem and Some Applications*, J. Geom. Symmetry Phys. **49** (2018) 1-61.

[5] Hestenes D., *New Foundations for Classical Mechanics*, 2nd Edn, Kluwer, New York 2002.

[6] Marle Ch.-M., *Géométrie symplectique et géométrie de Poisson*, Calvage et Mounet, Paris 2018.

Ramon González Calvet
Institut Pere Calders
Campus Universitat Autònoma de Barcelona
08193 Cerdanyola del Vallès, SPAIN
*E-mail address*: rgonzalezcalvet@gmail.com