JGSP **42** (2016) 95–98

JOURNAL OF
Geometry and Symmetry in Physics
ISSN 1312-5192

BOOK REVIEW

The Lorentz Group (in Russian), by Fedor I. Fedorov, Science, Moscow 1979, 384 pp, $\Phi\frac{20402-164}{053(02)-79}129-79$.

The book provides an original treatment of the Lorentz group in $3+1$ dimensions based on the author's extensive research on special relativity and elementary particle physics. His fascinating approach, however, shades light on many topics concerning the rotation group in $\mathbb{R}^3$, its spin cover $\mathsf{SU}(2)$ and the complex Lorentz group $\mathsf{SO}(4,\mathbb{C})$. The main tool used in the analysis of these groups is the so-called *vector-parameter* proposed earlier by Rodrigues in [9] and thus, often referred to as *Rodrigues' vector*. This construction, which may be obtained in a rather straightforward manner via central projection form the unit sphere, has proven useful and convenient in many practical situation concerning the three-dimensional rotations. It has been revived by Gibbs (and for this reason also called *Gibbs' vector*), but in recent years it seems to be less popular among the Western scientific community. Only lately, very few (and mostly informative) reports on it have appeared, such as [1, 8]. On the other hand, in the former Soviet block this quite handy construction has been in active research for decades (see [2–4, 6]). Papers like [5] and [7] make an exception in Western literature, but they actually originate on the East. There is no doubt that the popularity of vectorial parametrization in this region is mostly due to Fedorov's work and in particular, the book discussed here. And the applications span quite a wide range: from projective geometry and group theory to computer graphics and aerospace engineering. Thus, the Western mathematical and physical community may benefit greatly from the acquaintance with Fedorov's work.

The book begins with a brief introduction to orthogonal transformations and in particular, the three-dimensional case with its somewhat special features. Then, the vector-parameter is derived from the Cayley transform and its main properties are discussed: its composition law, the non-linear representation of $\mathsf{SO}(3)$ it provides in a natural way, as well as its relation to other classical parameterizations, e.g. the Euler and Bryan angles. Some more advanced topics, such as invariant integration and infinitesimal operators (including quantum-mechanical angular momenta and

Hamiltonians), are also discussed. The construction is then lifted to the spin cover $\mathsf{SU}(2)$ and the complex group $\mathsf{SO}(3,\mathbb{C})$, which allows for the study of its real forms (also referred to as *Wigner little groups*) in a natural and consistent manner.

Chapter two provides a classical treatment of the irreducible representations of the rotation group with the aid of the vector-parameter construction. Some of the results of the first chapter, e.g. infinitesimal operators and the invariant measure on $\mathsf{SO}(3)$, are used in this detailed analysis. The reader may find a great deal of representation theory in a rather compressed volume. The third chapter is dedicated to the Lorentz group of $3+1$ dimensional Minkowski space-time and thus, starts with a brief geometric discussion on the latter. The author exploits Wick's rotation from the very beginning and takes advantage of the local isomorphism $\mathsf{SO}(3,1)^{+} \cong \mathsf{SO}(3,\mathbb{C})$ that allows for using the complex vector-parameter construction in the relativistic case. Then, the *tensor-parameter* (interpreted as Maxwell's stress-energy tensor) is introduced with its composition law and some of its properties are carefully investigated. Using its characteristic equation, the invariant axis problem is approached and many features of the Lorentz group are thus derived. In particular, plane transformations and Wigner little groups are considered into more detail. Chapter four is dedicated to the representation theory of the proper Lorentz group and is organized similarly to chapter two. Apart from the standard topics, one may find much information about the infinite-dimensional representations as well as the complex Lorentz group $\mathsf{SO}(4,\mathbb{C})$. Finally, in the last, fifth chapter some applications of the previously derived results in elementary particle physics are considered. The author begins with the classical Dirac and Klein-Gordon free-particle equations and gradually builds up to quantization and coherent states, the inclusion of external fields and calculation of matrix elements in quantum electrodynamics. At the end, there is an Appendix dedicated to some standard topics in linear algebra and group representation theory that may be useful to the unexperienced reader.

The exposition in all five chapters is rather approachable - no extensive knowledge on the specific topics discussed in the monograph is assumed beyond the standard courses in linear algebra and calculus (and perhaps, group theory on a very basic level). All advanced techniques are explained in detail and even well-known constructions from abstract algebra and theoretical physics are derived consistently in a rather pedagogical manner. Therefore, the book may be quite useful not only to specialists, but to students as well. It contains introductory level notes in several disciplines that are vitally important for theoretical and mathematical physics. On the other hand, even experienced in this area scientists can benefit greatly from the alternative approach and in particular, the numerous advantages of the vector-parameter description. One thing that may appear a bit disappointing however, is that some of the derivations seem to be rather technical, while they could be made far more efficient in the proper algebraic frame, e.g. there is no mention of geometric algebras and quaternions in particular, or the Hopf fibration whatsoever, although the whole vector-parameter construction seems to originate in these concepts. Hence, the book provides also an implicit exercise to the abstract-thinkers, namely to reformulate the main results derived therein in terms of these more fundamental algebraic objects. Considering the importance of this relatively less familiar description of rotations and Lorentz transformations and the numerous advantages it provides in both fundamental physics and the engineering applications, it is highly recommended that a translation in English of this book finally appears. In this reviewer's humble opinion, such an opportunity would provide a tremendous boost for the development in the respective areas of the Western science and technology.

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