On the Maximum Crossing Number

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Abstract

Research about crossings is typically about minimization. In this paper, we consider *maximizing* the number of crossings over all possible ways to draw a given graph in the plane. Alpert et al. [Electron. J. Combin., 2009] conjectured that any graph has a *convex* straight-line drawing, that is, a drawing with vertices in convex position, that maximizes the number of edge crossings. We disprove this conjecture by constructing a planar graph on twelve vertices that admits a non-convex drawing with more crossings than any convex drawing. Bald et al. [Proc. COCOON, 2016] showed that it is NP-hard to compute the maximum number of crossings of a geometric graph and that the weighted geometric case is NP-hard to approximate. We strengthen these results by showing hardness of approximation even for the unweighted geometric case. We also prove that the unweighted topological case is NP-hard.

1   Introduction

While traditionally in graph drawing one wants to minimize the number of edge crossings, we are interested in the opposite problem. Specifically, given a graph G, what is the maximum number of edge crossings possible, and what do embeddings[1] of G that attain this maximum look like? Such questions have first been asked as early as in the 19th century [3, 23]. Perhaps due to the counterintuitive nature of the problem (as illustrated by the disproved conjecture below) and due to the lack of established tools and concepts, little is known about maximizing the number of crossings.

Besides its theoretical appeal, motivation for this problem can be found in analyzing the worst-case scenario when edge crossings are undesirable but the placement of vertices and edges cannot be controlled.

There are three natural variants of the crossing maximization problem in the plane. In the *topological* setting, edges can be drawn as curves, so that any pair of edges crosses at most once, and adjacent edges do not cross. In the straight-line variant (known for historical reasons as the *rectilinear* setting), edges must be drawn as straight-line segments. If we insist that the vertices are placed in convex position (e.g., on the boundary of a disk or on the vertices of a convex polygon) and the edges must be routed in the interior of their convex hull, the topological and rectilinear settings are equivalent, inducing the same number of crossings: the number only depends on the order of the vertices along the boundary of the disk. In this *convex setting*, a pair of edges crosses if and only if its endpoints alternate along the boundary of the convex hull.

The topological setting.   The maximum crossing number was introduced by Ringel [20] in 1963 and independently by Grünbaum [10] in 1972.

Definition 1 ([21]) *The* maximum crossing number *of a graph G, max-cr(G), is the largest number of crossings in any topological drawing of G in which no three distinct edges cross in one point and every pair of edges has at most one point in common (a shared endpoint counts, touching points are forbidden).*

In particular, max-cr(G) is the maximum number of crossings in the topological setting. Note that only independent pairs of edges, that is those edge pairs with no common endpoint, can cross. The number of independent pairs of edges in a graph $G = (V, E)$ is given by $M(G) := \binom{|E|}{2} - \sum_{v \in V} \binom{\deg(v)}{2}$, a parameter introduced by Piazza et al. [18]. For every graph G, we have max-cr$(G) \leq M(G)$, and graphs for which equality holds are known as *thrackles* or *thrackable* [26]. Conway's Thrackle Conjecture [15] states that max-cr$(G) = M(G)$ implies $|E(G)| \leq |V(G)|$. If true, thrackles are precisely the pseudoforests (graphs in which every connected component has at most one cycle) in which there is no cycle of length four and at most one odd cycle [26].

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[1] We consider only embeddings where vertices are mapped to distinct points in the plane and edges are mapped to continuous curves containing no vertex points other than those of their end vertices.

Another famous open problem is the Subgraph Problem posed by Ringeisen et al. [19]: Is it true that whenever H is a subgraph or induced subgraph of G, then we have max-cr$(H) \leq$ max-cr(G)? Note that this statement is not trivially true as a drawing of H with max-cr(H) crossings might not be extendable to a drawing of G satisfying the requirements in Definition 1 above.

Let us remark that allowing pairs of edges to only touch without properly crossing each other, would indeed change the problem. For example, the 4-cycle C_4 has two pairs of independent edges, and C_4 can be drawn with one pair crossing and the other pair touching, but C_4 is not thrackable; it is impossible to draw C_4 with both pairs crossing, i.e., max-cr(C_4) is 1 and not 2.

It is known that max-cr$(K_n) = \binom{n}{4}$ [20] and that every tree is thrackable, i.e., max-cr$(G) = M(G)$ whenever G is a tree [18]. We refer to Schaefer's survey [21] for further known results on the maximum crossing numbers of several graph classes.

The straight-line setting.   The maximum rectilinear crossing number was introduced by Grünbaum [10]; see also [7].

Definition 2 *The* maximum rectilinear crossing number *of a graph G,* max-$\overline{\mathrm{cr}}(G)$*, is the largest number of crossings in any straight-line drawing of G.*

For every graph G, we have max-$\overline{\mathrm{cr}}(G) \leq$ max-cr$(G) \leq M(G)$, where each inequality is strict for some graphs, while equality is possible for other graphs. For example, for the n-cycle C_n we have max-$\overline{\mathrm{cr}}(C_n) =$ max-cr$(C_n) = M(C_n) = n(n-3)/2$ for odd n [26], while max-$\overline{\mathrm{cr}}(C_n) = M(C_n) - n/2 + 1$ and max-cr$(C_n) = M(C_n)$ for even $n \neq 4$ [1, 24]. For further rectilinear crossing numbers of specific graphs we again refer to Schaefer's survey [21].

For several graph classes, such as trees, the maximum (topological) crossing number is known exactly, while little is known about the maximum rectilinear crossing number. Verbitsky [25] studied the maximum rectilinear crossing number, which he called the *obfuscation number*, restricted to planar graphs. He showed that max-$\overline{\mathrm{cr}}(G) < 3n^2$ for any n-vertex planar graph G. For maximally planar graphs, that is, triangulations, Kang et al. [13] gave a $(56/39 - \varepsilon)$-approximation for computing the maximum rectilinear crossing number.

The convex setting.   It is easy to see that in the convex setting we may assume, without loss of generality, that all vertices are placed on a circle and edges are drawn as straight-line segments. In fact, if the vertices are in convex position and edges are routed in the interior of the convex hull of all vertices, then a pair of edges is crossing if and only if the vertices of the two edges alternate in the circular order along the convex hull.

Definition 3 *The* maximum convex crossing number *of a graph G,* max-cr$^\circ(G)$, *is the largest number of crossings in any drawing of G where the vertices lie on the boundary of a disk and the edges in the interior.*

From the definitions we now have that, for every graph G,

$$\text{max-cr}^\circ(G) \leq \text{max-}\overline{\text{cr}}(G) \leq \text{max-cr}(G) \leq M(G), \tag{1}$$

but this time it is not clear whether or not the first inequality can be strict. It is tempting (and rather intuitive) to say that in order to get many crossings in the rectilinear setting, all vertices should always be placed in convex position. In other words, this would mean that the maximum rectilinear crossing number and maximum convex crossing number always coincide. Indeed, this has been conjectured by Alpert et al. in 2009.

Conjecture 1 (Alpert et al. [1]) *Any graph G has a drawing with vertices in convex position that has $\text{max-}\overline{\text{cr}}(G)$ crossings, that is, $\text{max-}\overline{\text{cr}}(G) = \text{max-cr}^\circ(G)$.*

Our contribution.   Our main result is that Conjecture 1 is false. We provide several counterexamples in Section 3. There we first present a rather simple analysis for a counterexample with 37 vertices. We then improve upon this by showing that the planar 12-vertex graphs shown in Figure 5 (d) are counterexamples as well. Before we get there, we discuss the four parameters in (1) and relations between them in more detail, and introduce some new problems in Section 2. Finally, in Section 4, we investigate the complexity and approximability of crossing maximization and show that the topological problem is NP-hard, while the rectilinear problem is even hard to approximate.

2   Preliminaries and Basic Observations

Here we discuss the chain of inequalities in (1) and extend it by several items. Recall that for a graph G, $M(G)$ denotes the number of independent pairs of edges in G. By (1), we have that $\text{max-cr}^\circ(G) \leq M(G)$. We will show that this inequality is tight up to a factor of 3. The first part of the next lemma is due to Verbitsky [25].

Lemma 1 *For every graph G, we have $\text{max-cr}^\circ(G) \geq M(G)/3$. Moreover, if G has chromatic number at most 3, then $\text{max-cr}^\circ(G) \geq M(G)/2$.*

Proof: First, let G be any graph. We place the vertices of G on a circle in a circular order chosen uniformly at random from the set of all their circular orders. Then each pair of independent edges of G is crossing with probability $1/3$ since among the six possible orders of the endpoints, only two yield a crossing. Hence, there must be an ordering witnessing $\text{max-cr}^\circ(G) \geq M(G)/3$.

Second, assume that G can be properly colored with at most three colors. In this case we place the vertices of G on a circle in such a way that the three color classes occupy three pairwise disjoint arcs. In each color class, we order the vertices randomly, choosing each linear order with the same probability. Doing this independently for each color class, each pair of independent edges is crossing with probability $1/2$. Hence, the expected number of crossings is
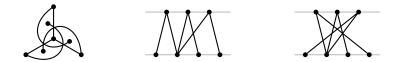
Figure 1: The smallest tree G that is not a caterpillar with a topological drawing with max-cr$(G) = M(G) = 9$ crossings (left), a 2-layer drawing with bcr$(G) = 1$ crossings (middle) and a 2-layer drawing with max-$\overline{\text{cr}}(G) = M(G) - \text{bcr}(G) = 8$ crossings (right).

$M(G)/2$, which, by the pigeon-hole principle, implies that there must be an ordering witnessing max-cr$^\circ(G) \geq M(G)/2$. □

By Lemma 1 we can extend the chain of inequalities in (1) as follows: For every graph G, we have

$$M(G)/3 \leq \text{max-cr}^\circ(G) \leq \text{max-}\overline{\text{cr}}(G) \leq \text{max-cr}(G) \leq M(G). \tag{2}$$

The constant $1/3$ in the first inequality in (2) cannot be improved: Consider the six edges connecting a 4-tuple of vertices in a rectilinear drawing of the complete graph K_n. There is exactly one crossing among them if the four vertices are in convex position, and there is no crossing among them otherwise. It follows that the maximum rectilinear crossing number of K_n is attained if and only if the vertices are in convex position, and in this case there are $M(K_n)/3 = \binom{n}{4}$ crossings. Since Ringel [20] proved that max-cr$(K_n) = \binom{n}{4}$, we get that max-cr$^\circ(K_n) = \text{max-}\overline{\text{cr}}(K_n) = \text{max-cr}(K_n) = M(K_n)/3 = \binom{n}{4}$.

We now introduce another item in the chain of inequalities (2). We say that a rectilinear drawing of a graph G is *separated* if there is a line ℓ that intersects every edge of G. Clearly, this is only possible if G is bipartite and in this case the line ℓ separates the vertices of the two color classes of G.

Particularly nice are *2-layer drawings* where the vertices of the two color classes are required to be placed on two parallel lines; see Fig. 1. It is easy to see that drawing bipartite graphs in the 2-layer model is equivalent to separated drawings with vertices in convex position. In this 2-layer model, the *minimum* number of crossings of a bipartite graph G has been studied under the name *bipartite crossing number*, denoted bcr(G).

Lemma 2 *For every bipartite graph G, the maximum number of crossings among all 2-layer drawings of G is exactly $M(G) - \text{bcr}(G)$.*

Proof: Consider any separated convex drawing of any bipartite graph G. A pair of independent edges is crossing if and only if their endpoints alternate along the convex hull. So if $e_1 = u_1 v_1$ and $e_2 = u_2 v_2$ with u_1, u_2 being above the separating line ℓ and v_1, v_2 below, then e_1 and e_2 are crossing if in the circular order we see $u_1 - u_2 - v_1 - v_2$, and non-crossing if we see $u_1 - u_2 - v_2 - v_1$. In particular, reversing the order of all vertices below the separating line ℓ

transforms crossings into non-crossings and vice versa. This shows that for a separated convex drawing with k crossings, reversing results in exactly $M(G)-k$ crossings, which concludes the proof. $\square$

Note that, for any bipartite graph G, the proof of Lemma 1 gives a 2-layered drawing of G with at least $M(G)/2$ crossings. Clearly, this is a lower bound for the maximum number of crossings over all two-layered drawings of G. This number is specified by Lemma 2 and trivially upperbounded by the maximum number of crossings over all convex drawings of G. Hence, we can extend the chain of inequalities (2) as follows for every bipartite graph G:

$$M(G)/2 \leq M(G) - \mathrm{bcr}(G) \leq \text{max-cr}^\circ(G) \leq \text{max-}\overline{\mathrm{cr}}(G) \leq \text{max-cr}(G) \leq M(G).$$
$$(3)$$

It remains open whether the new inequality $M(G) - \mathrm{bcr}(G) \leq \text{max-cr}^\circ(G)$ in (3) is attained with equality for every bipartite graph G. For example, for a tree G it is known, see e.g. [26], that $\text{max-cr}(G) = M(G)$, but it is not hard to see that $\text{max-}\overline{\mathrm{cr}}(G) = M(G)$ if and only if G is a caterpillar[2]. (Hence $\text{max-}\overline{\mathrm{cr}}(G) < \text{max-cr}(G)$ holds for every tree which is not a caterpillar.) Moreover, it is equally easy to see that a tree G has a crossing-free 2-layer drawing if and only if G is a caterpillar. Thus, for every tree G, we have that $M(G) - \mathrm{bcr}(G) = M(G)$ if and only if $\text{max-}\overline{\mathrm{cr}}(G) = M(G)$. We again refer to Fig. 1 for an illustration.

The extended chain of inequalities (3) leads to two natural questions:

Problem 1 *Does every bipartite graph G have a separated drawing with* $\text{max-}\overline{\mathrm{cr}}(G)$ *many crossings? Does every tree G have a separated convex drawing with* $\text{max-}\overline{\mathrm{cr}}(G)$ *crossings, i.e., is* $\text{max-}\overline{\mathrm{cr}}(G) = M(G) - \mathrm{bcr}(G)$*?*

It was Garey and Johnson [8] who showed that bipartite crossing minimization is NP-hard for multigraphs. Schaefer [21, Footnote 33] showed how to adjust the proof for simple graphs. The problem remains NP-hard if the ordering of the vertices on one side is prescribed [6]. On trees, bipartite crossing minimization can be solved efficiently [22]. For the one-sided two-layer crossing minimization, Nagamochi [16] gave an 1.47-approximation algorithm, improving upon the well-known median heuristic, which yields a 3-approximation [6]. The weighted case, which we define formally in Section 4, admits a 3-approximation algorithm [5].

3   Counterexamples for Conjecture 1

In this section we present counterexamples for the convexity conjecture. After some preliminary work we provide a counterexample $H(4)$ on 37 vertices. To show that this graph is a counterexample, we need to analyze only two cases. (To show that $H(2)$ with 19 vertices also is a counterexample would require more work. Instead, in Appendix A, we prove that a certain planar subgraph of $H(2)$ with only 12 vertices and 16 edges is already a counterexample.)

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[2]A caterpillar is a tree in which all non-leaf vertices lie on a common path.

A set of vertices $X \subset V$ in a graph $G = (V, E)$ is a set of *twins* if all vertices of X have the same neighborhood in G (in particular, since there are no self-loops, X is an independent set). The twin relation is an equivalence relation and thus the vertex set of any graph naturally partitions into its inclusion maximal vertex subsets that are pairwise twins. A *vertex split* of vertex v in G consists in adding a new vertex v' to G such that v' is a twin of v, that is, for any edge vu, there is an edge $v'u$, and these are all the edges at v'.

Lemma 3 *For any graph G there is a convex drawing of G maximizing the number of crossings among all convex drawings of G, such that each set of twins forms an interval of consecutive vertices along the convex hull of the drawing.*

Proof: Suppose $V_1, \ldots, V_s$ are the maximal sets of twins in G. Consider a convex drawing of G maximizing the number of crossings. It clearly suffices to show that for any set V_i we may move all the points of V_i next to one of the points of V_i without decreasing the number of crossings, since this procedure done iteratively s times, once for each of the sets $V_1, \ldots, V_s$, results in a desired convex drawing of G.

We call a crossing k-*rich* with respect to V_i if there are k vertices of V_i among the four vertices of the edges forming the crossing. For any fixed i, since V_i is independent, the set of all crossings partitions into 0-rich, 1-rich, and 2-rich crossings with respect to V_i. If we move only vertices of V_i then 0-rich crossings remain in the drawing. If the vertices of V_i appear in consecutive order along the convex hull of the drawing then the number of 2-rich crossings is maximized due to the following argument. For any two vertices u, v of V_i and for any two neighbors x, y of V_i, the 4-cycle $uxvy$ is self-crossing which gives rise to a 2-rich crossing. Since every 2-rich crossing appears in a single 4-cycle and every 4-cycle can give rise to at most one crossing, the number of 2-rich crossings is indeed maximized whenever the vertices of V_i appear in consecutive order along the convex hull. It remains to show that there is a vertex v in V_i such that we can move the other vertices next to v without decreasing the number of 1-rich crossings. Each 1-rich crossing involves exactly one vertex of V_i. The number of 1-rich crossings involving a given vertex of V_i is affected only by the position of that vertex and of the vertices of $V \setminus V_i$. Thus, if we choose $v \in V_i$ as the vertex involved in the largest number of 1-rich crossings and move all the other vertices of V_i next to v, every vertex will be involved in at least as many 1-rich crossings as it was before the vertices were moved. □

The construction of $H(k)$. For the construction of our example graphs $H(k)$, we start with a 9-cycle on vertices $v_0, \ldots, v_8$ with edges $v_i v_{i+1}$ where $i + 1$ is to be taken modulo 9. Add a 'central' vertex z adjacent to v_0, v_3, v_6. This graph on 10 vertices is the base graph H. The example graph $H(k)$ is obtained from H by considering each vertex v_i on the 9-cycle and applying $k - 1$ consecutive vertex splits of v_i. The graph $H(k)$ thus consists of nine independent sets V_i of size k and the central vertex z. In total it has $9k + 1$ vertices and $9k^2 + 3k$ edges. Figure 2 (left) shows a schematic drawing of $H(k)$, where
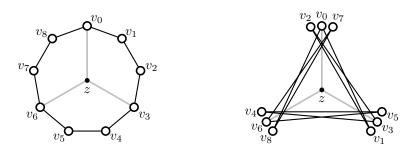
Figure 2: Left: A simplified version of the graph $H(k)$. Each white circle represents k independent vertices, each black line segment represents a bundle of k^2 edges, each gray line segment represents k edges. Right: A non-convex drawing of the simplified version of $H(k)$.

each black edge represents a "bundle" of k^2 edges of $H(k)$ and each gray edge represents a set of k edges. We will show that for $k \geq 4$ the drawing in Fig. 2 (right) has more crossings than any drawing with vertices in convex position.

From Lemma 3 we know that, in convex drawings of $H(k)$ with many crossings, the twin pairs of vertices can be assumed to be next to each other. Drawings of $H(k)$ of this kind are essentially determined by the corresponding drawings of H, in which each set of twins is represented just by one representative; see Fig. 2. This justifies that later on we only look at convex drawings of H with weighted crossings, and not of the full $H(k)$.

An independent set of edges of $H(k)$ is *weak* if the corresponding edges in the base graph H are not independent; it is *strong* otherwise. The next lemma shows that our non-convex drawing of $H(k)$ realizes as many crossings on weak pairs of independent edges as possible. This allows us to focus on strong pairs in the subsequent analysis. We remark that there are also convex drawings of $H(k)$ realizing that many crossings on weak edge pairs.

Lemma 4 *The drawing of $H(k)$ on the right side of Fig. 2 maximizes the number of crossings on weak pairs of independent edges.*

Proof: Each edge $v_i v_{i+1}$ of H maps to a $K_{k,k}$ in $H(k)$. In the given drawing the $K_{k,k}$ is represented by a black edge. Since V_i, V_{i+1} are in separated convex position the $K_{k,k}$ contributes $\binom{k}{2} \cdot \binom{k}{2}$ crossings.

A pair of adjacent edges $v_{i-1} v_i$ and $v_i v_{i+1}$ in H maps to a $K_{k,2k}$ in $H(k)$. We know that max-$\overline{\mathrm{cr}}(K_{k,2k}) = \binom{k}{2}\binom{2k}{2}$ and this number of crossings is realized with separated convex position. In the drawing V_i and $V_{i-1} \cup V_{i+1}$ are in separated convex position.

A pair of adjacent edges $v_i z$ and $v_i v_{i+1}$ in H maps to a $K_{k,k+1}$ in $H(k)$. Now we have max-$\overline{\mathrm{cr}}(K_{k,k+1}) = \binom{k}{2}\binom{k+1}{2}$, and this number of crossings is realized with separated convex position of the vertices. In the drawing $V_i, V_{i+1} \cup \{z\}$ are in separated convex position. The case of adjacent edges $v_i z$ and $v_{i-1} v_i$ is identical.   □

The remaining crossings of the drawing of $H(k)$ correspond to crossings of two independent edges of H. These are either two black edges or a black and a gray edge of H. Black edges represent a bundle of k^2 edges of $H(k)$ and gray edges a bundle of k edges of $H(k)$. Hence a crossing of two black edges represents k^4 individual crossing pairs and a crossing of a black and a gray edge represents k^3 individual crossing pairs. We devide by k^3 and speak about two independent black edges as a pair of weight k and of a black and a gray edge as a pair of weight 1. In the given non-convex drawing of $H(k)$ every pair of independent black edges is crossing but every black edge has a unique independent gray edge which it does not cross. Hence, the total weight of the independent non-crossing pairs of edges of H is 9. We summarize by saying that the given non-convex drawing of $H(k)$ has a weighted loss of 9 with respect to its strong pairs of independent edges.

The loss of convex drawings. We now study the weighted loss of convex drawings of H. In a convex drawing every black edge splits the 7 non-incident cycle vertices into those on one side and those on the other side. The *span* of a black edge is the number of vertices on the smaller side. Hence, the span of an edge is one of 0, 1, 2, 3.

First, consider the case where the 9-cycle is drawn with zero loss, i.e., each black edge has span 3 and contributes a crossing with 6 other black edges. The cyclic order of the cycle vertices is $v_0, v_2, v_4, v_6, v_8, v_1, v_3, v_5, v_7$. Any two neighbors of z have the same distance in this cyclic order. Therefore, we may assume that z is in the short interval spanned by v_0 and v_6. Every edge of the 9-cycle is disjoint from at least one of the two gray edges zv_0 and zv_6 and the edge v_7v_8 is disjoint from both. This shows that the weighted loss of this drawing is at least 10.

Second, consider the case where the 9-cycle is not drawn with zero loss, i.e., at least one black edge has span at most 2. A sequence of eight consecutive edges of span 3 forces the last edge to also have span 3. Hence, we have at least two black edges e and f of span at most 2. Each of these edges is disjoint from at least two independent black edges. Since the two edges may be disjoint they contribute a weighted loss of at least $3k$. For $k \geq 4$ this exceeds the weighted loss of the drawing of Fig. 2.

We summarize the above discussion as follows.

Theorem 1 *For any $k \geq 4$, it holds that* $\max\text{-}\overline{\mathrm{cr}}(H(k)) > \max\text{-}\mathrm{cr}^\circ(H(k))$, *that is, $H(k)$ has a straight-line drawing with more crossings than any straight-line drawing of $H(k)$ with vertices in convex position.*

4 Complexity

Very recently, Bald et al. [2] showed, by reduction from MaxCut, that it is NP-hard to compute the maximum rectilinear crossing number $\max\text{-}\overline{\mathrm{cr}}(G)$ of a given graph G. Their reduction also shows that it is hard to approximate the

weighted case better than ≈ 0.878 assuming the Unique Games Conjecture and better than 16/17 assuming $\mathcal{P} \neq \mathcal{NP}$. In the convex case, one can "guess" the permutation; hence, this special case is in $\mathcal{NP}$. Bald et al. also stated that rectlinear crossing maximization is similar to rectilinear crossing minimization in the sense that the former "inherits" the membership in the class of the existential theory of the reals ($\exists\mathbb{R}$), and hence in PSPACE, from the latter. They also showed how to derandomize Verbitsky's approximation algorithm [25] for max-$\overline{\mathrm{cr}}$, turning the expected approximation ratio of 1/3 into a deterministic one.

We now tighten the hardness results of Bald et al. by showing APX-hardness for the *unweighted* case. Recall that MAXCUT is NP-hard to approximate beyond a factor of 16/17 [12]. Under the Unique Games Conjecture, MAXCUT is hard to approximate even beyond a factor of ≈ 0.878 [14]—the approximation ratio of the famous semidefinite programming approach of Goemans and Williamson [9] for MAXCUT. For a graph G, let max-cut(G) be the maximum number of edges crossing a cut, over all cuts of G.

Theorem 2 *Given a graph G, max-$\overline{\mathrm{cr}}(G)$ cannot be approximated better than* MAXCUT.

Proof: As Bald et al., we reduce from MAXCUT. In their reduction, they add a large-enough set I of independent edges to the given graph G. They argue that max-$\overline{\mathrm{cr}}(G+I)$ is maximized if the edges in I behave like a single edge with high weight that is crossed by as many edges of G as possible. Indeed, suppose for a contradiction that, in a drawing with the maximum number of crossings, an edge $e \in I$ crosses fewer edges than another edge e' in I. Then e can be drawn such that its endpoints are so close to the endpoints of e' that both edges cross the same edges—and each other. This would increase the number of crossings; a contradiction. Without loss of generality, we can make the "heavy edge" so long that its endpoints lie on the convex hull of the drawing. This means that the heavy edge induces a cut of G. The cut is maximum since the heavy edge can be made arbitrarily heavy.

Instead of adding a set I of independent edges to G, we add a star S_t with $t = \binom{m}{2} + 1$ edges, where $m = |E(G)|$. Then, max-$\overline{\mathrm{cr}}(G) < t$. The advantage of the star is that all its edges are incident to the same vertex and, hence, cannot cross each other. Let $G' = G + S_t$ be the resulting graph. Exactly as for the set I above, we argue that all edges of S_t must be crossed by the same number of edges of G, and must in fact form a cut of G. Hence, we get

$$t \cdot \text{max-cut}(G) \leq \text{max-}\overline{\mathrm{cr}}(G') \leq t \cdot \text{max-cut}(G) + \text{max-}\overline{\mathrm{cr}}(G) < t \cdot (\text{max-cut}(G)+1).$$

This yields max-cut(G) = $\lfloor \text{max-}\overline{\mathrm{cr}}(G')/t \rfloor$. Hence, any α-approximation for maximum rectilinear crossing number yields an α-approximation for MAXCUT. $\square$

With the same argument, we also obtain hardness of approximation for max-cr$^\circ$, which Bald et al. [2] showed to be NP-hard. The reason is that in

the convex setting, too, the "heavy obstacle" splits the vertex set into a "left" and a "right" side.

Corollary 1 *Given a graph G, max-cr$^\circ(G)$ cannot be approximated better than* MAXCUT.

Next we consider the *weighted* topological case, which formally is defined as follows. For a graph G with positive edge weights $w\colon E \to \mathbb{Q}_{>0}$ and a drawing D of G, let $\mathcal{X}$ be the set of (unordered) pairs of edges that cross in D, let wt-cr$(D) = \sum_{\{e,e'\}\in\mathcal{X}} w(e) \cdot w(e')$ be the weighted crossing number of D, and let max-wt-cr(G) be the maximum weighted crossing number of G, that is, the maximum of wt-cr(D) over every drawing D of G. We refer to the problem of computing this number for a given graph as *weighted crossing maximization*.

Compared to the rectilinear and the convex case above, the difficulty of the topological case is that an obstacle (such as the heavy star above) does not necessarily separate the vertices into "left" and a "right" groups any more. Instead, our new obstacle separates the vertices into an "inner" group and an "outer" group, which allows us to reduce from a cut-based problem.
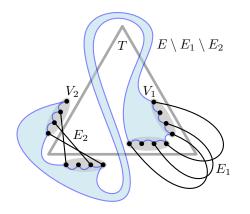
Our new starting point is the NP-hard problem 3MAXCUT [27], which is the special case of MAXCUT where the input graph is required to be 3-regular.

Theorem 3 *Given an edge-weighted graph G and a rational number $c > 0$, it is NP-complete to decide whether* max-wt-cr$(G) \geq c$.

Proof: Clearly, topological crossing maximization is in $\mathcal{NP}$ since we can guess a rotation system for the given graph and, for each edge, the ordered subset of the other edges that cross it. In polynomial time, we can then check whether (a) the weights of the crossings sum up to the given threshold c, and (b) the solution is feasible, simply by realizing the crossings via dummy vertices of degree 4 and testing for planarity of the so-modified graph.

To show NP-hardness, we reduce from 3MAXCUT. Given an instance of 3MAXCUT, that is, a 3-regular graph G and an integer $k > 0$, we construct an instance of topological crossing maximization, that is, a weighted graph G' and a rational number $c' > 0$ such that G has a cut crossed by at least k edges if and only if G' has a drawing with weighted crossing number at least c'. Let G' be the disjoint union of G with edges of weight 1 and a single triangle T with edges of (large) weight t. Let n be the number of vertices and m the number of edges of G. Due to the 3-regularity of G, we have $m = 3n/2$. We set $t = 9n^2/8$ and $c' = t(2m + k)$.

Let (V_1, V_2) be a solution of 3MAXCUT, that is, a cut of G crossed by k edges. We need to show that this implies max-wt-cr$(G') \geq c'$. We construct a drawing D' of G' as in Fig. 3. For $i \in \{1, 2\}$, let E_i be the edge set of $G[V_i]$. We can route the edges of G such that each of the k edges in the cut crosses all three edges of T and each of the $m - k$ edges in $E_1 \cup E_2$ crosses exactly two edges of T. Hence, max-wt-cr$(G') \geq$ max-wt-cr$(D') \geq t(3k + 2(m - k)) = c'$ as desired.
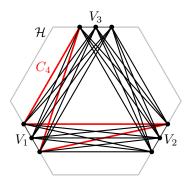
Figure 3: Given a 3-regular graph G, a drawing of $G' = G+T$ with many crossings yields a large cut of G if the edges of the triangle T have much larger weight than the edges of G. The edges (in the light blue region) that cross T three times are in the cut.

Figure 4: A crossing-maximal drawing of the complete tripartite graph $K_{k,k,k}$ for $k = 3$.

Conversely, let D' be any drawing of G' and let $c' = \text{wt-cr}(D')$. We need to show that $G = G' - T$ has a cut that is crossed by at least $\lfloor c'/t \rfloor - 2m$ edges. As incident edges cannot cross, the triangle T of G' must be drawn in D' without self-crossings. Since $\text{max-cr}(G) \leq \binom{m}{2} = \binom{3n/2}{2} < 9n^2/8 = t$, we have that $x = \lfloor c'/t \rfloor$ is the number of crossings between edges of G and edges of T. Let V_1 be the set of vertices of G in the interior of T, and let $V_2 = V \setminus V_1$. Consider the cut (V_1, V_2), and let k be the number of edges crossing this cut. Each of these k edges contributes at most 3 to x, and each of the $m - k$ edges that lie in $G[V_1]$ or $G[V_2]$ contributes at most 2 to x (as in Fig. 3). Hence, $x \leq 3k + 2(m - k) = k + 2m$, and $k \geq x - 2m = \lfloor c'/t \rfloor - 2m$ as desired.

Clearly, our reduction takes polynomial time.   □

In Appendix B we argue why it is unlikely that weighted crossing maximization admits a PTAS.

We now set out to strengthen the result of Theorem 3; we want to show that even the maximum *unweighted* crossing number is hard to compute. Observe that in the above proof, the given graph G from the 3MAXCUT instance remained unweighted, but we required a heavily weighted additional triangle T. Our goal is now, essentially, to substitute T with an unweighted structure that serves the same purpose. Unfortunately, due to the large number of crossings of this new structure, we cannot make any statement about non-approximability of the unweighted case. The naïve approach of simply adding multiple unweighted triangles does not easily work since already the entanglement of the triangles among each other is non-trivial to argue.

Theorem 4 *Given a graph G and an integer $c > 0$, it is NP-complete to decide*

whether max-cr$(G) \geq c$.

Proof: The membership in $\mathcal{NP}$ follows from Theorem 3. To argue hardness, given an instance G of 3MaxCut, we construct an unweighted graph G'—the instance for computing max-cr(G')—as the disjoint union of G and a complete tripartite graph $K := K_{k,k,k}$ with k vertices per partition set, $k > \sqrt{9/8 \cdot n}$. A result of Harborth [11] yields max-cr$(K) = \binom{3k}{4} - 3\binom{k}{4} - 6k\binom{k}{3} \in \Theta(k^4)$.

We first analyze a crossing-maximal drawing of K; see Fig. 4. Consider a straight-line drawing "on a regular hexagon $\mathcal{H}$". Let V_1, V_2, V_3 be the partition sets of K and label the edges of $\mathcal{H}$ cyclically $1, 2, \ldots, 6$. Place V_i, $1 \leq i \leq 3$, along edge $2i$ of $\mathcal{H}$. We claim that max-cr(K) is achieved by this drawing. In fact, the arguments are analogous to the maximality of the naïve drawing for complete bipartite graphs on two layers: a 4-cycle can have at most one crossing. In the above drawing, every 4-cycle has a crossing. On the other hand, any crossing in *any* drawing of K is contained in a 4-cycle.

Intuitively, when thinking about shrinking the sides 2, 4, 6 in $\mathcal{H}$, we obtain a drawing akin to T in the hardness proof for the maximum weighted crossing number. It remains to argue that there is an optimal drawing of the full graph G' where K is drawn as described. Consider a drawing realizing max-cr(G') and note that any triangle in K is formed by a vertex triple, with a vertex from each partition set. Pick a triple $\tau = (v_1, v_2, v_3) \in V_1 \times V_2 \times V_3$ that induces a triangle T_τ with the maximum number of crossings with G among all such triangles. Now, redraw K along T_τ according to the above drawing scheme such that, for $i = 1, 2, 3$, it holds that (a) all vertices of V_i are in a small neighborhood of v_i and (b) any edge $w_i w_j$ with $w_i \in V_i, w_j \in V_j$ for some $j \neq i$ crosses exactly the same edges of G as the edge $v_i v_j$. Our new drawing retains the same crossings within G', achieves the maximum number of crossings within K, and does not decrease the number of crossings between K and G; hence it is optimal. In this drawing, K plays the role of the heavy triangle T in the hardness proof of the weighted case, again yielding NP-hardness. $\square$

5   Conclusions and Open Problems

We have considered the crossing maximization problem in the topological, rectilinear, and convex settings. In particular, we disproved a conjecture of Alpert et al. [1] that the maximum crossing number in the latter two settings always coincide. On the other hand, we propose a new setting, the "separated drawing" setting, and ask whether for every bipartite graph the maximum rectilinear, maximum convex, maximum separated, and maximum separated convex crossing numbers coincide.

Concerning complexity, we have shown that the maximum rectilinear crossing number is APX-hard and the maximum topological crossing number is NP-hard. A natural question then is whether the maximum topological crossing number is also APX-hard. We have shown this to be true in the weighted topological case. It also remains open whether rectilinear crossing maximization is

in $\mathcal{NP}$, which would have followed if the rectilinear and convex setting were equivalent as conjectured by Alpert et al.. A reviewer of an earlier version of this paper was wondering about the complexity of maximum crossing number for planar graphs. For planar graphs, MaxCut is tractable and our hardness arguments no longer apply, leaving open the question of the complexity of computing the maximum crossing number for this graph class. Another reviewer recalled that minimizing crossings with a fixed rotation scheme is NP-hard [17] and wondered about the corresponding maximization problem.

Other intriguing crossing maximization problems remain open: apart from the two classic problems that we mentioned above—Conway's Thrackle Conjecture and Ringeisen's Subgraph Problem—we are interested in the separation of the rectilinear and the separated convex setting for bipartite graphs.

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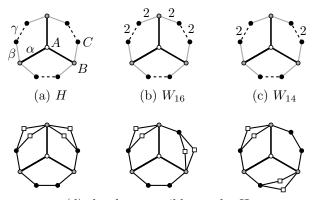
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Appendix

A Counterexamples with 12 Vertices

Here we provide three similar graphs with 12 vertices and 16 edges violating the convexity conjecture (Conjecture 1). Note that each graph is planar and has maximum degree 4 or 5. This shows that the convexity conjecture is false also for some natural graph classes such as planar graphs or graphs with maximum degree at most four. Our proof is based on a relatively long case-analysis. Manfred Scheucher independently verified by a computer search that these three graphs indeed violate the convexity conjecture. Moreover, his unsuccessful attempts to find a smaller counterexample with the use of computer search support our feeling that the convexity conjecture might hold for all graphs on at most 11 vertices.

Let H be the graph with 10 vertices and 12 edges from the previous subsections. We distinguish three types of vertices: A-vertices, B-vertices, and C-vertices. The central vertex is the only A-vertex. The three vertices v_0, v_3, v_6 of H connected to the central vertex are the B-vertices and the six vertices in H of degree 2 are C-vertices. The three edges adjacent to the A-vertex are called α-edges, the six edges connecting a B-vertex with a C-vertex are called β-edges and the remaining three edges connecting independent pairs of C-vertices are called γ-edges. The nine B- and C-vertices are *cycle vertices*, and the nine β- and γ-edges forming a 9-cycle are called *cycle edges*.

(a) H (b) W_{16} (c) W_{14}

(d) the three possible graphs H_{16}

Figure 5: (a) The graph H with one A-vertex (white), three B-vertices (gray), six C-vertices (black), three α-edges (thick), six β-edges (gray), and three γ-edges (dashed); (b) the weighted graph W_{16}; (c) the weighted graph W_{14}; (d) the three possible graphs H_{16} with split vertices as white squares.

We choose H_{16} as a graph obtained from H by selecting a pair of non-adjacent C-vertices and perfoming a single vertex split on each of these. Since the C-vertices have degree 2 in H, each split introduces one vertex and two

edges. Thus, the graph H_{16} has 12 vertices and 16 edges. Up to isomorphism, H_{16} is one of the three planar graphs depicted in Fig. 5 (d). It corresponds to the weighted graph W_{16} (see Fig. 5 (b)), which is the graph H with edge weights, where two of the β-edges have weight two, two of the γ-edges have weight two, and the remaining eight edges have weight one. Further, let W_{14} be the same weighted graph with the exception that all the β-edges have weight one; see Fig. 5 (c). Thus, only two γ-edges have weight two, otherwise the edges in W_{14} have weight one. Up to isomorphism, the graph W_{14} is uniquely determined regardless of the graph H_{16}.

We now give two lemmas used in the proof that H_{16} is a counterexample for the convexity conjecture. We say that in a drawing an edge e *avoids* another edge f if e and f are independent and non-crossing.

Lemma 5 *In any drawing of H, any cycle edge avoids another edge.*

Proof: Let e be a cycle edge. Then there is a 5-cycle Z consisting of edges non-adjacent to e (the cycle Z contains two α-edges, two β-edges and one γ-edge). There must be two consecutive vertices of Z lying on the same side of the edge e in the considered drawing. The edge connecting these two vertices is avoided by e. $\square$

Lemma 6 *In any convex drawing of H, any cycle edge of span $s \in \{0, 1, 2\}$ avoids at least $6 - 2s$ cycle edges.*

Proof: Let e be a cycle edge of span s. We first give an upper bound on the number of edges incident to e. The edge e is incident to exactly one cycle edge at each of its two vertices. Since every cycle edge intersecting e is incident to one of the s cycle vertices of the "span interval" of e, at most $2s$ cycle edges intersect e. Altogether, at most $2 + 2s$ cycle edges different from e have a point in common with e. Since there are eight cycle edges different from e, the edge e avoids at least $8 - (2 + 2s) = 6 - 2s$ cycle edges. $\square$

We now fix a convex drawing D of H_{16} maximizing the number of crossings and with twins placed next to each other. It gives a convex drawing of the
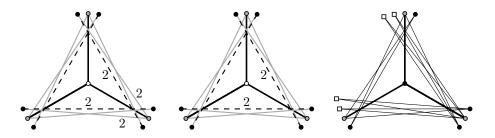

Figure 6: Non-convex drawing of W_{16} of weighted loss 13 (left), of W_{14} of weighted loss 11 (middle), and of H_{16} with loss 13 with split vertices as white squares (right).

weighted graph W_{14} in the way described above. Since there is a non-convex drawing of H_{16} with loss 13 as shown in Fig. 6, we need to show that the loss of the drawing D of H_{16} is at least 14. From Lemma 5, applied on the drawing D, the loss of H_{16} and the weighted loss of the corresponding drawing of W_{14} differ by at least two. Thus, it suffices to show that the weighted loss of the drawing of W_{14} given by the drawing D is at least 12. Before proving it, we fix some notation.

The nine B- and C-vertices of W_{14} are denoted by $1, 2, \ldots, 9$ in the counterclockwise order in which they appear in the drawing D. Without loss of generality we may assume that the B-vertices are 1, j, k, where $1 < j < k \leq 9$ and the vertex A lies in the counterclockwise interval $(k, 1)$. In other words, the three vertices $k, A, 1$ appear in this counterclockwise order along the convex hull of the vertex set of D.

In the following, if a β-edge avoids a γ-edge in D, we say that there is a *$\beta\gamma$-avoidance*. Similarly we define *$\beta\beta$-avoidances* as avoidances of pairs of the β-edges, and *$\gamma\gamma$-avoidances* as avoidances of pairs of the γ-edges. Finally, *$\alpha*$-avoidances* are avoidances of pairs of edges that contain an α-edge.

Lemma 7 *There are at least $2(k-2)$ $\alpha*$-avoidances.*

Proof: Let X be the set of the $k - 2$ vertices $2, 3, \ldots, k - 1$. If a cycle edge connects two vertices of X then it avoids the α-edges $A1$ and Ak. If a cycle edge is incident to one of the vertices of X then it avoids one of the α-edges $A1$ and Ak. Thus, for each cycle edge e, the number of α-edges avoided by e is at least as big as the number of incidences of e with X. Since the total number of incidences of the vertices in X with the cycle edges is exactly $2|X| = 2(k - 2)$, the number of $\alpha*$-avoidances is at least $2(k - 2)$. □

We now distinguish six cases.

Case 1: $k = 3$ and there is no $\gamma\gamma$-avoidance.   In this case the β-vertices are 1, 2, 3 and the three γ-edges are 47, 58, and 69. Each of them has span 2 and therefore, by Lemma 6, it avoids at least two of the β-edges. Since the total weight of the γ-edges is 5, the $\beta\gamma$-avoidances have total weight at least 10. Since there are at least two $\alpha*$-avoidances by Lemma 7, we get that the weighted loss of the drawing of W_{14} (i.e., the total weighted number of avoidances) is at least 12 in Case 1.

Case 2: $k = 3$ and there is a $\gamma\gamma$-avoidance.   The β-edge β_4 containing the vertex 4 has the five C-vertices 5, 6, 7, 8, 9 on the same side and therefore avoids two γ-edges. Since any two γ-edges have total weight three or four, it follows that β_4 appears in $\beta\gamma$-avoidances of total weight at least three. By symmetry, β_9 also appears in $\beta\gamma$-avoidances of total weight at least three.

The edge β_5 has the four C-vertices 6, 7, 8, 9 on the same side and therefore avoids at least one γ-edge. By symmetry, β_8 also avoids at least one γ-edge.

Summarizing, the edges $\beta_5, \beta_6, \beta_8, \beta_9$ appear in $\beta\gamma$-avoidances of total weight at least $3 + 1 + 1 + 3 = 8$. Additionally, there are two $\alpha*$-avoidances and there

is a $\gamma\gamma$-avoidance which is necessarily of weight two or four. It follows that the avoidances have total weight at least $8 + 2 + 2 = 12$.

Case 3: $k = 4$ and there is no $\gamma\gamma$-avoidance. Without loss of generality, we assume that the B-vertices are $1, 3, 4$. Then the γ-edges are 27, 58, 69. The edge 58 avoids the β-edges β_2 and β_9. Similarly, the edge 69 avoids the β-edges β_2 and β_5. Since the edges 58 and 69 have total weight three or four, they appear in $\beta\gamma$-avoidances of total weight at least $3 \cdot 2 = 6$.

The edge β_2 avoids either the two β-edges incident to the C-vertex 1 or the two β-edges incident to the C-vertex 4. Thus, there are at least two $\beta\beta$-avoidances. Also, there are at least four $\alpha*$-avoidances by Lemma 7. Altogether, the avoidances have total weight at least $6 + 2 + 4 = 12$.

Case 4: $k = 4$ and there is a $\gamma\gamma$-avoidance. As in Case 3, we assume that the B-vertices are $1, 3, 4$. The edge β_2 avoids two of the three γ-edges, which gives two $\beta\gamma$-avoidances of total weight three or four. The edge β_2 also avoids at least one β-edge connecting one of the vertices 1 and 4 with one of the vertices in the interval $[5, 9]$.

Since there is a $\gamma\gamma$-avoidance, the interval $[5, 9]$ contains the vertices of a γ-edge γ_0 of span at most 1. (Note that there could additionally be $\gamma\gamma$-avoidances with γ-edges of span at least 2.) The edge γ_0 avoids at least one γ-edge and at least two β-edges different from β_2 (for example, if γ_0 connects vertices 6 and 8, it avoids the β-edges β_5 and β_9). The $\gamma\gamma$-avoidance has weight two or four, and the two $\beta\gamma$-avoidances have total weight at least two.

Summarizing, avoidances involving no α-edge have total weight at least $3 + 1 + 2 + 2 = 8$. Since there are at least four $\alpha*$-avoidances by Lemma 7, all avoidances have total weight at least $8 + 4 = 12$.

Case 5: $k = 5$. The two β-edges with both vertices in the interval $[1, 5]$ have span at most 2, and therefore appear in at least four avoidances among cycle edges. There are at least six $\alpha*$-avoidances by Lemma 7. It follows that there are at least ten avoidances.

Since each of the two γ-edges of weight two avoids another edge, there are at least two avoidances of weight two or an avoidance of weight four. We conclude that all the avoidances have total weight at least $10 + 2 = 12$.

Case 6: $k \geq 6$. Suppose first that all nine cycle edges have span three. Then the cycle edges form the cycle 162738495, The B-vertices are 1, 4, 7, the γ-edge 26 avoids the two α-edges $A1$ and $A7$, and each of the other eight cycle edges avoids exactly one of the α-edges $A1$, $A4$, $A7$. Thus, there are ten $\alpha*$-avoidances. Since each of the two γ-edges of weight two appears in at least two avoidances, the total weight of avoidances is at least $10 + 2 = 12$.

Suppose now that there is a cycle edge with span smaller than three. Then this edge avoids at least two cycle edges. Additionally there are at least eight $\alpha*$-avoidances. Altogether there are at least $2 + 8 = 10$ avoidances. Since each

of the two γ-edges of weight two appears in some avoidance, all the avoidances have total weight at least $10 + 2 = 12$.

B   It is Unlikely that Weighted Crossing Maximization Admits a PTAS

Due to the additive term $2m$ between the size of a cut and max-wt-cr(G) (see the definition of c' in the proof of Theorem 3), the existence of a PTAS for weighted crossing maximization does not directly imply a PTAS for 3MaxCut. A PTAS for weighted crossing maximization would, however, give us a very good estimation of the quantity $q = 2m + \text{max-cut}(G)$. Since G is 3-regular, we know that $2m/3 \leq \text{max-cut}(G) \leq m$. Hence, assuming a $(1 - \varepsilon)$-approximation for max-cut(G) some $\varepsilon > 0$, the ratio between the smallest and the largest possible value of q is $(8m/3 - \varepsilon)/(3m) = 8/9 - \varepsilon' = 0.\overline{8} - \varepsilon'$ for some $\varepsilon' \leq \varepsilon$. This would be the approximation ratio of an algorithm for 3MaxCut based on a hypothetical PTAS for weighted crossing maximization.

3MaxCut is APX-hard, but with 0.997 the best known inapproximability ratio [4] is too large to yield a contradiction to the existence of a PTAS for weighted crossing maximization. However, to the best of our knowledge, the best approximation algorithm for 3MaxCut is the semidefinite program of Goemans and Williamson [9] for general MaxCut. Its approximation ratio is ≈ 0.878, and any improvement beyond this factor, even for the special case of 3-regular graphs, would be rather unexpected.