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BOOK REVIEW

An Introduction to Lie Groups and Lie Algebras, by Alexander Kirillov, Jr. Cambridge University Press, Cambridge, 2008, xi + 222pp., ISBN 978-0-521-88969-8.

The author, of the book under review Alexander Kirillov, Jr. is a faculty member of the Institute for Mathematical Sciences and at the Mathematical Department of Stony Brook University, New York, USA.

The introduction briefly explains the philosophy of the author on how Lie groups and Lie algebras should be taught. It stresses more on the ideas and on the exercises, rather than on the rigorous proofs.

Chapter 2 deals with the basic definitions in Lie groups. The complex Lie group G is introduced as a set with two compatible structures: G is a group and G is a complex analytic manifold. Next in a rather smooth and natural way the important notion of orbits and homogeneous spaces are introduced as well as the definitions of the classical groups.

Chapter 3 deals with the interrelations between Lie groups and is Lie algebras $\mathfrak{g}$. Of course it starts by introducing the main notions of the theory: the exponential map, the Lie group generators and their commutators, the Jacobi identity and the adjoint action of the Lie group G onto its Lie algebra $\mathfrak{g}$. These are naturally followed by the notions of a subalgebra, ideal and center, by the realization of the Lie algebras in terms of vector fields, and by the formulation of the fundamental theorems in Lie theory. At the end the complex and real forms are introduced.

In Chapter 4 the representation theory of Lie algebras is outlined. The author introduces the notion of representation and irreducible representation, followed by the intertwining operators and Schur's lemma. Next the representation theory for finite groups is briefly formulated, which allows one to explain the character formula and Peter-Weyl theorem. The important example, the representations of the $\mathfrak{sl}(2,\mathbb{C})$ algebra are constructed and then used to analyze the eigenfunctions of the spherical Laplace operator, which are crucial for understanding the spectrum of the hydrogen atom.

Chapter 5 is devoted to the structure theory of the Lie algebras. Naturally it starts with the important notion of the universal enveloping algebra and Poincare-Birkhoff-Witt theorem, followed by the definitions of the solvable, nilpotent and semi-simple Lie algebras. Next it is shown that every representation of solvable Lie algebra can be realized by upper triangular matrices (Lie's theorem). Another essential result is that if $\mathfrak{g}$ is nilpotent then every adjoint operator $\mathrm{ad}_x, \quad x \in \mathfrak{g}$ is also nilpotent (Engel's theorem). Then Levy's theorem stating that any Lie algebra $\mathfrak{g}$ can be written down as a direct sum of its radical and a semi-simple subalgebra is formulated. An important tool in what follows is the invariant bilinear form K - the Killing form. It can be used as a criterium (Cartan's criterium) of solvability and semisimplicity of $\mathfrak{g}$. Namely, $\mathfrak{g}$ is solvable if $K([\mathfrak{g},\mathfrak{g}],\mathfrak{g}) = 0$ and $\mathfrak{g}$ is semisimple if $K(\mathfrak{g},\mathfrak{g})$ is non-degenerate. At the end of this chapter the Jordan decomposition of linear operators is defined.

Chapters 6 and 7 deal with the theory of complex semi-simple Lie algebras. Here the notion of Cartan subalgebra is introduced, which then along with the notion of the root systems. After explaining the properties of the root systems the author naturally introduces the Weyl group as their symmetry group and the Dynkin diagrams. To each representation of the semisimple Lie algebra $\mathfrak{g}$ one can relate a weight system with the same symmetry properties. The chapter ends with the Serre relations which play important role in recovering the simple Lie algebra $\mathfrak{g}$ from its Dynkin diagram. Thus the classification of the semi-simple Lie algebras is reduced to the classification of all possible Dynkin diagrams, or equivalently, to classification of their sets of simple roots.

The last Chapter 8 is devoted to the representation theory of the semi-simple Lie algebras. It explains Verma modules (i.e., representations with highest weights) and states the classification theorem for all finite dimensional representations of the Lie algebra $\mathfrak{g}$. This requires the classification of all possible highest weights and dominant weights of $\mathfrak{g}$. Next, the author explains the method of decomposing of any representations of $\mathfrak{g}$ into direct sum of irreducible representation labeled by the dominant weights, each taken with a certain multiplicity. He shows also how it is possible to calculate these multiplicities using the Weyl character formulae.
In the next subsection the representations of $\mathfrak{sl}(N,\mathbb{C})$ are described using the Young diagrams. The chapter ends by analyzing the Harish-Chandra isomorphism.

The Appendix A summarizes the properties of the root systems for the classical series of simple Lie algebras. The next Appendix B provides a sample syllabus for a one-semester graduate course on Lie algebras and Lie groups based on this monograph. Finally a brief overview of the literature is given, which helps students wishing to get deeper knowledge of the subject and to extend it to the theory of infinite-dimensional Lie groups and Lie algebras, or to quantum groups.

In all, the exposition is very clear and logical. It has the advantage of giving the basic facts about Lie algebra theory with enough arguments but skipping the tedious technical details of proofs. Another excellent feature of the book is that many of the basic notions, properties and results are illustrated by a great number of exercises and examples. In my opinion this book is a nice addition to the landmarks in the field, see [1–3]. I strongly recommend it to anyone wishing to enter into the beautiful and exciting field of Lie algebras and their applications.

References

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