Increasing-Chord Graphs On Point Sets

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Abstract

We tackle the problem of constructing increasing-chord graphs spanning point sets. We prove that, for every point set $P$ with $n$ points, there exists an increasing-chord planar graph with $O(n)$ Steiner points spanning $P$. The main intuition behind this result is that Gabriel triangulations are increasing-chord graphs, a fact which might be of independent interest. Further, we prove that, for every convex point set $P$ with $n$ points, there exists an increasing-chord graph with $O(n \log n)$ edges (and with no Steiner points) spanning $P$.
1 Introduction

A proximity graph is a geometric graph that can be constructed from a point set by connecting points that are “close”, for some local or global definition of proximity. Proximity graphs constitute a topic of research in which the areas of graph drawing and computational geometry nicely intersect. A typical graph drawing question in this topic asks to characterize the graphs that can be represented as a certain type of proximity graphs. A typical computational geometry question asks to design an algorithm to construct a proximity graph spanning a given point set.

Euclidean minimum spanning trees and Delaunay triangulations are famous examples of proximity graphs. Given a point set $P$, a Euclidean minimum spanning tree of $P$ is a geometric tree with $P$ as vertex set and with minimum total edge length; the Delaunay triangulation of $P$ is a triangulation $T$ such that no point in $P$ lies inside the circumcircle of any triangle of $T$. From a computational geometry perspective, given a point set $P$ with $n$ points, a Euclidean minimum spanning tree of $P$ with maximum degree five exists and can be constructed in $O(n \log n)$ time [5]; also, the Delaunay triangulation of $P$ exists and can be constructed in $O(n \log n)$ time [5]. From a graph drawing perspective, every tree with maximum degree five admits a representation as a Euclidean minimum spanning tree [16] and it is NP-hard to decide whether a tree with maximum degree six admits such a representation [9]; also, characterizing the class of graphs that can be represented as Delaunay triangulations is a deeply studied question, which still eludes a clear answer; see, e.g., [7, 8]. Refer to the excellent survey by Liotta [14] for more on proximity graphs.

While proximity graphs have constituted a frequent topic of research in graph drawing and computational geometry, they gained a sudden peak in popularity even outside these communities in 2004, when Papadimitriou et al. [19] devised an elegant routing protocol that works effectively in all the networks that can be represented as a certain type of proximity graphs, called greedy graphs. For two points $p$ and $q$ in the plane, denote by $pq$ the straight-line segment having $p$ and $q$ as end-points, and by $|pq|$ the length of $pq$. A geometric path $(v_1, \ldots, v_n)$ is greedy if $|v_{i+1}v_i| < |v_{i+1}v_n|$, for every $1 \leq i \leq n-1$. A geometric graph $G$ is greedy if, for every ordered pair of vertices $u$ and $v$, there exists a greedy path from $u$ to $v$ in $G$. A lot is known about the existence of greedy graphs spanning given point sets and about the possibility of representing graphs as greedy graphs; see, e.g., [3, 11, 13, 18, 19]. A result strictly related to our paper is that, for every point set $P$, the Delaunay triangulation of $P$ is a greedy graph [18].

In this paper we study self-approaching and increasing-chord graphs, that are types of proximity graphs defined by Alamdari et al. [2].

A geometric path $P = (v_1, \ldots, v_n)$ is self-approaching from $v_1$ to $v_n$ if, for every three points $a$, $b$, and $c$ in this order on $P$ from $v_1$ to $v_n$ (possibly $a$, $b$, and $c$ are internal to segments of $P$), we have $|bc| < |ac|$. The geometric path in Figure 1 is self-approaching from $v_1$ to $v_n$. A geometric graph $G$ is self-approaching if, for every ordered pair of vertices $u$ and $v$, $G$ contains a self-approaching path from $u$ to $v$.
A geometric path $P = (v_1, \ldots, v_n)$ is increasing-chord between $v_1$ and $v_n$ if it is self-approaching both from $v_1$ to $v_n$ and from $v_n$ to $v_1$; equivalently, $P$ is increasing-chord if, for every four points $a$, $b$, $c$, and $d$ in this order on $P$ from $v_1$ to $v_n$, we have $|bc| < |ad|$ (from which the name increasing-chord). A geometric graph $G$ is increasing-chord if, for every pair of vertices $u$ and $v$, $G$ contains an increasing-chord path between $u$ and $v$. Observe that, by definition, an increasing-chord graph is also self-approaching.

The study of self-approaching and increasing-chord graphs is motivated by their relationship with greedy graphs (by definition, a self-approaching graph is also greedy), and by the fact that such graphs have a small geometric dilation, namely at most 5.3332 for self-approaching graphs [12] and at most 2.094 for increasing-chord graphs [20].

Alamdari et al. [2] considered three types of problems about self-approaching and increasing-chord graphs.

1. **Complexity of recognizing self-approaching and increasing-chord graphs:** Alamdari et al. [2] showed how to test in $O(n)$ time (in $O(n \log^2 n / \log \log n)$ time) whether an $n$-vertex path in $\mathbb{R}^2$ (resp. in $\mathbb{R}^3$) is self-approaching. They also exhibit an $\Omega(n \log n)$ lower bound for the same problem in $\mathbb{R}^3$. Further, they proved that it is NP-hard to test the existence of a self-approaching path between two given vertices in a geometric graph in $\mathbb{R}^3$ and left open the intriguing problem of determining the complexity of testing whether a geometric graph is self-approaching or increasing-chord in two or more dimensions.

2. **Realizability of a given abstract graph as a self-approaching or increasing-chord graph:** Alamdari et al. [2] characterized the class of trees that can be realized as self-approaching graphs; recently, Nöllenburg et al. [17] proved that planar triangulations can be realized as increasing-chord graphs, that planar 3-trees can be realized as increasing-chord planar graphs, and that
triconnected planar graphs can be realized as increasing-chord graphs in the hyperbolic plane.

3. **Existence of a self-approaching and increasing-chord graph spanning a given point set:** Alamdari et al. [2] showed how to construct, for every point set \( P \) with \( n \) points in \( \mathbb{R}^2 \), an increasing-chord graph that spans \( P \) and uses \( O(n) \) Steiner points (which are extra points that are added to the input point set). They also proved that the Delaunay triangulation of a point set is not always a self-approaching graph.

In this paper we focus our attention on the third type of problem above, i.e., on the problem of constructing self-approaching and increasing-chord graphs spanning given point sets in \( \mathbb{R}^2 \). We prove two main results.

- We show how to construct, for every point set \( P \) with \( n \) points, an increasing-chord planar graph with \( O(n) \) Steiner points spanning \( P \). This answers a question of Alamdari et al. [2] and improves upon their result mentioned above, since our increasing-chord graphs are planar (while the increasing-chord graphs constructed in [2] are not, although they have thickness at most two) and contain increasing-chord paths between every pair of points, including the Steiner points (which is not the case for the graphs in [2]). It is interesting that our result is achieved by studying Gabriel triangulations, which are proximity graphs strongly related to Delaunay triangulations (the Gabriel graph of a point set \( P \) is a subgraph of the Delaunay triangulation of \( P \)). On the way to proving our main result, we show that Gabriel triangulations are increasing-chord graphs, which is not the case, in general, for Delaunay triangulations [2].

- We show that, for every convex point set \( P \) with \( n \) points, there exists an increasing-chord graph that spans \( P \) and that has \( O(n \log n) \) edges (and no Steiner points).

The rest of the paper is organized as follows. In Section 2 we give some preliminaries. In Section 3 we show how to construct increasing-chord planar graphs with few Steiner points spanning given point sets. In Section 4 we show how to construct increasing-chord graphs with few edges spanning given convex point sets. Finally, in Section 5 we conclude and suggest some open problems.

## 2 Definitions and Preliminaries

A **geometric graph** \( (P, S) \) consists of a point set \( P \) in the plane and of a set \( S \) of straight-line segments (called **edges**) between points in \( P \). A geometric graph is **planar** if no two of its edges cross. A planar geometric graph partitions the plane into connected regions called **faces**. The bounded faces are internal and the unbounded face is the **outer face**. A geometric planar graph is a **triangulation** if every internal face is delimited by a triangle and the outer face is delimited by a convex polygon.
Let $p$, $q$, and $r$ be points in the plane. We denote by $\angle pqr$ the angle defined by a clockwise rotation around $q$ bringing $pq$ to coincide with $qr$.

A convex combination of a set of points $P = \{p_1, \ldots, p_k\}$ is a point $\sum \alpha_i p_i$ where $\sum \alpha_i = 1$ and $\alpha_i \geq 0$ for each $1 \leq i \leq k$. The convex hull $H_P$ of $P$ is the set of points that can be expressed as a convex combination of the points in $P$. A convex point set $P$ is such that no point is a convex combination of the others. Let $P$ be a convex point set and $\vec{d}$ be a directed straight line not orthogonal to any line through two points of $P$. Order the points in $P$ as their projections appear on $\vec{d}$; then the minimum point and the maximum point of $P$ with respect to $\vec{d}$ are the first and the last point in such an ordering. We say that $P$ is one-sided with respect to $\vec{d}$ if the minimum and the maximum point of $P$ with respect to $\vec{d}$ are consecutive along the boundary of $H_P$. See Figure 2. A one-sided convex point set is a convex point set that is one-sided with respect to some directed straight line $\vec{d}$.

The proof of our first lemma gives an algorithm to construct an increasing-chord planar graph spanning a one-sided convex point set.

Figure 2: A convex point set that is one-sided with respect to a directed straight line $\vec{d}$.

**Lemma 1** Let $P$ be any one-sided convex point set with $n$ points. There exists an increasing-chord planar graph spanning $P$ with $2n - 3$ edges.

**Proof:** Assume that $P$ is one-sided with respect to the positive $x$-axis $\vec{x}$. Such a condition can be met after a suitable rotation of the Cartesian axes. Let $p_1, p_2, \ldots, p_n$ be the points in $P$, ordered as their projections appear on $\vec{x}$. Assume that $p_2, p_3, \ldots, p_{n-1}$ are above the straight line through $p_1$ and $p_n$, as the case in which they are below such a line is symmetric.

We show by induction on $n$ that an increasing-chord planar graph $G$ spanning $P$ exists, in which all the edges on the boundary of $H_P$ are in $G$.

If $n = 2$ then the graph with a single edge $\overrightarrow{p_1p_2}$ is an increasing-chord planar graph spanning $P$.

Next, assume that $n > 2$ and let $p_j$ be a point with largest $y$-coordinate in $P$ (possibly $j = 1$ or $j = n$). Point set $Q = P \setminus \{p_j\}$ is convex, one-sided with respect to $\vec{x}$, and has $n-1$ points. By induction, there exists an increasing-chord planar graph $G'$ spanning $Q$ in which all the edges on the boundary of $H_Q$ are in $G'$. Let $G$ be the graph obtained by adding vertex $p_j$ and edges $\overrightarrow{p_{j-1}p_j}$ and $\overrightarrow{p_jp_{j+1}}$ to $G'$ (where $p_{n+1} = p_0$ and $p_{-1} = p_n$). We have that $G$ is planar, given
that \(G'\) is planar and that edges \(p_{j-1}p_j\) and \(p_jp_{j+1}\) are on the boundary of \(\mathcal{H}_P\). Further, all the edges on the boundary of \(\mathcal{H}_P\) are in \(G\). Moreover, \(G\) contains an increasing-chord path between every pair of points in \(Q\), by induction; also, \(G\) contains an increasing-chord path between \(p_j\) and every point \(p_i\) in \(Q\), as one of the two paths on the boundary of \(\mathcal{H}_P\) connecting \(p_j\) and \(p_i\) is both \(x-\) and \(y\)-monotone, and hence increasing-chord, as proved in [2]. Finally, \(G\) is a maximal outerplanar graph, hence it has \(2n - 3\) edges. □

The Gabriel graph of a point set \(P\) is the geometric graph that has an edge \(pq\) between two points \(p\) and \(q\) if and only if the closed disk whose diameter is \(pq\) contains no point of \(P \setminus \{p, q\}\) in its interior or on its boundary. A Gabriel triangulation is a triangulation that is the Gabriel graph of its point set \(P\). We say that a point set \(P\) admits a Gabriel triangulation if the Gabriel graph of \(P\) is a triangulation. A triangulation is a Gabriel triangulation if and only if every angle of a triangle delimiting an internal face is acute [10]. See [10, 14, 15] for more properties about Gabriel graphs.

In Section 3 we will prove that every Gabriel triangulation is increasing-chord. A weaker version of the converse is also true, as proved in the following.

Lemma 2 Let \(P\) be a set of points and let \(G(P, S)\) be an increasing-chord graph spanning \(P\). Then all the edges of the Gabriel graph of \(P\) are in \(S\).

Proof: Suppose, for a contradiction, that there exists an increasing-chord graph \(G(P, S)\) and an edge \(\overline{pq}\) of the Gabriel graph of \(P\) such that \(\overline{pq} \notin S\). Then consider any increasing-chord path \(P = (u = w_1, w_2, \ldots, w_k = v)\) in \(G\). Since \(\overline{pq} \notin S\), it follows that \(k > 2\). Assume w.l.o.g. that \(w_1, w_2,\) and \(w_k\) appear in this clockwise order on the boundary of triangle \((w_1, w_2, w_k)\). Since the closed disk with diameter \(\overline{pq}\) does not contain any point in its interior or on its boundary, it follows that \(\angle w_kw_2w_1 < 90^\circ\). If \(\angle w_kw_1w_k \geq 90^\circ\), then \(|w_kw_k| < |w_kw_k|\), a contradiction to the assumption that \(P\) is increasing-chord. If \(\angle w_kw_1w_k < 90^\circ\), then the altitude of triangle \((w_1, w_2, w_k)\) incident to \(w_k\) hits \(\overline{w_1w_2}\) in a point \(h\). Hence, \(|hw_k| < |w_kw_k|\), a contradiction to the assumption that \(P\) is increasing-chord which proves the lemma. □

3 Increasing-Chord Planar Graphs with Few Steiner Points Spanning Point Sets

In this section we show that, for any point set \(P\), one can construct an increasing-chord planar graph \(G(P', S)\) such that \(P \subseteq P'\) and \(|P'| \in O(|P|)\).

Our proof consists of two main ingredients. The first one is that Gabriel triangulations are increasing-chord graphs. The second one is a result of Bern et al. [4] stating that, for any point set \(P\), there exists a point set \(P'\) such that \(P \subseteq P'\), \(|P'| \in O(|P|)\), and \(P'\) admits a Gabriel triangulation. Combining these two facts proves our main result.

The proof that Gabriel triangulations are increasing-chord graphs consists of two parts. In the first one, we prove that geometric graphs having a \(\theta\)-path...
between every pair of points are increasing-chord. In the second one, we prove that every Gabriel triangulation contains a $\theta$-path between every pair of points.

We introduce some definitions. The slope of a straight-line segment $\overline{uv}$ is the angle spanned by a clockwise rotation around $u$ that brings $\overline{uv}$ to coincide with the positive $x$-axis. Thus, if $\theta$ is the slope of $\overline{uv}$, then $\theta + k \cdot 360^\circ$ is also the slope of $\overline{uv}$, $\forall k \in \mathbb{Z}$. A straight-line segment $\overline{uv}$ is a $\theta$-edge if its slope is in the interval $[\theta - 45^\circ; \theta + 45^\circ]$. Also, a geometric path $P = (p_1, \ldots, p_k)$ is a $\theta$-path from $p_1$ to $p_k$ if $p_ip_{i+1}$ is a $\theta$-edge, for every $1 \leq i \leq k - 1$. Consider a point $a$ on a $\theta$-path $P$ from $p_1$ to $p_k$. Then the subpath $P_a$ of $P$ from $a$ to $p_k$ is also a $\theta$-path. Moreover, denote by $W_\theta(a)$ the closed wedge with an angle of $90^\circ$ incident to $a$ and whose delimiting lines have slope $\theta - 45^\circ$ and $\theta + 45^\circ$; then we have that $P_a$ is contained in $W_\theta(a)$, which easily follows from the fact that $p_ip_{i+1}$ is a $\theta$-edge, for every $1 \leq i \leq k - 1$ (see Figure 3). We have the following:

Lemma 3 Let $P$ be a $\theta$-path from $p_1$ to $p_k$, for some angle $\theta$. Then $P$ is increasing-chord.

Proof: Lemma 3 in [12] states the following (see also [1]): A curve $C$ with endpoints $p$ and $q$ is self-approaching from $p$ to $q$ if and only if, for every point $a$ on $C$, there exists a closed wedge with an angle of $90^\circ$ incident to $a$ and containing the part of $C$ between $a$ and $q$. As observed before the lemma, for every point $a$ on $P$, the closed wedge $W_\theta(a)$ with an angle of $90^\circ$ incident to $a$ and whose delimiting lines have slope $\theta - 45^\circ$ and $\theta + 45^\circ$ contains the subpath $P_a$ of $P$ from $a$ to $p_k$. Hence, by Lemma 3 in [12], $P$ is self-approaching from $p_1$ to $p_k$.

An analogous proof shows that $P$ is self-approaching from $p_k$ to $p_1$, given that $P$ is a $(\theta + 180^\circ)$-path from $p_k$ to $p_1$. \hfill $\Box$

We now prove that Gabriel triangulations contain $\theta$-paths.

Lemma 4 Let $G$ be a Gabriel triangulation on a point set $P$. For every two points $s, t \in P$, there exists an angle $\theta$ such that $G$ contains a $\theta$-path from $s$ to $t$.

Proof: Consider any two points $s, t \in P$. Rotate $G$ clockwise of an angle $\phi$ so that $y(s) = y(t)$ and $x(s) < x(t)$. Observe that, if there exists a $\theta$-path from $s$ to $t$ after the rotation, then there exists a $(\theta + \phi)$-path from $s$ to $t$ before the rotation.
A $\theta$-path $(p_1, \ldots, p_k)$ in $G$ is maximal if there is no $z \in P$ such that $p_{k+1}z$ is a $\theta$-edge. For every maximal $\theta$-path $P = (p_1, \ldots, p_k)$ in $G$, $p_k$ lies on the boundary of $H_P$. To prove this, assume the converse, for a contradiction. Since $G$ is a Gabriel triangulation, the angle between any two consecutive edges incident to an internal vertex of $G$ is smaller than 90°, thus there is a $\theta$-edge incident to $p_k$. This contradicts the maximality of $P$. A maximal $\theta$-path $(s = p_1, \ldots, p_k)$ is high if either (a) $y(p_k) > y(t)$ and $x(p_k) < x(t)$, or (b) $p_ip_{i+1}$ intersects the vertical line through $t$ at a point above $t$, for some $1 \leq i \leq k-1$. Symmetrically, a maximal $\theta$-path $(s = p_1, \ldots, p_k)$ is low if either (a) $y(p_k) < y(t)$ and $x(p_k) < x(t)$, or (b) $p_ip_{i+1}$ intersects the vertical line through $t$ at a point below $t$, for some $1 \leq i \leq k-1$. High and low $(\theta + 180\degree)$-paths starting at $t$ can be defined analogously. The proof of the lemma consists of two main claims.

Claim A. If a maximal $\theta$-path $P_s$ starting at $s$ and a maximal $(\theta + 180\degree)$-path $P_t$ starting at $t$ exist such that $P_s$ and $P_t$ are both high or both low, for some $-45\degree \leq \theta \leq 45\degree$, then there exists a $\theta$-path in $G$ from $s$ to $t$.

Claim B. For some $-45\degree \leq \theta \leq 45\degree$, there exist a maximal $\theta$-path $P_s$ starting at $s$ and a maximal $(\theta + 180\degree)$-path $P_t$ starting at $t$ that are both high or both low.

Observe that Claims A and B imply the lemma.

We now prove Claim A. Suppose that $G$ contains a maximal high $\theta$-path $P_s$ starting at $s$ and a maximal high $(\theta + 180\degree)$-path $P_t$ starting at $t$, for some $-45\degree \leq \theta \leq 45\degree$. If $P_s$ and $P_t$ share a vertex $v \in P$, then the subpath of $P_s$ from $s$ to $v$ and the subpath of $P_t$ from $v$ to $t$ form a $\theta$-path in $G$ from $s$ to $t$. Thus, it suffices to show that $P_s$ and $P_t$ share a vertex. For a contradiction assume the converse. Let $p_s$ and $p_t$ be the end-vertices of $P_s$ and $P_t$ different from $s$ and $t$, respectively. Recall that $p_s$ and $p_t$ lie on the boundary of $H_P$. Denote by $\hat{l}_s$ and $\hat{l}_t$ the vertical half-lines starting at $s$ and $t$, respectively, and directed toward increasing $y$-coordinates; also, denote by $q_s$ and $q_t$ the intersection points of $\hat{l}_s$ and $\hat{l}_t$ with the boundary of $H_P$, respectively. Finally, denote by $Q$ the curve obtained by following the boundary of $H_P$ clockwise from $q_s$ to $q_t$.

Assume that $x(p_s) \geq x(t)$, as in Figure 4(a). Path $P_s$ starts at $s$ and passes through a point $r_s$ on $\hat{l}_s$ (possibly $r_s = q_s$), given that $x(p_s) \geq x(t)$. Path $P_t$ starts at $t$ and either passes through a point $r_t$ on $\hat{l}_t$, or ends at a point $q_t$ on $Q$, depending on whether $x(p_t) \leq x(s)$ or $x(p_t) > x(s)$, respectively. Since $P_s$ is $x$-monotone and lies in $H_P$, it follows that $r_t$ and $p_t$ are above or on $P_s$; also, $t$ is below $P_s$ given that $P_s$ is a high path. It follows $P_s$ and $P_t$ intersect, hence they share a vertex given that $G$ is planar.

Analogously, if $x(p_t) \leq x(s)$, then $P_s$ and $P_t$ share a vertex.

If $x(p_t) = x(p_s)$, then $p_s$ and $p_t$ are the same point, hence $P_s \cup P_t$ is a $\theta$-path from $s$ to $t$.

Next, if $x(s) < x(p_t) < x(p_s) < x(t)$, as in Figure 4(b), then the end-points of $P_s$ and $P_t$ alternate along the boundary of the region $R$ that is the intersection of $H_P$, of the half-plane to the right of $\hat{l}_s$, and of the half-plane to the left of $\hat{l}_t$. Since $P_s$ and $P_t$ are $x$-monotone, they lie in $R$, thus they intersect, and hence they share a vertex.
Finally, assume that \( x(s) < x(p_s) < x(p_t) < x(t) \), as in Figure 4(c). Let \( a_1, \ldots, a_h \) be the clockwise order of the points along \( Q \), starting at \( p_s = a_1 \) and ending at \( a_h = p_t \). By the assumption \( x(p_s) < x(p_t) \) we have \( h \geq 2 \). We prove that \( \overline{a_1a_2} \) is a \( \theta \)-edge. Suppose, for a contradiction, that \( \overline{a_1a_2} \) is not a \( \theta \)-edge. Since the slope of \( \overline{a_1a_2} \) is larger than \(-90^\circ\) and smaller than \(90^\circ\), it is either larger than \( \theta + 45^\circ \) and smaller than \(90^\circ\), or it is larger than \(-90^\circ\) and smaller than \( \theta - 45^\circ \). First, assume that the slope of \( \overline{a_1a_2} \) is larger than \( \theta + 45^\circ \) and smaller than \(90^\circ\), as in Figure 5(a). Since the slope of \( \overline{a_1a_2} \) is between \( \theta - 45^\circ \) and \( \theta + 45^\circ \), it follows that \( a_1 \) is below the line composed of \( \overline{a_2t} \) and \( \overline{a_2l} \), which contradicts the assumption that \( a_1 \) is on \( Q \). Second, if the slope of \( \overline{a_1a_2} \) is larger than \(-90^\circ\) and smaller than \( \theta - 45^\circ \), then we distinguish two further cases. In the first case, represented in Figure 5(b), the slope of \( \overline{a_1l} \) is larger than \( \theta - 45^\circ \), hence \( a_2 \) is below the line composed of \( \overline{a_1l} \) and \( \overline{a_1l} \), which contradicts the assumption that \( a_2 \) is on \( Q \). In the second case, represented in Figure 5(c), the slope of \( \overline{a_1l} \) is in the interval \([-90^\circ; \theta - 45^\circ]\). It follows that the slope of \( \overline{a_1l} \) is in the interval \([90^\circ; \theta + 135^\circ]\); since the slope of \( \overline{a_1h} \) is smaller than the one of \( \overline{a_1l} \), we have that \( P_l \) is not a \((\theta + 180^\circ)\)-path. This contradiction proves that \( \overline{a_1a_2} \) is a \( \theta \)-edge. However, this contradicts the assumption that \( P_s \) is a maximal \( \theta \)-path, and hence concludes the proof of Claim A.

We now prove Claim B. First, we prove that, for every \( \theta \) in the interval \([-45^\circ; 45^\circ]\), there exists a maximal \( \theta \)-path starting at \( s \) that is low or high. Indeed, it suffices to prove that there exists a \( \theta \)-edge incident to \( s \), as such an edge is also a \( \theta \)-path starting at \( s \), and the existence of a \( \theta \)-path starting at \( s \)
implies the existence of a maximal \( \theta \)-path starting at \( s \). Consider a straight-line segment \( e_\theta \) that is the intersection of a directed half-line incident to \( s \) with slope \( \theta \) and of a disk of arbitrarily small radius centered at \( s \). If \( e_\theta \) is internal to \( \mathcal{H}_P \), then consider the two edges \( e_1 \) and \( e_2 \) of \( G \) that are encountered when rotating \( e_\theta \) around \( s \) counter-clockwise and clockwise, respectively. Then \( e_1 \) or \( e_2 \) is a \( \theta \)-edge, as the angle spanned by a clockwise rotation bringing \( e_1 \) to coincide with \( e_2 \) is smaller than 90°, given that \( G \) is a Gabriel triangulation, and \( e_\theta \) is encountered during such a rotation. If \( e_\theta \) is outside \( \mathcal{H}_P \), which might happen if \( s \) on the boundary of \( \mathcal{H}_P \), then assume that the slope of \( e_\theta \) is in the interval \( [0°; 45°] \) (the case in which the slope of \( e_\theta \) is in the interval \( [-45°; 0°] \) is analogous). Then the angle spanned by a clockwise rotation bringing \( e_\theta \) to coincide with \( \mathcal{M} \) is at most 45°. Since \( \mathcal{M} \) is in interior or on the boundary of \( \mathcal{H}_P \), an edge \( e_1 \) of \( G \) is encountered during such a rotation, hence \( e_1 \) is a \( \theta \)-edge. An analogous proof shows that, for every \( \theta \) in the interval \( [-45°; 45°] \), there exists a maximal \( (\theta + 180°) \)-path starting at \( t \) that is low or high.

Second, we prove that, for some \( \theta \in [-45°; 45°] \), there exist a maximal low \( \theta \)-path and a maximal high \( \theta \)-path both starting at \( s \). All the maximal \(( -45°) \)-paths (all the maximal \(( 45°) \)-paths) starting at \( s \) are low (resp. high), given that every edge on these paths has slope in the interval \([ -90°; 0°] \) (resp. \([ 0°; 90°] \)). Thus, let \( \theta \) be the smallest constant in the interval \([ -45°; 45°] \) such that a maximal high \( \theta \)-path exists. We prove that there also exists a maximal low \( \theta \)-path starting at \( s \). Consider an arbitrarily small \( \epsilon > 0 \). By assumption, there exists no high \(( \theta - \epsilon) \)-path. Hence, from the previous argument there exists a low \(( \theta - \epsilon) \)-path \( P \). If \( \epsilon \) is sufficiently small, then no edge of \( P \) has slope in the interval \([ \theta - 45° - \epsilon; \theta - 45°] \). Thus every edge of \( P \) has slope in the interval \([ \theta - 45°; \theta + 45° - \epsilon] \), hence \( P \) is a maximal low \( \theta \)-path starting at \( s \).

Since there exist a maximal high \( \theta \)-path starting at \( s \), a maximal low \( \theta \)-path starting at \( s \), and a maximal \(( \theta + 180°) \)-path starting at \( t \) that is low or high, it follows that there exist a maximal \( \theta \)-path \( P_s \) starting at \( s \) and a maximal \( (\theta + 180°) \)-path \( P_t \) starting at \( t \) that are both high or both low. This proves Claim B and hence the lemma. \( \square \)

Lemma 3 and Lemma 4 immediately imply the following.

**Corollary 1** Any Gabriel triangulation is increasing-chord.

We are now ready to state the main result of this section.

**Theorem 1** Let \( P \) be a point set with \( n \) points. One can construct in \( O(n \log n) \) time an increasing-chord planar graph \( G(P', S) \) such that \( P \subseteq P' \) and \(|P'| \in O(n) \).

**Proof:** Bern, Eppstein, and Gilbert \cite{Bern1999} proved that, for any point set \( P \), there exists a point set \( P' \) with \( P \subseteq P' \) and \(|P'| \in O(n) \) such that \( P' \) admits a Gabriel triangulation \( G \). Both \( P' \) and \( G \) can be computed in \( O(n \log n) \) time \cite{Bern1999}. By Corollary 4 \( G \) is increasing-chord, which concludes the proof. \( \square \)
We remark that $o(|P|)$ Steiner points are not always enough to augment a point set $P$ to a point set that admits a Gabriel triangulation. Namely, consider any point set $B$ with $O(1)$ points that admits no Gabriel triangulation. Construct a point set $P$ out of $|P|/|B|$ copies of $B$ placed “far apart” from each other, so that any triangle with two points in different copies of $B$ is obtuse. Then a Steiner point has to be added inside the convex hull of each copy of $B$ to obtain a point set that admits a Gabriel triangulation.

4 Increasing-Chord Graphs with Few Edges
Spanning Convex Point Sets

In this section we prove the following theorem:

**Theorem 2** For every convex point set $P$ with $n$ points, there exists an increasing-chord geometric graph $G(P, S)$ such that $|S| \in O(n \log n)$.

The main idea behind the proof of Theorem 2 is that any convex point set $P$ can be decomposed into some one-sided convex point sets $P_1, \ldots, P_k$ (which by Lemma 7 admit increasing-chord spanning graphs with linearly many edges) in such a way that every two points of $P$ are part of some $P_i$ and that $\sum |P_i|$ is small. In order to perform such a decomposition, we introduce the concept of balanced $(\vec{d}_1, \vec{d}_2)$-partition.

Let $P$ be a convex point set and let $\vec{d}$ be a directed straight line not orthogonal to any line through two points of $P$. See Figure 6. Let $p_{\min}(\vec{d})$ and $p_{\max}(\vec{d})$ be the minimum and maximum point of $P$ with respect to $\vec{d}$, respectively. Let $P_1(\vec{d})$ be composed of those points in $P$ that are encountered when walking clockwise along the boundary of $H_P$ from $p_{\min}(\vec{d})$ to $p_{\max}(\vec{d})$, where $p_{\min}(\vec{d}) \in P_1(\vec{d})$ and $p_{\max}(\vec{d}) \notin P_1(\vec{d})$. Analogously, let $P_2(\vec{d})$ be composed of those points in $P$ that are encountered when walking clockwise along the boundary of $H_P$ from $p_{\max}(\vec{d})$ to $p_{\min}(\vec{d})$, where $p_{\max}(\vec{d}) \in P_2(\vec{d})$ and $p_{\min}(\vec{d}) \notin P_2(\vec{d})$.

Let $\vec{d}_1$ and $\vec{d}_2$ be two directed straight lines not orthogonal to any line through two points of $P$, where the clockwise rotation that brings $\vec{d}_1$ to coincide with $\vec{d}_2$ is at most $180^\circ$. The $(\vec{d}_1, \vec{d}_2)$-partition of $P$ partitions $P$ into subsets $P_a = P_1(\vec{d}_1) \cap P_2(\vec{d}_2)$, $P_b = P_1(\vec{d}_1) \cap P_2(\vec{d}_2)$, $P_c = P_2(\vec{d}_1) \cap P_1(\vec{d}_2)$, and $P_d = P_2(\vec{d}_1) \cap P_2(\vec{d}_2)$. Note that every point in $P$ is contained in one of $P_a$, $P_b$, $P_c$, and $P_d$. A $(\vec{d}_1, \vec{d}_2)$-partition of $P$ is balanced if $|P_a| + |P_d| \leq \frac{|P|}{2} + 1$ and $|P_b| + |P_c| \leq \frac{|P|}{2} + 1$. We now argue that, for every point set $P$, a balanced $(\vec{d}_1, \vec{d}_2)$-partition of $P$ always exists, even if $\vec{d}_1$ is arbitrarily prescribed.

**Lemma 5** Let $P$ be a convex point set and let $\vec{d}_1$ be a directed straight line not orthogonal to any line through two points of $P$. Then there exists a directed straight line $\vec{d}_2$ that is not orthogonal to any line through two points of $P$ such that the $(\vec{d}_1, \vec{d}_2)$-partition of $P$ is balanced.
Figure 6: Subsets $P_1(\vec{d})$ and $P_2(\vec{d})$ of a point set $P$ determined by a directed straight line $\vec{d}$.

**Proof:** Denote by $q_1 = p_{\text{min}}(\vec{d}_1), q_2, \ldots, q_l, q_{l+1} = p_{\text{max}}(\vec{d}_1)$ the points of $P$ encountered when walking clockwise on the boundary of $\mathcal{H}_P$ from $p_{\text{min}}(\vec{d}_1)$ to $p_{\text{max}}(\vec{d}_1)$. Also, denote by $r_1 = p_{\text{max}}(\vec{d}_1), r_2, \ldots, r_m, r_{m+1} = p_{\text{min}}(\vec{d}_1)$ the points of $P$ encountered when walking clockwise on the boundary of $\mathcal{H}_P$ from $p_{\text{max}}(\vec{d}_1)$ to $p_{\text{min}}(\vec{d}_1)$.

Initialize $\vec{d}_2$ to be a directed straight line coincident with $\vec{d}_1$. When $\vec{d}_2 = \vec{d}_1$, we have $P_a = \{q_1, q_2, \ldots, q_l\}$, $P_d = \{r_1, r_2, \ldots, r_m\}$, $P_b = \emptyset$, and $P_c = \emptyset$. We now rotate $\vec{d}_2$ clockwise until it is opposite to $\vec{d}_1$ (that is, parallel and pointing in the opposite direction). As we rotate $\vec{d}_2$, sets $P_1(\vec{d}_2)$ and $P_2(\vec{d}_2)$ change, hence sets $P_a$, $P_b$, $P_c$, and $P_d$ change as well. When $\vec{d}_2$ is opposite to $\vec{d}_1$, we have $P_a = \emptyset$, $P_d = \emptyset$, $P_b = \{q_1, q_2, \ldots, q_l\}$, and $P_c = \{r_1, r_2, \ldots, r_m\}$. We will argue that there is a moment during such a rotation of $\vec{d}_2$ in which the corresponding $(\vec{d}_1, \vec{d}_2)$-partition of $P$ is balanced. Assume that at any time instant during the rotation of $\vec{d}_2$ the following hold (see Figs. 7(a)--(b)):

- $P_b = \{q_1, q_2, \ldots, q_{j+1}\}$ (possibly $P_b$ is empty);
- $P_a = \{q_{j+1}, q_{j+2}, \ldots, q_l\}$ (possibly $P_a$ is empty);
- $P_c = \{r_1, r_2, \ldots, r_k\}$ (possibly $P_c$ is empty);
- $P_d = \{r_{k+1}, r_{k+2}, \ldots, r_m\}$ (possibly $P_d$ is empty); and
- $q_{j+1}$ and $r_{k+1}$ are the minimum and maximum point of $P$ with respect to $\vec{d}_2$, respectively.

The assumption is indeed true when $\vec{d}_2$ starts moving, with $j = 0$ and $k = 0$. As we keep on rotating $\vec{d}_2$ clockwise, at a certain moment $\vec{d}_2$ becomes orthogonal to $\vec{q}_{j+1}q_{j+2}$ or to $\vec{r}_{k+1}r_{k+2}$ (or to both if $\vec{q}_{j+1}q_{j+2}$ and $\vec{r}_{k+1}r_{k+2}$ are
Figure 7: (a) Sets $P_a$, $P_b$, $P_c$, and $P_d$ at a certain time instant during the rotation of $\vec{d}_2$. (b) The slope of $\vec{d}_2$ with respect to the slopes of the lines orthogonal to $q_jq_{j+1}$, to $q_{j+1}q_{j+2}$, to $r_kr_{k+1}$, and to $r_{k+1}r_{k+2}$.

parallel). Thus, as we keep on rotating $\vec{d}_2$ clockwise, sets $P_a$, $P_b$, $P_c$, and $P_d$ change. Namely:

If $\vec{d}_2$ becomes orthogonal first to $q_{j+1}q_{j+2}$ and then to $r_{k+1}r_{k+2}$, then as $\vec{d}_2$ rotates clockwise after the position in which it is orthogonal to $q_{j+1}q_{j+2}$, we have

- $P_b = \{q_1, q_2, \ldots, q_j, q_{j+1}\}$;
- $P_a = \{q_{j+2}, q_{j+3}, \ldots, q_l\}$ (possibly $P_a$ is empty);
- $P_c = \{r_1, r_2, \ldots, r_k\}$ (possibly $P_c$ is empty);
- $P_d = \{r_{k+1}, r_{k+2}, \ldots, r_m\}$ (possibly $P_d$ is empty); and
- $q_{j+2}$ and $r_{k+1}$ are the minimum and maximum point of $P$ with respect to $\vec{d}_2$, respectively.

If $\vec{d}_2$ becomes orthogonal first to $r_{k+1}r_{k+2}$ and then to $q_{j+1}q_{j+2}$, then as $\vec{d}_2$ rotates clockwise after the position in which it is orthogonal to $r_{k+1}r_{k+2}$, we have that $P_a$ and $P_b$ stay unchanged, that $r_{k+1}$ passes from $P_d$ to $P_c$, and that $q_{j+1}$ and $r_{k+2}$ are the minimum and maximum point of $P$ with respect to $\vec{d}_2$, respectively.

If $\vec{d}_2$ becomes orthogonal to $q_{j+1}q_{j+2}$ and $r_{k+1}r_{k+2}$ simultaneously, then as $\vec{d}_2$ rotates clockwise after the position in which it is orthogonal to $q_{j+1}q_{j+2}$, we have that $q_{j+1}$ passes from $P_a$ to $P_b$, that $r_{k+1}$ passes from $P_d$ to $P_c$, and that $q_{j+2}$ and $r_{k+2}$ are the minimum and maximum point of $P$ with respect to $\vec{d}_2$, respectively.

Observe that:

1. whenever sets $P_a$, $P_b$, $P_c$, and $P_d$ change, we have that $|P_a| + |P_d|$ and $|P_b| + |P_c|$ change at most by two;

2. whenever sets $P_a$, $P_b$, $P_c$, and $P_d$ change, we have that $|P_a| + |P_d|$ and $|P_b| + |P_c|$ change at most by two;
2. when \( \vec{d}_2 \) starts rotating we have that \(|P_a| + |P_d| = |P|\), and when \( \vec{d}_2 \) stops rotating we have that \(|P_a| + |P_d| = 0\);

3. when \( \vec{d}_2 \) starts rotating we have that \(|P_b| + |P_c| = 0\), and when \( \vec{d}_2 \) stops rotating we have that \(|P_b| + |P_c| = |P|\); and

4. \(|P_a| + |P_b| + |P_c| + |P_d| = |P|\) holds at any time instant.

By continuity, there is a time instant in which \(|P_a| + |P_d| = \lfloor |P|/2 \rfloor\) and \(|P_b| + |P_c| = \lceil |P|/2 \rceil\), or in which \(|P_a| + |P_d| = \lceil |P|/2 \rceil + 1\) and \(|P_b| + |P_c| = \lfloor |P|/2 \rfloor - 1\). This completes the proof of the lemma. \(\square\)

We now show how to use Lemma 5 in order to prove Theorem 2.

Let \( P \) be any point set. Assume that no two points of \( P \) have the same \( y \)-coordinate. Such a condition is easily met after rotating the Cartesian axes. Denote by \( \vec{l} \) a vertical straight line directed toward increasing \( y \)-coordinates. Each of \( P_1(\vec{l}) \) and \( P_2(\vec{l}) \) is convex and one-sided with respect to \( \vec{l} \). By Lemma 1 there exist increasing-chord graphs \( G_1 = (P_1(\vec{l}), S_1) \) and \( G_2 = (P_2(\vec{l}), S_2) \) with \( |S_1| < 2|P_1(\vec{l})| \) and \( |S_2| < 2|P_2(\vec{l})| \). Then graph \( G(P, S_1 \cup S_2) \) has less than \( 2(|P_1(\vec{l})| + |P_2(\vec{l})|) = 2|P| \) edges and contains an increasing-chord path between every pair of vertices in \( P_1(\vec{l}) \) and between every pair of vertices in \( P_2(\vec{l}) \). However, \( G \) does not have increasing-chord paths between any pair \((a, b)\) of vertices such that \( a \in P_1(\vec{l}) \) and \( b \in P_2(\vec{l}) \).

We now present and prove the following claim.

**Claim 1** Consider a convex point set \( Q \) and a directed straight line \( \vec{d}_1 \) not orthogonal to any line through two points of \( Q \). Then there exists a geometric graph \( H(Q, R) \) that contains an increasing-chord path between every point in \( Q_1(\vec{d}_1) \) and every point in \( Q_2(\vec{d}_1) \), such that \(|R| \in O(|Q| \log |Q|)\).

The application of the claim with \( Q = P \) and \( \vec{d}_1 = \vec{l} \) provides a graph \( H(P, R) \) that contains an increasing-chord path between every pair \((a, b)\) of vertices such that \( a \in P_1(\vec{l}) \) and \( b \in P_2(\vec{l}) \). Thus, the union of \( G \) and \( H \) is an increasing-chord graph with \( O(|P| \log |P|) \) edges spanning \( P \). Therefore, the above claim implies Theorem 2.

We give an inductive algorithm to construct \( H \). Let \( f(Q, \vec{d}_1) \) be the number of edges that \( H \) has as a result of the application of our algorithm on a point set \( Q \) and a directed straight-line \( \vec{d}_1 \). Also, let \( f(n) = \max\{f(Q, \vec{d}_1)\} \), where the maximum is among all point sets \( Q \) with \( n = |Q| \) points and among all the directed straight-lines \( \vec{d}_1 \) that are not orthogonal to any line through two points of \( Q \).

Let \( Q \) be any convex point set with \( n \) points and let \( \vec{d}_1 \) be any directed straight line not orthogonal to any line through two points of \( Q \). By Lemma 5 there exists a directed straight line not orthogonal to any line through two points of \( Q \) such that the \((\vec{d}_1, \vec{d}_2)\)-partition of \( Q \) is balanced.

Let \( Q_a = Q_1(\vec{d}_1) \cap Q_1(\vec{d}_2) \), let \( Q_b = Q_1(\vec{d}_1) \cap Q_2(\vec{d}_2) \), let \( Q_c = Q_2(\vec{d}_1) \cap Q_1(\vec{d}_2) \), and let \( Q_d = Q_2(\vec{d}_1) \cap Q_2(\vec{d}_2) \).
Point set $Q_a \cup Q_c$ is convex and one-sided with respect to $\vec{d}_2$. By Lemma 1 there exists an increasing-chord graph $H_1(Q_a \cup Q_c, R_1)$ with $|R_1| < 2(|Q_a| + |Q_c|)$ edges. Analogously, by Lemma 1 there exists an increasing-chord graph $H_2(Q_b \cup Q_d, R_2)$ with $|R_2| < 2(|Q_b| + |Q_d|)$ edges.

Hence, there exists a graph $H_3(Q, R_1 \cup R_2)$ with $|R_1 \cup R_2| < 2(|Q_a| + |Q_c| + |Q_b| + |Q_d|) = 2|Q| = 2n$ edges containing an increasing-chord path between every point in $Q_a$ and every point in $Q_c$, and between every point in $Q_b$ and every point in $Q_d$. However, $G$ does not have an increasing-chord path between any point in $Q_a$ and any point in $Q_d$, and does not have an increasing-chord path between any point in $Q_b$ and any point in $Q_c$.

By Lemma 5, we have $|Q_a| + |Q_d| \leq \frac{n}{2} + 1$ and $|Q_b| + |Q_c| \leq \frac{n}{2} + 1$. By definition, we have $f(Q_a \cup Q_d, \vec{d}_1) \leq f(|Q_a| + |Q_d|) \leq f(\frac{n}{2} + 1)$. Analogously, we have $f(Q_b \cup Q_c, \vec{d}_1) \leq f(|Q_b| + |Q_c|) \leq f(\frac{n}{2} + 1)$. Hence, $f(n) \leq 2n + 2f(\frac{n}{2} + 1) \in O(n \log n)$. This proves the claim and hence Theorem 2.

5 Conclusions

We considered the problem of constructing increasing-chord graphs spanning point sets. We proved that, for every point set $P$, there exists a planar increasing-chord graph $G(P', S)$ with $P \subseteq P'$ and $|P'| \in O(|P|)$. We also proved that, for every convex point set $P$, there exists an increasing-chord graph $G(P, S)$ with $|S| \in O(|P| \log |P|)$.

Despite our research efforts, the main question on this topic remains open:

Open Problem 1 Is it true that, for every (convex) point set $P$, there exists an increasing-chord planar graph $G(P, S)$?

One of the directions we took in order to tackle this problem is to assume that the points in $P$ lie on a constant number of straight lines. While a simple modification of the proof of Lemma 1 allows us to prove that an increasing-chord planar graph always exists spanning a set of points lying on two straight lines, it is surprising and disheartening that we could not prove a similar result for sets of points lying on three straight lines. The main difficulty seems to lie in the construction of planar increasing-chord graphs spanning sets of points lying on the boundary of an acute triangle.

Open Problem 2 Is it true that, for every set $P$ of points lying on the boundary of an acute triangle, there exists an increasing-chord planar graph $G(P, S)$?

Gabriel graphs naturally generalize to higher dimensions, where empty balls replace empty disks. In Section 3 we showed that, for points in $\mathbb{R}^2$, every Gabriel triangulation is increasing-chord. Can this result be generalized to higher dimensions?

Open Problem 3 Is it true that, for every point set $P$ in $\mathbb{R}^d$, any Gabriel triangulation of $P$ is increasing-chord?
Finally, it would be interesting to understand if increasing-chord graphs with few edges can be constructed for any (possibly non-convex) point set:

**Open Problem 4** Is it true that, for every point set $P$, there exists an increasing-chord graph $G(P,S)$ with $|S| \in o(|P|^2)$?
References


