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Pole Dancing: 3D Morphs for Tree Drawings

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Abstract

We study the question whether a crossing-free 3D morph between two straight-line drawings of an n-vertex tree T can be constructed consisting of a small number of linear morphing steps. We look both at the case in which the two given drawings are two-dimensional and at the one in which they are three-dimensional. In the former setting we prove that a crossing-free 3D morph always exists with $O(rpw(T)) \subseteq O(\log n)$ steps, where $rpw(T)$ is the rooted pathwidth or Strahler number of T, while for the latter setting $\Theta(n)$ steps are always sufficient and sometimes necessary.

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We here refer to pole dancing as a fitness and competitive sport. The authors hope that many of our readers try this activity themselves, and will in return introduce many pole dancers to Graph Drawing, thereby alleviating the gender imbalance in both communities. The authors do not condone any pole activity used for sexual exploitation or abuse of women or men. A preliminary version of this paper appeared at the 26th International Symposium on Graph Drawing and Network Visualization.

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1 Introduction

A *morph* between two drawings of the same graph is a continuous transformation from one drawing to the other. Thus, any time instant of the morph defines a different drawing of the graph. Ideally, the morph should preserve the properties of the initial and final drawings throughout. As the most notable example, a morph between two planar graph drawings should guarantee that every intermediate drawing is also planar; in this case the morph is called *planar*.

Planar morphs have been studied for decades and nowadays find applications in animation, modeling, and computer graphics; see, e.g., [16, 17]. A planar morph between any two topologically-equivalent[†] planar straight-line[‡] drawings of the same planar graph always exists; this was proved for maximal planar graphs by Cairns [10] back in 1944, and then for all planar graphs by Thomassen [25] almost forty years later. Note that a planar morph between two planar graph drawings that are not topologically equivalent does not exist.

Several research efforts have been spent lately to study the question whether a planar morph between any two topologically-equivalent planar straight-line drawings of the same planar graph always exists such that the vertex trajectories have low complexity. This is usually formalized as follows. Let Γ and Γ' be two topologically-equivalent planar straight-line drawings of the same planar graph G. Then a morph $\mathcal{M}$ is a sequence $\langle \Gamma_1, \Gamma_2, \ldots, \Gamma_k \rangle$ of planar straight-line drawings of G such that $\Gamma_1 = \Gamma$, $\Gamma_k = \Gamma'$, and $\langle \Gamma_i, \Gamma_{i+1} \rangle$ is a planar linear morph, for each $i = 1, \ldots, k-1$. A *linear morph* $\langle \Gamma_i, \Gamma_{i+1} \rangle$ is such that each vertex moves along a straight-line segment at constant speed; that is, assuming that the morph happens between time $t = 0$ and time $t = 1$, the position of a vertex v at any time $t \in [0, 1]$ is $(1-t)\Gamma_i(v) + t\Gamma_{i+1}(v)$. The complexity of a morph $\mathcal{M}$ is then measured by the number of its *morphing steps*, i.e., by the number of linear morphs it consists of. In the following, a morphing step is sometimes simply called a *step*.

A recent sequence of papers [3, 4, 5, 6] culminated in a proof [2] that a planar morph between any two topologically-equivalent planar straight-line drawings of the same n-vertex planar graph can always be constructed consisting of $\Theta(n)$ steps. This bound is asymptotically optimal in the worst case, even for paths.

The question we study in this paper is whether morphs with sub-linear complexity can be constructed if a third dimension is allowed to be used. That is: Let Γ and Γ' be two topologically-equivalent planar straight-line drawings of the same n-vertex planar graph G (throughout the paper, whenever we talk about *planar* drawings, we always mean crossing-free 2D drawings in the xy-plane). Does a morph $\mathcal{M} = \langle \Gamma = \Gamma_1, \Gamma_2, \ldots, \Gamma_k = \Gamma' \rangle$ exist such that: (i) for

[†]Two planar drawings of a connected graph are *topologically equivalent* if they define the same clockwise order of the edges around each vertex and their outer faces are delimited by the same walk.

[‡]A *straight-line drawing* Γ of a graph G maps vertices to points in a Euclidean space and edges to open straight-line segments between the images of their end-vertices. We denote by $\Gamma(v)$ the image of a vertex v and by $\Gamma(G')$ the image of a subgraph G' of G.

$i = 1, \ldots, k$, the drawing Γ_i is a crossing-free straight-line 3D drawing of G, i.e., a straight-line drawing of G in $\mathbb{R}^3$ such that no two edges cross; (ii) for $i = 1, \ldots, k-1$, the step $\langle \Gamma_i, \Gamma_{i+1} \rangle$ is a crossing-free linear morph, i.e., no two edges cross throughout the transformation; and (iii) $k = o(n)$? A morph $\mathcal{M}$ satisfying properties (i) and (ii) is a *crossing-free 3D morph.*

Our main result is a positive answer to the above question for trees. Namely, we prove that, for any two planar straight-line drawings Γ and Γ' of an n-vertex tree T, there is a crossing-free 3D morph with $O(rpw(T))$ steps between Γ and Γ', where $rpw(T)$ is the rooted pathwidth or Strahler number of T (a definition of this parameter will be given later); this provides a sub-linear bound with respect to the number of vertices of T, indeed $rpw(T) \in O(\log n)$ [8]. Notably, our morphing algorithm works even if Γ and Γ' are not topologically equivalent, hence the use of a third dimension overcomes another important limitation of planar two-dimensional morphs. Our algorithm morphs both Γ and Γ' to an intermediate suitably-defined *canonical 3D drawing*; in order to do that, a root-to-leaf path H of T is moved to a vertical line and then the subtrees of T rooted at the children of the vertices in H are moved around that vertical line, thus resembling a pole dance, which inspires the title of our paper.

We also look at whether our result can be generalized to morphs of crossing-free straight-line 3D drawings of trees. That is, the drawings Γ and Γ' now live in $\mathbb{R}^3$, and the question is again whether a crossing-free 3D morph between Γ and Γ' exists with $o(n)$ steps. We prove that this is not the case: Two crossing-free straight-line 3D drawings of a path might require $\Omega(n)$ steps to be morphed one into the other. The matching upper bound can always be achieved: For any two crossing-free straight-line 3D drawings Γ and Γ' of the same n-vertex tree T there is a crossing-free 3D morph between Γ and Γ' with $O(n)$ steps.

Finally, we consider morphs of tree drawings in Euclidean spaces with more than three dimensions. In particular, we show that, for any integer $d \geq 2$, a $(d+2)$-dimensional morph with a constant number of steps exists between any two crossing-free straight-line tree drawings in the same d-dimensional space.

The rest of the paper is organized as follows. In Section 2 we deal with crossing-free 3D morphs of 3D tree drawings. In Section 3 we show how to construct 2-step crossing-free 3D morphs between planar straight-line drawings of a path. In Section 4 we present our main result about crossing-free 3D morphs of planar tree drawings. In Section 5 we deal with morphs in $\mathbb{R}^d$, with $d \geq 4$. Finally, in Section 6 we conclude and present some open problems.

Throughout the rest of the paper, whenever we talk about a *vertical* straight line in $\mathbb{R}^3$, we always mean a line parallel to the z-axis; further, whenever we talk about a *horizontal* plane, we always mean a plane parallel to the xy-plane.

We remark that morphs of crossing-free straight-line drawings have also been studied under additional restrictions, such as keeping the edge lengths constant, especially for paths and trees. See, e.g., [13] for results in 2D, [9] for results in 3D, and [12] for results in 4D.

2 Morphs of 3D drawings of trees

In this section we give a tight $\Theta(n)$ bound on the number of steps in a crossing-free 3D morph between two crossing-free straight-line 3D tree drawings.

We start with the upper bound.

Theorem 1 *For any two crossing-free straight-line 3D drawings Γ and Γ' of an n-vertex tree T, there exists a crossing-free 3D morph from Γ to Γ' that consists of $O(n)$ steps.*

Proof: We prove, by induction on n, that a crossing-free 3D morph $\mathcal{M}$ between Γ and Γ' exists with $3n-2$ steps. The base case, in which $n=1$, is trivial.

For the inductive case, in which $n>1$, consider any leaf v and let u be its only neighbor in T. Remove v and the edge uv from T, Γ, and Γ', obtaining an $(n-1)$-vertex tree T' and two crossing-free straight-line 3D drawings Δ and Δ' of it. By induction, there is a crossing-free 3D morph $\mathcal{M}' = \langle \Delta = \Delta_0, \Delta_1, \dots, \Delta_{3n-5} = \Delta' \rangle$ with $3n-5$ steps between Δ and Δ'. Let $\varepsilon > 0$ be the smallest distance from u to any vertex or any non-incident edge throughout the morph $\mathcal{M}'$.

We show how to construct $\mathcal{M}$ starting from $\mathcal{M}'$. In particular, we will determine a placement for v in each of $\Delta_0, \Delta_1, \dots, \Delta_{3n-5}$, thus constructing a sequence $\Gamma_0, \Gamma_1, \dots, \Gamma_{3n-5}$ of straight-line 3D drawings of T. In particular, we will place v at distance less than ε from u in each of $\Gamma_0, \Gamma_1, \dots, \Gamma_{3n-5}$. This implies that v is at distance less than ε from u throughout the morph $\langle \Gamma_0, \Gamma_1, \dots, \Gamma_{3n-5} \rangle$. As a consequence, any crossing in $\langle \Gamma_0, \Gamma_1, \dots, \Gamma_{3n-5} \rangle$ involves uv and a different edge incident to u.

We start from Γ_0, in which we place v at a point $\Gamma_0(v)$ along the straight-line segment $\Gamma(u)\Gamma(v)$ at distance less than ε from $\Gamma_0(u)$. The linear morph $\langle \Gamma, \Gamma_0 \rangle$ is crossing-free; in particular, only v moves during such a morph. We define the last step $\langle \Gamma^*, \Gamma' \rangle$ of our 3D morph analogously: Only v moves and $\Gamma^*(v)$ lies along the straight-line segment $\Gamma'(u)\Gamma'(v)$ at distance less than ε from $\Gamma^*(u)$.

Suppose that a crossing-free straight-line 3D drawing Γ_i of T has been defined, for some integer $i \in \{0, \dots, 3n-6\}$. Since the existence of a crossing in a linear morph $\langle \Delta_i, \Delta_{i+1} \rangle$ is invariant under a translation of Δ_{i+1} of any vector, we can assume without loss of generality that $\Delta_i(u) = \Delta_{i+1}(u)$. Then the motion of each edge incident to u in $\langle \Delta_i, \Delta_{i+1} \rangle$ defines a triangle with one end-vertex in u. Since the number of such triangles is finite, there is a point $\Gamma_{i+1}(v)$ at distance less than ε from u such that the straight-line segment $\Gamma_i(v)\Gamma_{i+1}(v)$ does not pass through u and has no intersection with any of such triangles (which implies that the edge uv does not intersect any other edge incident to u during $\langle \Delta_i, \Delta_{i+1} \rangle$), except possibly at $\Gamma_i(v)$. However, by assumption, Γ_i is crossing-free; hence $\langle \Gamma_i, \Gamma_{i+1} \rangle$ is a crossing-free 3D morph.

During the linear morph $\langle \Gamma_{3n-5}, \Gamma^* \rangle$ the vertex v might overlap with u, or the edge uv might overlap with another edge incident to u. However, only v moves during such a morph, hence a sufficiently small perturbation of $\Gamma_{3n-5}(v)$ makes $\langle \Gamma_{3n-5}, \Gamma^* \rangle$ crossing-free without introducing any crossings in $\langle \Gamma_{3n-6}, \Gamma_{3n-5} \rangle$.

Hence, $\mathcal{M} = \langle \Gamma, \Gamma_0, \Gamma_1, \dots, \Gamma_{3n-5}, \Gamma^*, \Gamma' \rangle$ is the desired crossing-free 3D morph with $3n-2$ steps. This concludes the induction and the proof of the theorem. □

The following lemma will be useful in order to prove the lower bound.

Lemma 1 *During a linear morph between two straight-line 3D drawings of a graph G, any two edges of G intersect $O(1)$ times.*

Proof: At any time instant t of the morph, the segments representing two non-adjacent edges of G define a tetrahedron. If the segments intersect, then the tetrahedron degenerates and its volume is 0. This volume can be expressed as the Cayley-Menger determinant [21], and since each coordinate forming this determinant depends linearly on t, the moments of time when the volume is 0 are zeros of a univariate polynomial of degree 6, hence there are $O(1)$ of them. Analogously, at any time instant of the morph, the segments representing two adjacent edges of G define a triangle. If the segments intersect, then the triangle degenerates and its area is 0. This area can be expressed as a univariate polynomial of degree 4, hence there are $O(1)$ moments of time when it is 0. □

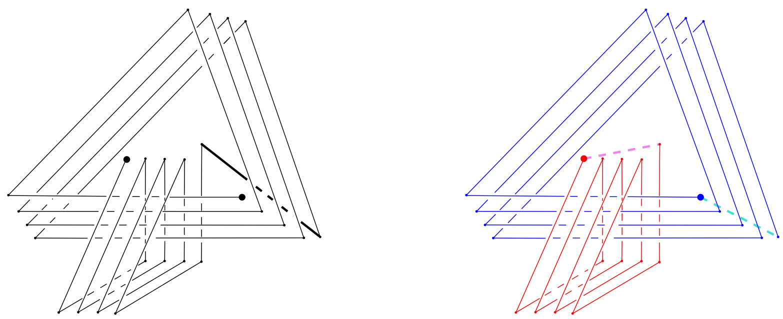
The following lower bound matches the upper bound of Theorem 1.

Theorem 2 *There exist two crossing-free straight-line 3D drawings Γ and Γ' of an n-vertex path P such that any crossing-free 3D morph from Γ to Γ' consists of $\Omega(n)$ steps.*

Before proving Theorem 2, we review some definitions and facts from knot theory; refer, e.g., to the book by Adams [1]. A *knot* is an embedding of a circle S^1 in $\mathbb{R}^3$. A *link* is a collection of knots which do not intersect each other. Note that different knots of a link may be linked together. For links of two knots, the (absolute value of the) *linking number* is an invariant that classifies links with respect to ambient isotopies. Intuitively, the linking number is the number of times that each knot winds around the other. The linking number is known to be invariant with respect to different projections of the same link [1]. Given a projection of a link $\mathcal{K}$ consisting of two knots, the linking number of $\mathcal{K}$ can be determined as follows: first, orient the two knots of $\mathcal{K}$ arbitrarily; then, for every crossing between the two knots in the projection, add $+1$ or -1 if rotating the understrand respectively clockwise or counterclockwise lines it up with the overstrand (taking into account the direction); finally, divide the obtained number by 2.

Proof: [of Theorem 2] The drawing Γ of P is defined as follows. Embed the first $\lfloor n/2 \rfloor$ edges of P in 3D as a spiral of monotonically-decreasing height. Embed the rest of P as a spiral of the same type affinely transformed so that it goes around one of the sides of the former spiral. See Figure 1(a). The drawing Γ' places the vertices of P in order along the unit parabola in the plane $y = 0$.

Cut the edge joining the two spirals (the bold edge in Figure 1(a)). Removing an edge makes morphing easier so any lower bound would still apply. Now close



(a) (b)

Figure 1: Illustration for the proof of Theorem 2. (a) The drawing Γ of P, with $n = 26$. (b) The link $\mathcal{K}$ obtained from Γ; the invisible edges are dashed.

the two open curves using two *invisible* edges to obtain a *link* $\mathcal{K}$ of two knots; see Figure 1(b). It is easy to verify that the (absolute value of the) linking number of $\mathcal{K}$ is $\Omega(n^2)$: indeed, determining it by the above procedure for the projection given by Figure 1(b) results in the linking number being equal to the number of crossings between the two links in this projection. On the other hand, in the drawing Γ', the two knots of $\mathcal{K}$ are separated by a plane and so the linking number of $\mathcal{K}$ in Γ' is 0.

In any linear morph of $\mathcal{K}$, the edges of P cannot cross each other, but they can cross invisible edges. However, by Lemma 1, during a linear morph between two straight-line 3D drawings of $\mathcal{K}$ any two edges of $\mathcal{K}$ intersect $O(1)$ times. Thus each invisible edge can only be crossed $O(n)$ times during a linear morph. A single crossing can only change the linking number by 1. Therefore the linking number can only decrease by $O(n)$ in a linear morph of $\mathcal{K}$. It follows that the morph from Γ to Γ' consists of $\Omega(n)$ linear morphs. □

3 Morphing two planar drawings of a path in 3D

In this section we show how to morph two planar straight-line drawings Γ and Γ' of an n-vertex path $P := (v_0, \dots v_{n-1})$ into each other in two steps.

The *canonical 3D drawing* of P, denoted by $\mathcal{C}(P)$, is the crossing-free straight-line 3D drawing of P that maps each vertex v_i to the point $(0, 0, i) \in \mathbb{R}^3$, as shown in Figure 2. We now prove the following.

Theorem 3 *For any two planar straight-line drawings Γ and Γ' of an n-vertex path P, there exists a crossing-free 3D morph $\mathcal{M} = \langle \Gamma, \mathcal{C}(P), \Gamma' \rangle$ with 2 steps.*

Proof: It suffices to prove that the linear morph $\langle \Gamma, \mathcal{C}(P) \rangle$ is crossing-free, since the morph $\langle \mathcal{C}(P), \Gamma' \rangle$ is just the morph $\langle \Gamma', \mathcal{C}(P) \rangle$ played backwards.

Since $\langle \Gamma, \mathcal{C}(P) \rangle$ is linear, the speed at which the vertices of P move is constant (though it might be different for different vertices). Thus the speed at which

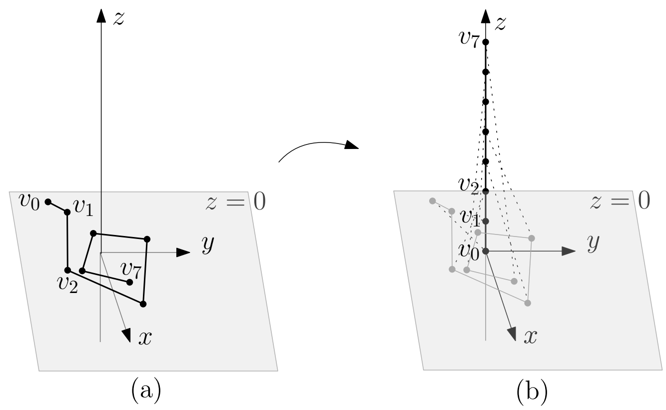


Figure 2: (a) A straight-line planar drawing Γ of an n-vertex path P and (b) a morph from Γ to $\mathcal{C}(P)$. The vertex trajectories are represented by dotted lines.

their projections on the z-axis move is constant as well. For each $i = 0, \ldots, n-1$, the vertex v_i moves at constant speed from its position $(x_i, y_i, 0)$ in Γ to its position $(0, 0, i)$ in $\mathcal{C}(P)$; it follows that at any time during the motion (except at the initial time $t = 0$) we have $z(v_0) < z(v_1) < \ldots < z(v_{n-1})$. Therefore, in any intermediate drawing of $\langle \Gamma, \mathcal{C}(P) \rangle$ any edge $v_i v_{i+1}$ is separated from any other edge by the horizontal plane through one of its end-points. Hence, no crossing happens during $\langle \Gamma, \mathcal{C}(P) \rangle$. $\square$

4 Morphing two planar drawings of a tree in 3D

Let T be a tree with n vertices, arbitrarily rooted at any vertex. In this section we show that any two planar straight-line drawings of T can be morphed into one another by means of a crossing-free 3D morph whose number of steps is linear in the rooted pathwidth of T; this number is hence in $O(\log n)$ [8].

Similarly to Section 3, we first define a canonical 3D drawing $\mathcal{C}(T)$ of T (see Section 4.1), and then show how to construct a crossing-free 3D morph from any planar straight-line drawing of T to $\mathcal{C}(T)$ (see Section 4.2).

Before proceeding, we introduce some necessary definitions and notation. By a *cylinder* we always mean a right cylinder having a horizontal circle as a base. By a *cone* we always mean a straight circular cone generated by a ray rotated around a fixed vertical line (the *axis*) while keeping its origin fixed at a point (the *apex*) on this line. The *slope* $\phi(C)$ of a cone C is the slope of the generating ray as determined in the vertical plane containing the ray.

Let C^{in} and C^{out} be two cones with the same apex and with $\phi(C^{\text{in}}) > \phi(C^{\text{out}}) > 0$; further, let $\mathcal{P}^*$ be a horizontal plane that is higher than the apex of C^{in} and C^{out}. Then the *funnel* $\mathcal{F}$ determined by C^{in}, C^{out}, and $\mathcal{P}^*$ is the closed bounded region of $\mathbb{R}^3$ that is delimited by $\mathcal{P}^*$ from above, by C^{in} from the inside, and by C^{out} from the outside; see Figure 3.

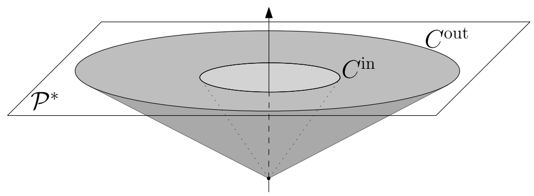


Figure 3: The funnel $\mathcal{F}$ determined by the cones C^{in} and C^{out}, and by the plane $\mathcal{P}^*$.

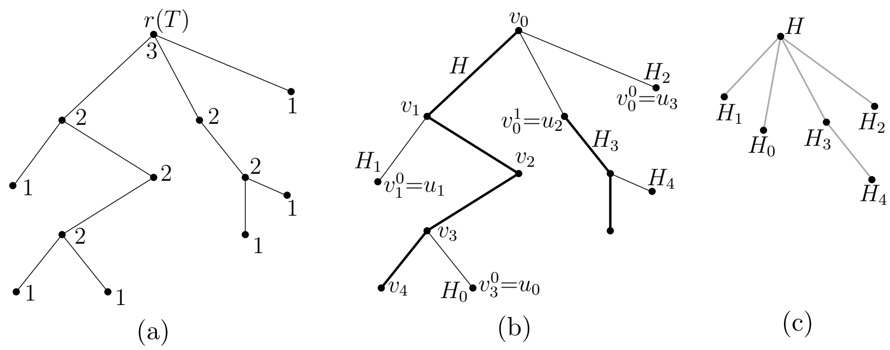


Figure 4: The illustration (a) shows, for each vertex v of a tree T, the number $rpw(T(v))$. In particular, $rpw(T) = 3$. The illustration (b) shows with bold lines the heavy edges of T forming the heavy paths $H, H_0, \ldots, H_4$, and the labeling of the vertices in H and of their light children. The illustration (c) shows the path tree of T.

For a tree T, let $T(v)$ denote the subtree of T rooted at a vertex v. Also, let $|T|$ denote the number of vertices in T and let $r(T)$ denote the root of T.

The *Strahler number* or *Horton-Strahler number* of a tree is a parameter which was introduced by Horton and Strahler [19, 23, 24]. The same parameter was recently rediscovered by Biedl [8] with the name of *rooted pathwidth* when addressing the problem of computing upward tree drawings with optimal width. The rooted pathwidth of a tree T, which we denote by $rpw(T)$, is defined as follows. If $|T| = 1$, then $rpw(T) = 1$. Otherwise, let k be the maximum rooted pathwidth of any subtree rooted at a child of $r(T)$. Then $rpw(T) = k$ if exactly one subtree rooted at a child of $r(T)$ has rooted pathwidth equal to k, and $rpw(T) = k + 1$ if more than one subtree rooted at a child of $r(T)$ has rooted pathwidth equal to k; see Figure 4(a).

The *heavy-rooted-pathwidth decomposition* of a tree T is defined as follows; refer to Figure 4(b). For each non-leaf vertex v of T, let c^* be the child of v in T such that $rpw(T(c^*))$ is maximum (ties are broken arbitrarily). Then (v, c^*) is a *heavy edge*; further, each child $c \neq c^*$ of v is a *light child* of v, and the edge (v, c) is a *light edge*. Connected components of heavy edges form paths,

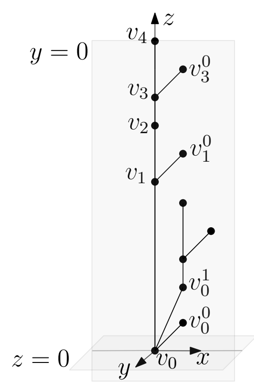


Figure 5: The canonical 3D drawing $\mathcal{C}(T)$ for the tree T in Figure 4.

called *heavy paths*, which may have many incident light edges. Each path has a vertex, called the *head*, that is the closest vertex to $r(T)$. The *path tree* of T is a tree whose vertices correspond to heavy paths in T; see Figure 4(c). The parent of a heavy path P in the path tree is the heavy path that contains the parent of the head of P. The root of the path tree is the heavy path containing $r(T)$. We denote by H the root of the path tree of T; let $v_0, \dots, v_{k-1}$ be the ordered sequence of the vertices of H, where $v_0 = r(T)$. For $i = 0, \dots, k-1$, we let $v_i^0, \dots, v_i^{t_i}$ be the light children of v_i in any order. Let $L = u_0, u_1, \dots, u_{l-1}$ be the sequence of the light children of H ordered so that: (i) any light child of a vertex v_j precedes any light child of a vertex v_i, if $i < j$; and (ii) the light child v_i^{j+1} of a vertex v_i precedes the light child v_i^j of v_i. For a vertex $u_i \in L$, we denote by $p(u_i)$ its parent; note that $p(u_i) \in H$.

It is known [8] that the height of the path tree of an n-vertex tree T is at most $rpw(T) \in O(\log n)$. Note that the heavy-rooted-pathwidth decomposition is slightly different from the well-known heavy-path decomposition [22] which we used in an earlier version of this paper.

4.1 Canonical 3D drawing of a tree

Given a tree T and a heavy-rooted-pathwidth decomposition of T, we define the *canonical 3D drawing* $\mathcal{C}(T)$ of T as the crossing-free straight-line 3D drawing of T that maps each vertex v of T to its *canonical position* $\mathcal{C}(v)$ defined as follows; refer to Figure 5. Note that our canonical drawing is equivalent to the "standard" straight-line upward drawing of a tree [8, 11, 14].

- First, we set $\mathcal{C}(v_0) = (0, 0, 0)$ for the root v_0 of T.
- Second, for each $i = 1, \dots, k-1$, we set $\mathcal{C}(v_i) = (0, 0, z_{i-1} + |T(v_{i-1})| - |T(v_i)|)$, where z_{i-1} is the z-coordinate of $\mathcal{C}(v_{i-1})$.
- Third, for each $i = 1, \dots, k-1$ and for each $j = 0, \dots, t_i$, we determine $\mathcal{C}(v_i^j)$ as follows. If $j = 0$, then we set $\mathcal{C}(v_i^j) = (1, 0, 1 + z_i)$, where z_i is the

z-coordinate of $\mathcal{C}(v_i)$; otherwise, we set $\mathcal{C}(v_i^j) = (1, 0, z_i^{j-1} + |T(v_i^{j-1})|)$, where z_i^{j-1} is the z-coordinate of $\mathcal{C}(v_i^{j-1})$.

- Finally, in order to determine the canonical positions of the vertices in $T(v_i^j) \setminus \{v_i^j\}$, for each $i = 0, \ldots, k-2$ and each $j = 0, \ldots, t_i$, we recursively construct the canonical 3D drawing $\mathcal{C}(T(v_i^j))$ of $T(v_i^j)$, and translate all the vertices by the same vector so that v_i^j is sent to $\mathcal{C}(v_i^j)$.

Remark 1 The canonical position $\mathcal{C}(v)$ of any vertex v of T is $(dpt(v), 0, dfs(v))$. Here $dpt(v)$ is the depth, in the path tree of T, of the vertex that corresponds to the heavy path of T that contains v, and $dfs(v)$ is the position of v in a depth-first search on T in which the children of any vertex are visited as follows: first visit the light children in reverse order with respect to L, and then visit the child incident to the heavy edge.

Remark 2 The canonical 3D drawing $\mathcal{C}(T)$ of T lies on a rectangular grid with height n and width $rpw(T)$ in the plane $y = 0$.

4.2 The procedure *Canonize*(Γ)

Let Γ be a planar straight-line drawing of T. Below we give a recursive procedure *Canonize*(Γ) that constructs an $O(rpw(T))$-step crossing-free 3D morph from Γ to the canonical 3D drawing $\mathcal{C}(T)$ of T. This is enough to prove that, for any two planar straight-line drawings Γ and Γ′ of T, there exists a crossing-free 3D morph from Γ to Γ′ with $O(rpw(T))$ steps, since a morph from $\mathcal{C}(T)$ to Γ′ can be obtained by playing the morph from Γ′ to $\mathcal{C}(T)$ backwards.

The procedure *Canonize*(Γ) consists of seven *phases*. Some of these phases, namely Phase 1, Phase 2, Phase 3, Phase 6, and Phase 7 are single linear morphs; Phase 5 consists of a constant number of morphing steps; finally, Phase 4 performs some (simultaneous) recursive calls to the procedure *Canonize*(), hence it consists of a number of morphing steps which is linear in the rooted pathwidth of some subtree on which the procedure is recursively invoked.

We assume that the root v_0 of T is placed at $(0, 0, 0)$ in Γ. This is not a loss of generality, up to a suitable modification of the reference system. We fix a constant $k^* \in \mathbb{R}$ with $k^* > 1$, which we consider global to the procedure *Canonize*(Γ) and its recursive calls. The global constant k^* will help us to realize Phase 5 of *Canonize*(Γ) in $O(1)$ morphing steps.

The procedure *Canonize*(Γ) maintains the following "steady-low-root" invariant: The root v_0 of T never moves and is on the lower base of the smallest cylinder that bounds the volume used by the morph *Canonize*(Γ).

Phase 1 (set the pole). The first phase of the procedure *Canonize*(Γ) aims to construct a linear morph $\langle \Gamma, \Gamma_1 \rangle$, where Γ_1 is such that the heavy path $(v_0, \ldots, v_{k-1})$ of T lies on the vertical line $x = y = 0$ and the subtrees of T rooted at the light children of each vertex v_i lie on the horizontal plane through v_i.

More precisely, the vertices of T are placed in Γ_1 as follows. For $i = 0, \dots, k-1$, place v_i at the point $\mathcal{C}(v_i)$. Every vertex that belongs to a subtree rooted at a light child of v_i is placed at a point such that its trajectory in the morph defines the same vector as the trajectory of v_i.* Below we refer to $\Gamma_1(H)$ as the *pole*. The pole will remain stationary throughout the rest of the procedure *Canonize*(Γ). We have the following.

Lemma 2 *Phase 1 of the procedure Canonize*(Γ) *is a crossing-free linear morph.*

Proof: For each vertex $v_i \in H$, all the vertices in $T(v_i) \setminus T(v_{i+1})$, i.e., v_i and all the vertices in the subtrees rooted at the light children of v_i, are translated by the same vector during the morph. Since Γ is planar, there is no collision between distinct vertices and edges of $T(v_i) \setminus T(v_{i+1})$ during the morph.

Further, for distinct i and j, the drawings of $T(v_i) \setminus T(v_{i+1})$ and $T(v_j) \setminus T(v_{j+1})$ lie on horizontal planes at different heights throughout the morph, except at the initial time instant, hence they do not collide with each other.

Finally, as in the proof of Theorem 3, we have that each edge (v_i, v_{i+1}) of H is separated from each $T(v_j) \setminus T(v_{j+1})$ by the horizontal plane through v_i or v_{i+1}, depending on whether $j \leq i$ or $j > i$, respectively. The lemma follows. □

Phase 2 (displace). The second phase of the procedure *Canonize*(Γ) aims to construct a linear morph $\langle \Gamma_1, \Gamma_2 \rangle$ which moves each subtree $T(v_i^j)$ to a different horizontal plane. The movement is small enough so that each vertex v_i^j is still below the vertex v_{i+1}.

More precisely, let $0 < \varepsilon_{l-1} < \varepsilon_{l-2} < \cdots < \varepsilon_1 < \varepsilon_0 < 1$ be real numbers to be determined later, where l is the number of right children of H. We define Γ_2 as follows. For each $i = 0, \dots, k-1$, let $\Gamma_2(v_i) = \Gamma_1(v_i)$; further, for each $i = 0, \dots, l-1$ and for each vertex v in $T(u_i)$, let $\Gamma_2(v)$ have the same x- and y-coordinates as $\Gamma_1(v)$ and let $\Gamma_2(v)$ have a z-coordinate equal to the z-coordinate of $\Gamma_1(v)$ plus ε_i. We have the following.

Lemma 3 *Phase 2 of the procedure Canonize*(Γ) *is a crossing-free linear morph, provided that the numbers $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{l-1}$ are sufficiently small.*

Proof: The lemma directly follows from a standard continuity argument, like the one used in the proof of Fáry's theorem [15]: Any sufficiently small perturbation of the vertex positions in a crossing-free drawing maintains the drawing crossing-free. Now, for any drawing of the morph $\langle \Gamma_1, \Gamma_2 \rangle$, the distance between the position of any vertex v and $\Gamma_1(v)$ is at most ε_0, hence the morph $\langle \Gamma_1, \Gamma_2 \rangle$ is crossing-free, provided that ε_0 is sufficiently small. □

*Since the morph $\langle \Gamma, \Gamma_1 \rangle$ is linear, the trajectory of any vertex v is simply the vector whose initial and terminal points are $\Gamma(v)$ and $\Gamma_1(v)$, respectively.

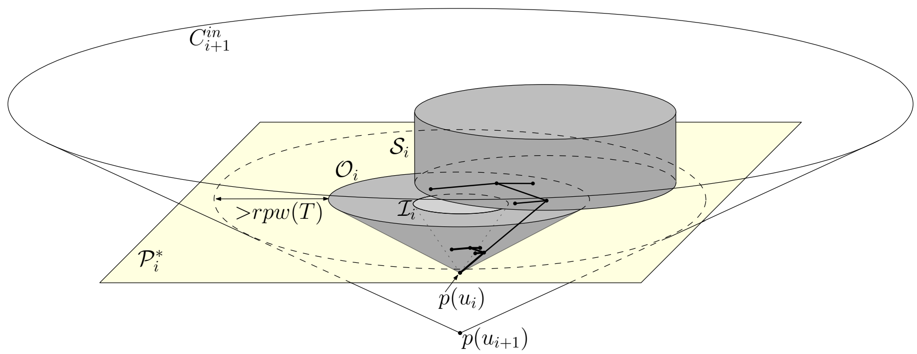


Figure 6: Illustration for the Properties (F1)–(F6). The fat small drawing is $\Gamma_2(T(u_i))$, while the lighter larger drawing is $\Gamma_3(T(u_i))$. The intersection of the cone C_{i+1}^{in} with the plane $\mathcal{P}_i^*$ is dashed.

Phase 3 (lift). The aim of the third phase of the procedure *Canonize*(Γ) is to construct a linear morph $\langle\Gamma_2, \Gamma_3\rangle$, where Γ_3 is such that the drawings of any two subtrees $T(u_i)$ and $T(u_j)$ rooted at different light children u_i and u_j of vertices in H are vertically and horizontally separated. In particular, the vertical separation between $\Gamma_3(T(u_i))$ and $\Gamma_3(T(u_j))$ needs to be large enough so that the recursively computed morphs *Canonize*$(\Gamma_3(T(u_i)))$ and *Canonize*$(\Gamma_3(T(u_j)))$, which are going to be performed in the next phase of the procedure, do not interfere with each other.

We describe how to construct Γ_3. While doing so, we also determine the values $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{l-1}$ from Phase 2. As anticipated, $\Gamma_3(v_i) = \Gamma_2(v_i)$, for each vertex v_i in H. For $i = 0, \ldots, l-1$, we construct $\Gamma_3(T(u_i))$ together with:

- a cylinder $\mathcal{S}_i$ that bounds the volume used by the recursively constructed morph *Canonize*$(\Gamma_3(T(u_i)))$ – this morph will be part of Phase 4; and
- a funnel $\mathcal{F}_i$ determined by two cones C_i^{in} and C_i^{out} with their apexes at $p(u_i)$ and by a horizontal plane $\mathcal{P}_i^*$ with equation $z = z_i^*$.

Denote by $\mathcal{I}_i$ and $\mathcal{O}_i$ the circles obtained as the intersections of C_i^{in} and C_i^{out} with $\mathcal{P}_i^*$. Further, denote by $\mathcal{A}_i$ the annulus delimited by $\mathcal{I}_i$ and $\mathcal{O}_i$ on $\mathcal{P}_i^*$. Our construction ensures that the following properties are satisfied for each $i = 0, 1, \ldots, l-1$ (refer to Figs. 6 and 7):

(F1) the ratio between the radii of $\mathcal{O}_i$ and $\mathcal{I}_i$ is greater than the global constant k^*;

(F2) if $i > 0$, then $\phi(C_{i-1}^{\text{out}}) > \phi(C_i^{\text{in}})$, where $\phi(C)$ denotes the slope of a cone C; moreover, the distance between any point of $\mathcal{O}_{i-1}$ and any point of the circle obtained as the intersection of C_i^{in} with $\mathcal{P}_{i-1}^*$ is larger than $rpw(T)$;

(F3) the funnel $\mathcal{F}_i$ contains $\Gamma_2(T(u_i))$ and $\Gamma_3(T(u_i))$; in particular, $\Gamma_3(T(u_i))$ lies on $\mathcal{A}_i$;

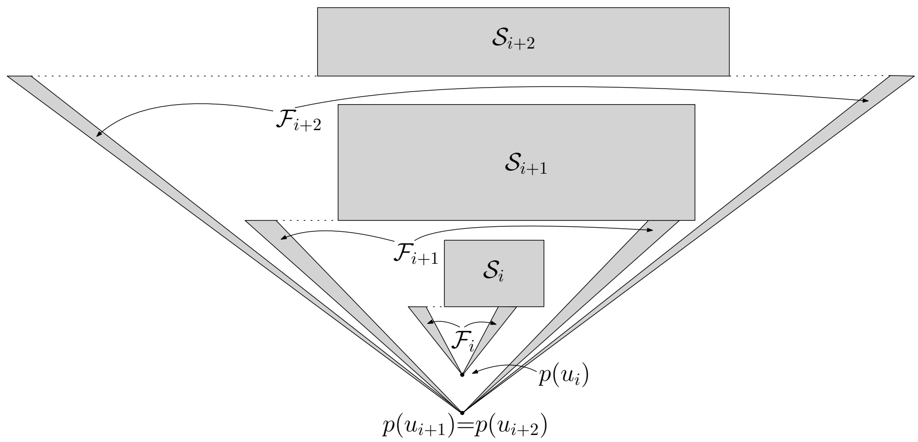


Figure 7: Cross section of three funnels and cylinders with the xz-plane.

(F4) the plane $\mathcal{P}_i^*$ is higher than every vertex in H and than every cylinder $\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{i-1}$;

(F5) the cylinder $\mathcal{S}_i$ has its lower base on the plane $\mathcal{P}_i^*$; and

(F6) the cylinders $\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{i-1}$ are above the cone C_i^{in}.

Before describing how to construct $\Gamma_3(T(u_i))$ and its associated cylinder $\mathcal{S}_i$ and funnel $\mathcal{F}_i$, we comment on Properties (F1)–(F6); these properties are exploited in Phases 3–7 to guarantee that the constructed morphs are crossing-free. Property (F1) ensures that the annulus $\mathcal{A}_i$ is "sufficiently thick"; during Phase 5, the vertex u_i needs to move across $\mathcal{A}_i$, and the thickness of $\mathcal{A}_i$ ensures that this movement can be realized in $O(1)$ morphing steps. Property (F2) ensures that any two funnels $\mathcal{F}_i$ and $\mathcal{F}_j$ are disjoint except, possibly, for the apexes of their cones, which might coincide; moreover, the second part of the property provides a lower bound on the minimum distance between the intersections of the funnels $\mathcal{F}_{i-1}$ and $\mathcal{F}_i$ with the plane $\mathcal{P}_{i-1}^*$. Property (F3) is used to prove that the lift of each subtree $T(u_i)$ which is performed in Phase 3 of the procedure *Canonize*() happens inside the corresponding funnel $\mathcal{F}_i$; this, together with the disjointness of the funnels, ensures that the lifts of distinct subtrees do not interfere with each other. Properties (F4) and (F5) guarantee a vertical separation between the cylinders $\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{l-1}$; this ensures that, in the upcoming Phase 4, the recursively constructed morphs *Canonize*($\Gamma_3(T(u_0))$), *Canonize*($\Gamma_3(T(u_1))$), ..., *Canonize*($\Gamma_3(T(u_{l-1}))$) can be executed simultaneously while guaranteeing the absence of crossings between the edges of any two distinct subtrees $T(u_i)$ and $T(u_j)$. Finally, Property (F6) provides a separation between cylinders and funnels, which is used to guarantee that the edge $p(u_i)u_i$ does not cross any edge of a tree $T(u_j)$ with $j < i$ during Phase 4.

Assume that, for some $i \in \{0, \dots, l-1\}$, the drawings $\Gamma_3(T(u_0))$, $\Gamma_3(T(u_1))$, ..., $\Gamma_3(T(u_{i-1}))$ have been constructed already, together with the cylinders

$\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{i-1}$ and the funnels $\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_{i-1}$; assume also that the numbers $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{i-1}$ have been determined. We show how to construct the drawing $\Gamma_3(T(u_i))$, the cylinder $\mathcal{S}_i$, and the funnel $\mathcal{F}_i$, and how to determine the number ε_i, so that Properties (F1)–(F6) are satisfied. Clearly, when $i = 0$, no drawing, cylinder, or funnel has been constructed yet. We proceed as follows.

1. If $i = 0$, then we let z_i^* be equal to one plus the z-coordinate of v_{k-1} (recall that the plane $\mathcal{P}_i^*$ has equation $z = z_i^*$); in this case Property (F4) is trivially satisfied, as every vertex of H has a z-coordinate smaller than or equal to the one of v_{k-1}. If $i > 0$, then we let $z_i^* = 1 + z_{i-1}^T$, where $z = z_{i-1}^T$ is the horizontal plane containing the upper base of $\mathcal{S}_{i-1}$; this ensures that Property (F4) is satisfied. Namely, the plane $\mathcal{P}_i^*$ is higher than $\mathcal{S}_{i-1}$, by construction, and is higher than the plane $\mathcal{P}_{i-1}^*$ containing the lower base of $\mathcal{S}_{i-1}$, which in turn is higher than every vertex in H and than every cylinder $\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{i-2}$.

2. Next, we define a cone C_i^{in} with apex at $p(u_i)$ so that the cylinders $\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{i-1}$ are all above C_i^{in}. This is trivial if $i = 0$. Otherwise, note that all such cylinders are above the horizontal plane through $p(u_i)$, given that they satisfy Properties (F4) and (F5); further, C_i^{in} coincides with this plane in the limit as $\phi(C_i^{\text{in}})$ goes to 0. By continuity, it suffices to choose $\phi(C_i^{\text{in}}) > 0$ small enough to ensure that C_i^{in} has all the cylinders $\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{i-1}$ above, hence Property (F6) is satisfied; the slope $\phi(C_i^{\text{in}})$ is also chosen so that it is smaller than $\phi(C_{i-1}^{\text{out}})$ and so that the distance between any point of $\mathcal{O}_{i-1}$ and any point of the circle obtained as the intersection of C_i^{in} with $\mathcal{P}_{i-1}^*$ is larger than $rpw(T)$. This ensures that Property (F2) is satisfied.

3. We now choose ε_i to be sufficiently small so that $\Gamma_2(T(u_i))$ is below C_i^{in}. Observe that, as ε_i goes to 0, the horizontal plane containing $\Gamma_2(T(u_i))$ approaches the horizontal plane containing $p(u_i)$; by continuity, it suffices to choose ε_i small enough to ensure that $\Gamma_2(T(u_i))$ is below C_i^{in}. The value of ε_i is also chosen small enough so that the morph $\langle \Gamma_1, \Gamma_2 \rangle$ is crossing-free, as in Lemma 3.

4. We now define a cone C_i^{out} with apex at $p(u_i)$; again by continuity, it suffices to choose $\phi(C_i^{\text{out}}) > 0$ small enough to ensure that C_i^{out} has $\Gamma_2(T(u_i))$ above, that $\phi(C_i^{\text{out}}) < \phi(C_i^{\text{in}})$, and that the ratio between the radii of $\mathcal{O}_i$ and $\mathcal{I}_i$ is greater than k^*. It follows that Property (F1) is satisfied and that $\Gamma_2(T(u_i))$ is inside $\mathcal{F}_i$.

5. Next, for any vertex $v \in T(u_i)$, we define $\Gamma_3(v)$ as the intersection point between the horizontal plane $\mathcal{P}_i^*$ and the ray from $p(u_i)$ through $\Gamma_2(v)$; note that such a ray is inside $\mathcal{F}_i$, given that $p(u_i)$ is the apex of $\mathcal{F}_i$ and that $\Gamma_2(v)$ is inside $\mathcal{F}_i$. Hence, Property (F3) is satisfied. This completes the construction of $\Gamma_3(T(u_i))$.

6. Finally, we recursively compute the morph *Canonize*$(\Gamma_3(T(u_i)))$ and we let $\mathcal{S}_i$ be the smallest cylinder enclosing such a morph. Note that the steady-low-root invariant on *Canonize*$(\Gamma_3(T(u_i)))$ ensures that u_i is on the lower base of $\mathcal{S}_i$. Then place $\mathcal{S}_i$ in the space so that the position of u_i on the lower base of $\mathcal{S}_i$ coincides with $\Gamma_3(u_i)$. This implies that Property (F5) is satisfied.

This concludes the description of the construction of Γ_3. Observe that, although such a construction is defined by looking at the subtrees $T(u_i)$ one at a time, Phase 3 actually consists of a single morphing step $\langle \Gamma_2, \Gamma_3 \rangle$, which we now prove to be crossing-free.

Lemma 4 *Phase 3 of the procedure Canonize*(Γ) *is a crossing-free linear morph.*

Proof: First, for any vertex $u_i \in L$, any two elements (vertices and edges) of $T(u_i)$ do not cross each other during the morph $\langle \Gamma_2, \Gamma_3 \rangle$, because the horizontal component of their motion is a scaling around the pole, and the vertical component is a lift up to the same height.

Second, consider any two vertices $u_i, u_j \in L$ with $i \neq j$. By Property (F3), any element (vertex or edge) of the tree $T(u_i)$ lies inside the funnel $\mathcal{F}_i$ throughout the morph $\langle \Gamma_2, \Gamma_3 \rangle$ and any element of $T(u_j)$ lies inside $\mathcal{F}_j$ throughout $\langle \Gamma_2, \Gamma_3 \rangle$. By Property (F2), we have that $\mathcal{F}_i$ and $\mathcal{F}_j$ are disjoint, except possibly at their apexes. Hence, no crossing occurs during the morph $\langle \Gamma_2, \Gamma_3 \rangle$ between the elements of $T(u_i)$ and $T(u_j)$. Analogously, no two edges $p(u_i)u_i$ and $p(u_j)u_j$ with $i \neq j$ cross each other, and no edge $p(u_i)u_i$ crosses the elements of a tree $T(u_j)$ if $i \neq j$. □

Phase 4 (recurse). For each $i = 0, \dots, l-1$, we make a recursive call *Canonize*$(\Gamma_3(T(u_i)))$. The resulting morphs are combined into a unique morph $\langle \Gamma_3, \dots, \Gamma_4 \rangle$, whose number of steps is equal to the maximum number of steps in any of the recursively computed morphs. Indeed, the first step of $\langle \Gamma_3, \dots, \Gamma_4 \rangle$ consists of the first steps of all the recursively computed morphs that have at least one step; the second step of $\langle \Gamma_3, \dots, \Gamma_4 \rangle$ consists of the second steps of all the recursively computed morphs that have at least two steps; and so on.

Lemma 5 *Phase 4 of the procedure Canonize*(Γ) *is a crossing-free morph.*

Proof: First, no two elements (vertices or edges) of the same tree $T(u_i)$ cross each other during $\langle \Gamma_3, \dots, \Gamma_4 \rangle$, as such a morph is recursively computed.

Second, by construction, the cylinder $\mathcal{S}_i$ bounds the volume used by the morph *Canonize*$(\Gamma_3(T(u_i)))$. Further, by Properties (F4) and (F5) the cylinders $\mathcal{S}_0, \mathcal{S}_1, \dots, \mathcal{S}_{l-1}$ are pairwise disjoint, hence no element of a tree $T(u_i)$ crosses an element of a distinct tree $T(u_j)$ during $\langle \Gamma_3, \dots, \Gamma_4 \rangle$. Properties (F4) and (F5) and the steady-low-root invariant also imply that no element of a tree $T(u_i)$ crosses an edge of H or an edge $p(u_j)u_j$ with $j \leq i$ during $\langle \Gamma_3, \dots, \Gamma_4 \rangle$.

Third, an edge $p(u_j)u_j$ does not cross an element of a tree $T(u_i)$ with $j > i$ during $\langle\Gamma_3, \dots, \Gamma_4\rangle$, since by Property (F6) the cylinder $\mathcal{S}_i$ is above C_j^{in}, while by Property (F3) the edge $p(u_j)u_j$ is below C_j^{in} in Γ_3 and it does not move during $\langle\Gamma_3, \dots, \Gamma_4\rangle$ by the steady-low-root invariant.

Finally, no two edges $p(u_i)u_i$, $p(u_j)u_j$, or $v_h v_{h+1}$ cross each other during $\langle\Gamma_3, \dots, \Gamma_4\rangle$, as none of such edges moves during this morph. □

Phase 5 (rotate). The next morph transforms Γ_4 into a drawing Γ_5 such that each vertex $u_i \in L$ is mapped to the point p_i^* which is the intersection of $\mathcal{I}_i$ with the plane $y = 0$ in the half-space $x > 0$ (recall that $\mathcal{I}_i$ is the circle obtained as the intersection of C_i^{in} with $\mathcal{P}_i^*$). Hence, after Phase 5, the whole drawing lies on the plane $y = 0$. Going from Γ_4 to Γ_5 in one crossing-free linear morph is not always possible, however we show how to accomplish such a morph in $O(k^*) \subseteq O(1)$ steps.

Refer to Figure 8. The morph performed in Phase 5 consists of a sequence of linear morphs; in each of these morphs all the vertices of $T(u_i)$ are translated by the same vector. This is done so that u_i stays inside $\mathcal{A}_i$ throughout the morph. Thus, the trajectory of u_i during Phase 5 defines a polygonal chain inside $\mathcal{A}_i$. By Property (F1), the ratio between the outer and the inner radius of $\mathcal{A}_i$ is at least the global constant k^*, hence we can inscribe a regular $O(k^*) \subseteq O(1)$-gon P in $\mathcal{A}_i$, and the trajectory of u_i can be defined so that it follows P plus at most two extra line segments, one from $\Gamma_4(u_i)$ to a vertex of P, and one from a vertex of P to p_i^*. We get the following.

Lemma 6 *Phase 5 of the procedure Canonize*(Γ) *is a crossing-free morph with* $O(1)$ *steps.*

Proof: First, since in each linear morph of Phase 5 all the vertices of $T(u_i)$ are translated by the same vector, it follows that at any time instant the drawing of $T(u_i)$ is a translation of the canonical 3D drawing $\mathcal{C}(T(u_i))$, hence there are no crossings between two elements of $T(u_i)$ during Phase 5.

Second, by Property (F5), the tree $T(u_i)$ lies in the closed half-space $z \geq z_i^*$ in Γ_4 and hence throughout Phase 5, as no vertex changes its z-coordinate during Phase 5. By Property (F4), the edges of H, the edges $p(u_j)u_j$ with $j < i$, and the elements of the trees $T(u_j)$ with $j < i$ lie in the open half-space $z < z_i^*$ in Γ_4 and hence throughout Phase 5, thus they do not cross any element of $T(u_i)$. Analogously, the edge $p(u_i)u_i$ does not cross any element of $T(u_i)$ during Phase 5, since by Property (F4) and by the steady-low-root invariant it lies in the open half-space $z < z_i^*$, except for the vertex u_i which is shared with $T(u_i)$.

Third, we prove that the elements of each tree $T(u_i)$ do not cross with any edge $p(u_j)u_j$ with $j > i$. Since $T(u_i)$ lies in the closed half-space $z \geq z_i^*$ throughout Phase 5, any crossing between an element of $T(u_i)$ and $p(u_j)u_j$ can only occur in the closed half-space $z \geq z_i^*$. By construction, the vertex u_i lies inside $\mathcal{A}_i$ and the drawing of $T(u_i)$ is a canonical drawing throughout Phase

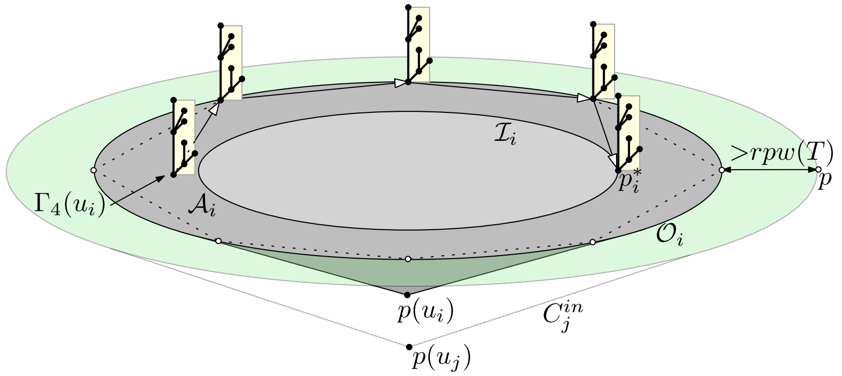


Figure 8: Illustration for Phase 5 of the procedure *Canonize*(Γ). The polygon with dashed boundary is the regular $O(1)$-gon P inscribed in $\mathcal{O}_i$. The arrows with white heads represent the movements of u_i in the morphing steps of Phase 5. The part of the disk delimited by the intersection of C_j^{in} with $\mathcal{P}_i^*$ outside $\mathcal{O}_i$ is light green.

5. Since the width of a canonical drawing is at most $rpw(T)$, it follows that, during Phase 5, any point of the projection of the drawing of $T(u_i)$ on the plane $\mathcal{P}_i^*$ is at distance at most $rpw(T)$ from $\mathcal{O}_i$. Hence, by Property (F2), the disk δ_j delimited by the intersection of C_j^{in} with $\mathcal{P}_i^*$ contains the projection of the drawing of $T(u_i)$ on $\mathcal{P}_i^*$ in its interior. On the other hand, since the edge $p(u_j)u_j$ lies inside $\mathcal{F}_j$ and hence below C_j^{in}, we have that the projection of the part of the edge $p(u_j)u_j$ in the closed half-space $z \geq z_i^*$ on $\mathcal{P}_i^*$ lies outside δ_j. It follows that the edge $p(u_j)u_j$ does not cross any element of $T(u_i)$ throughout Phase 5.

Finally, no two edges $p(u_i)u_i$ and $p(u_j)u_j$ with $i \neq j$ cross each other during Phase 5, since such edges lie inside $\mathcal{F}_i$ and $\mathcal{F}_j$, respectively, and such funnels are disjoint except, possibly, at their apexes, by Property (F2). □

The last two phases of the procedure *Canonize*(Γ) are *unidirectional morphs*, where a unidirectional morph is a linear 2D morph in which all the vertices move along parallel lines; see [2, 4, 6]. The following property of unidirectional morphs is going to be useful.

Corollary 1 *[2] Let $\langle \Gamma_A, \Gamma_B \rangle$ be a unidirectional morph between two planar straight-line drawings Γ_A and Γ_B of a graph G. Let u be a vertex of G, let vw be an edge of G and, for any drawing of G, let l_{vw} be the line through the edge vw oriented from v to w. Suppose that u is to the left of l_{vw} both in Γ_A and in Γ_B. Then v is to the left of l_{vw} throughout $\langle \Gamma_A, \Gamma_B \rangle$.*

Phase 6 (go down). This phase consists of a unidirectional morph $\langle \Gamma_5, \Gamma_6 \rangle$, where Γ_6 is defined as follows. For every vertex v_i in H, $\Gamma_6(v_i) = \Gamma_5(v_i)$; further, for each $i = 0, \dots, l-1$, and for each vertex v in $T(u_i)$, let $\Gamma_6(v)$ have the same x- and y-coordinates as $\Gamma_5(v)$ and let $\Gamma_6(v)$ have a z-coordinate equal to the z-coordinate of the canonical position $\mathcal{C}(v)$. Note that the morph $\langle \Gamma_5, \Gamma_6 \rangle$ happens on the plane $y = 0$. We have the following.

Lemma 7 *Phase 6 of the procedure Canonize*(Γ) *is a crossing-free linear morph.*

Proof: During the morph $\langle \Gamma_5, \Gamma_6 \rangle$ all the vertices $v_0, v_1, \ldots, v_{k-1}$ of H remain stationary, while all the other vertices move vertically downwards; indeed, by Properties (F4) and (F5), all the vertices in the subtrees $T(u_i)$ are higher than v_{k-1} in Γ_4 (and hence in Γ_5), while they are lower than v_{k-1} in Γ_6.

First, for $i = 0, \ldots, l-1$, all the vertices in the tree $T(u_i)$ are translated downwards by the same vector, hence no two elements (vertices or edges) in the same tree $T(u_i)$ cross each other.

Second, we prove that, for any $i, j \in \{0, 1, \ldots, l-1\}$ with $i < j$, all the vertices of $T(u_j)$ have x-coordinates greater than every vertex of $T(u_i)$ throughout $\langle \Gamma_5, \Gamma_6 \rangle$. Similarly to the proof of Lemma 6, the width of $\Gamma_5(T(u_i))$ is at most $rpw(T)$, hence, by Property (F2), the intersection point p of C_j^{in} with the planes $\mathcal{P}_i^*$ and $y = 0$ in the half-space $x > 0$ has x-coordinate greater than the x-coordinate of every point of $\Gamma_5(T(u_i))$. Since $\Gamma_5(u_j)$ coincides with p_j^*, whose x-coordinate is greater than the one of p, and since every point of $\Gamma_5(T(u_j))$ has x-coordinate greater than or equal to the one of $\Gamma_5(u_j)$, it follows that all the vertices of $T(u_j)$ have x-coordinates greater than every vertex of $T(u_i)$ in Γ_5, and hence throughout Phase 6. It follows that no element of $T(u_j)$ crosses an element of $T(u_i)$ or an edge $p(u_i)u_i$ if $i < j$.

Third, an edge $p(u_i)u_i$ does not cross any element of $T(u_i)$ as they are separated by the horizontal plane through u_i throughout Phase 6.

Finally, consider any edge $p(u_j)u_j$. Any tree $T(u_i)$ with $i < j$ is entirely above the line through $p(u_j)u_j$ both in Γ_5 and in Γ_6, which implies that it is entirely above the line through $p(u_j)u_j$ throughout Phase 6, by Corollary 1; hence $p(u_j)u_j$ does not cross any element of $T(u_i)$. The same argument also proves that $p(u_j)u_j$ does not cross any edge $p(u_i)u_i$ with $i < j$ (where possibly $p(u_i)$ and $p(u_j)$ are the same vertex). □

Phase 7 (go left). The final phase of our morphing procedure consists of a unidirectional morph $\langle \Gamma_6, \Gamma_7 \rangle$, where Γ_7 is the canonical 3D drawing $\mathcal{C}(T)$ of T. Note that this linear morph only moves the vertices horizontally; indeed, all the vertices lie on the plane $y = 0$ (already after Phase 5) and they have the same z-coordinate as in $\mathcal{C}(T)$ (as a result of Phase 6). We have the following.

Lemma 8 *Phase 7 of the procedure Canonize*(Γ) *is a crossing-free linear morph.*

Proof: During this morph all the vertices of H remain stationary, while all the other vertices move leftwards. Similarly to the proof of Lemma 7, we have that no two elements (vertices or edges) in the same tree $T(u_i)$ cross each other, as such elements are translated by the same vector; further, no element of a tree

$T(u_i)$ crosses an element of a distinct tree $T(u_j)$, as these trees are vertically separated throughout Phase 7. Any edge $p(u_i)u_i$ is vertically separated from all the trees $T(u_j)$ with $j \leq i$, and horizontally separated from all the trees $T(u_j)$ with $j \geq i$, hence $p(u_i)u_i$ does not cross any element of a tree $T(u_j)$ throughout Phase 7. Finally, no two edges $p(u_i)u_i$ and $p(u_j)u_j$ with $i < j$ (where possibly $p(u_i)$ and $p(u_j)$ are the same vertex) cross, as $p(u_i)u_i$ is above the line through $p(u_j)u_j$ both in Γ_6 and in Γ_7, and hence throughout Phase 7 by Corollary 1. □

We finally get the following.

Theorem 4 *For any two plane straight-line drawings* Γ *and* Γ' *of an* n*-vertex tree* T*, there exists a crossing-free 3D morph from* Γ *to* Γ' *with* $O(rpw(T)) \subseteq O(\log n)$ *steps.*

Proof: A 3D morph from Γ to Γ' can be constructed as the concatenation of *Canonize*(Γ) with the reverse of *Canonize*(Γ'). Hence, it suffices to prove that *Canonize*(Γ) is a crossing-free 3D morph with $O(rpw(T))$ steps. That $rpw(T) \in O(\log n)$ has been proved by Biedl [8].

Lemmas 2–8 ensure that *Canonize*(Γ) is a crossing-free 3D morph and that each of Phases 1, 2, 3, 5, 6, and 7 has $O(1)$ steps. Since the number of morphing steps of Phase 4 is equal to the maximum number of steps of any recursively computed morph and since, by definition of heavy path, each tree $T(u_i)$ for which a recursive call *Canonize*($\Gamma_3(T(u_i))$) is made has $rpw(T(u_i)) \leq rpw(T) - 1$, it follows that *Canonize*(Γ) requires $O(rpw(T))$ steps. □

5 Morphs in Higher-Dimensional Spaces

In this section we show that any two straight-line crossing-free drawings of a tree can be morphed into one another in two steps if the morph is allowed to use two dimensions more than the space where the input drawings lie.

We start by formally defining, for any $d \geq 2$, a *d-dimensional crossing-free morph* between two crossing-free straight-line d-dimensional drawings Γ and Γ' of the same tree T as a sequence $\mathcal{M} = \langle \Gamma = \Gamma_1, \Gamma_2, \ldots, \Gamma_k = \Gamma' \rangle$ such that: (i) for $i = 1, \ldots, k$, the drawing Γ_i is a crossing-free straight-line d-dimensional drawing of T; and (ii) for $i = 1, \ldots, k-1$, the step $\langle \Gamma_i, \Gamma_{i+1} \rangle$ is a crossing-free linear morph, i.e., no two edges cross throughout the transformation. We have the following.

Theorem 5 *Let* $d \geq 2$ *be an integer. For any two crossing-free straight-line* d*-dimensional drawings* Γ *and* Γ' *of a tree* T*, there exists a crossing-free* $(d+2)$*-dimensional morph from* Γ *to* Γ' *with* 2 *steps.*

Proof: We define the *canonical* $(d+2)$*-dimensional drawing* of T, denoted by $\mathcal{C}^{d+2}(T)$, as the crossing-free straight-line $(d+2)$-dimensional drawing of T that maps each vertex v to its *canonical position* $\mathcal{C}^{d+2}(v)$ defined as follows. For any $i \in \{1, 2, \ldots, d+2\}$, let $\mathcal{C}_i^{d+2}(v)$ denote the i-th coordinate of a vertex v in its canonical position. We use the same notation and terminology as in Section 4. In particular, let $(v_0, \dots, v_{k-1})$ be a heavy path of T, where v_0 is the root of T; further, for $i = 0, \dots, k-1$, let $v_i^0, \dots, v_i^{t_i}$ be the light children of v_i in any order.

First, we set $\mathcal{C}^{d+2}(v_0) = (0, \dots, 0)$. Second, for $i = 1, \dots, k-1$, we set $\mathcal{C}^{d+2}(v_i) = (0, \dots, 0, \mathcal{C}^{d+2}_{d+2}(v_{i-1}) + |T(v_{i-1})| - |T(v_i)|)$. Third, for $i = 1, \dots, k-1$ and for each light child v_i^j of v_i, we determine $\mathcal{C}^{d+2}(v_i^j)$ as follows. We set $\mathcal{C}^{d+2}(v_i^0) = (0, \dots, 0, 1, 1 + \mathcal{C}^{d+2}_{d+2}(v_i))$ and, for $i = 1, \dots, t_i$, we set $\mathcal{C}^{d+2}(v_i^j) = (0, \dots, 0, 1, \mathcal{C}^{d+2}_{d+2}(v_i^{j-1}) + |T(v_i^{j-1})|)$. Finally, in order to determine the canonical positions of the vertices in $T(v_i^j) \setminus \{v_i^j\}$, we recursively construct the canonical $(d+2)$-dimensional drawing $\mathcal{C}^{d+2}(T(v_i^j))$ of $T(v_i^j)$, and we translate all the vertices by the same vector so that v_i^j is sent to $\mathcal{C}^{d+2}(v_i^j)$.

We prove that $\langle \Gamma, \mathcal{C}^{d+2}(T), \Gamma' \rangle$ is a d-dimensional crossing-free morph. It suffices to prove that the linear morph $\langle \Gamma, \mathcal{C}^{d+2}(T) \rangle$ is crossing-free, since the morph $\langle \mathcal{C}^{d+2}(T), \Gamma' \rangle$ is just the morph $\langle \Gamma', \mathcal{C}^{d+2}(T) \rangle$ played backwards.

Label the vertices of T as $w_0, \dots, w_{n-1}$ in such a way that $\mathcal{C}^{d+2}_{d+2}(w_i) = i$. Recall that $\Gamma(w_j)$ denotes the position of w_j in Γ and let $\Gamma_i(w_j)$ denote the i-th coordinate of w_j in Γ. Then $\Gamma(w_j) = (\Gamma_1(w_j), \Gamma_2(w_j), \dots, \Gamma_d(w_j), 0, 0)$. At any time $t \in [0,1]$ of the morph $\langle \Gamma, \mathcal{C}^{d+2}(T) \rangle$, the position of w_j is hence $w_j(t) = (\Gamma_1(w_j) \cdot (1-t), \Gamma_2(w_j) \cdot (1-t), \dots, \Gamma_d(w_j) \cdot (1-t), \mathcal{C}^{d+2}_{d+1}(w_j) \cdot t, j \cdot t)$.

Consider any two edges (w_i, w_j) and (w_k, w_l). It comes directly from the definition of $(d+2)$-dimensional canonical drawing that (w_i, w_j) and (w_k, w_l) do not cross in $\mathcal{C}^{d+2}(T)$. Now suppose, for a contradiction, that at some time $t \in [0,1)$ of the morph $\langle \Gamma, \mathcal{C}^{d+2}(T) \rangle$ the edges (w_i, w_j) and (w_k, w_l) cross, possibly at their end-vertices. This implies that there exist values $\lambda, \gamma \in [0,1]$ such that $w_i(t) \cdot \lambda + w_j(t) \cdot (1-\lambda) = w_k(t) \cdot \gamma + w_l(t) \cdot (1-\gamma)$. Hence, for $x = 1, \dots, d$, we have that

$$\begin{aligned}
&\Gamma_x(w_i) \cdot (1-t) \cdot \lambda + \Gamma_x(w_j) \cdot (1-t) \cdot (1-\lambda) = \\
&\Gamma_x(w_k) \cdot (1-t) \cdot \gamma + \Gamma_x(w_l) \cdot (1-t) \cdot (1-\gamma),
\end{aligned}$$

and thus, since $t < 1$, we have that $\Gamma_x(w_i) \cdot \lambda + \Gamma_x(w_j) \cdot (1-\lambda) = \Gamma_x(w_k) \cdot \gamma + \Gamma_x(w_l) \cdot (1-\gamma)$. However, this implies that the two edges (w_i, w_j) and (w_k, w_l) cross at time $t = 0$ as well, a contradiction to the fact that Γ has no crossings. This concludes the proof of the theorem. □

We also believe that a generalization of Theorems 3 and 4 to higher dimension is possible, that is, that for any integer $d \geq 2$ and for any two crossing-free straight-line d-dimensional drawings Γ and Γ' of an n-vertex path P (of an n-vertex tree T), there exists a crossing-free $(d+1)$-dimensional morph $\mathcal{M}$ with 2 steps (resp. with $O(\log n)$ steps) from Γ to Γ'. While it is straightforward to extend the proof of Theorem 3 to prove the above statement for paths, it is not obvious how to modify the morphing algorithm for tree drawings presented in Section 4 to work in higher-dimensional spaces.

6 Conclusions and Open Problems

In this paper we studied crossing-free 3D morphs of tree drawings. We proved that, for any two planar straight-line drawings of the same n-vertex tree, there is a crossing-free 3D morph between them with a number of steps which is linear in the rooted pathwidth or Strahler number of T, hence it is in $O(\log n)$. While there exist n-vertex trees (e.g. the complete binary trees) whose rooted pathwidth is $\Omega(\log n)$, our bound is sub-logarithmic for several notable classes of trees, for example it is constant for paths and caterpillars.

This result gives rise to two natural questions. First, is it possible to bring our logarithmic upper bound down to constant? Second, does a crossing-free 3D morph exist with $o(n)$ steps for any two planar straight-line drawings of the same n-vertex planar graph? The latter question is interesting to us even for subclasses of planar graphs, like outerplanar graphs and planar 3-trees.

A major question in the topic of graph drawing morphs is whether, for any two planar straight-line drawings of a graph, a planar morph between them can always be constructed that guarantees bounded resolution and small area throughout the transformation; see, e.g., [7]. It would be interesting to understand whether the use of a third dimension simplifies this otherwise elusive problem. That is, given two planar straight-line drawings of a graph with bounded resolution and small area, does a crossing-free 3D morph always exists between them with bounded resolution and small volume? The morphing algorithms for tree drawings we presented in Sections 2 and 4 do not guarantee bounded resolution and small volume throughout the transformation, hence the above question is interesting even for trees.

We also proved that every two crossing-free straight-line 3D drawings of an n-vertex tree can be morphed into each other in $O(n)$ steps; such a bound is asymptotically optimal in the worst case. An easy extension of our results to graphs containing cycles seems unlikely. Indeed, the existence of a deterministic algorithm to construct a crossing-free 3D morph with a polynomial number of steps between two crossing-free straight-line 3D drawings of a cycle would imply that the unknot recognition problem is polynomial-time solvable. The *unknot recognition* problem asks whether a given knot is equivalent to a circle in the plane under an ambient isotopy. This problem has been the subject of investigation for decades; it is known to be in NP [18] and in co-NP [20], however determining whether it is in P has been an elusive goal so far.

Finally, we considered morphs in higher-dimensional spaces. In particular, we proved that, for any integer $d \geq 2$, a constant number of linear morphing steps are sufficient to construct a crossing-free $(d + 2)$-dimensional morph between any two crossing-free straight-line d-dimensional drawings of a tree. It would be interesting to understand whether similar results can be obtained for graph classes richer than trees.

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