Journ@l Electronique d'Histoire des Probabilités et de la Statistique

Electronic Journ@l for History of Probability and Statistics

Vol 4, n°1; Juin/June 2008

www.jehps.net

# Alfred J. Lotka and the Mathematics of Population

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**Abstract**

Alfred J. Lotka, considered as one of the founders of mathematical demography, wrote seminal articles and books from 1907 to his death in 1949. He elaborated in particular the concepts of stable age-distribution of a population and of natural rate of increase of a population. His scientific work is extensive and various: he wrote on malaria epidemiology, on population forecasts, on sterility in American marriages, on the extinction of families, on longevity, on the law of urban concentration, etc. With Louis I. Dublin, who was then also statistician at the Metropolitan Life Insurance Company, in New York, he published in 1930 the first edition of *The money value of a man* and, in 1936, the first edition of *Length of Life. A Study of the Life Table*. Lotka's contribution to the mathematical demography is summarized in his *Théorie analytique des associations biologiques*, a book he published in French.

**Résumé**

Considéré comme un des fondateurs de la démographie mathématique, Alfred J. Lotka, écrivit des articles et des livres majeurs de 1907 à sa mort en 1949. Il élabora en particulier les concepts de distribution par âge stable d'une population et de taux naturel d'accroissement de la population. Son œuvre scientifique est étendue et variée : il écrivit sur l'épidémiologie de la malaria, sur les projections de population, sur la stérilité dans les mariages américains, sur l'extinction des familles, sur la longévité, sur la loi de concentration urbaine, etc. Avec Louis I. Dublin, qui était alors également statisticien à la Metropolitan Life Insurance Company, à New York, il publia en 1930 la première édition de *The money value of a man* et, en 1936, la première édition de *Length of Life. A Study of the Life Table*. La contribution de Lotka à la démographie mathématique est résumée dans sa *Théorie analytique des associations biologiques*, livre qu'il a publié en français.

A demographer and statistician, Alfred J. Lotka is considered as one of the founders of mathematical demography. All authors who have investigated this field refer systematically to the work of Lotka. For example, Nathan Keyfitz [1977], who was at that time professor at Havard University, begins the preface of his *Introduction to the Mathematics of Population*, by a reference to Lotka and his book published in French in 1939. Ansley J. Coale [1972], who was for sixteen years director of the Office of Population Research in Princeton, declared in his book *The Growth and Structure of Human Populations. A Mathematical Investigation* (1972) that "[his] greatest intellectual debt [was] to A. J. Lotka".

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Lotka spent most of his working life as a statistician at the Metropolitan Life Insurance Company; in spite of a non academic career, his work was very influential in the academic world. The mathematical relations he demonstrated seemed essentially theoretical but they were used to estimate the unknown demographic parameters of populations, in developing countries.

## 1. A Career as a Statistician

Alfred James Lotka was born in 1880 in Lemberg, located at that time in Austria, and now in Ukraine (Lviv or Lvov). He studied physics and chemistry at the University of Birmingham and then in Leipzig, where a "center of the new field of physical chemistry, a union of physics and chemistry emphasizing the use of thermodynamic principles in chemical systems" [Kingsland, 1995] had recently been created. He studied physics at Cornell University and became assistant in that field. In 1912, he obtained a D. Sc. in science from the University of Birmingham. Thanks to Raymond Pearl, who was interested by "his mathematical and demographic approach to biological systems", he was linked with the Department of Biometry and Vital Statistics at Johns Hopkins University, but without any salary. In 1924 he was appointed as supervisor of Mathematical Research and as General Supervisor of the Statistical Bureau at the Metropolitan Life Insurance Company in New York. From 1934 to his retirement in 1947 he served in the Metropolitan Life Insurance Company as assistant statistician. He died very soon after his retirement, in December 1949.

Lotka published almost a hundred articles on very various themes, chemistry, physics, epidemiology, biology, etc. About half of the articles were devoted to population issues. He also wrote six books. The first, *Elements of Physical Biology*, was published in 1925. It became then *Elements of Mathematical Biology*. In 1930, as joint author with Louis I. Dublin, who was also a statistician at the Metropolitan Insurance Company, he published *The money value of a man.* In 1934 he published in French the first part of his *Théorie analytique des associations biologiques*, the part named "Principles". The second part, "Analyse démographique avec application particulière à l'espèce humaine" was published five years later, in 1939. In this publication, Alfred J. Lotka presented a synthesis of his contribution to the mathematics of population. It is interesting to note that this fundamental book was translated into Spanish and published in 1969 by the Latin American Center of Demography (Celade), a center operating under the auspices of the United Nations based in Santiago, Chile. An English version of this text has been available only since the year 1998, when the book was translated under the title *Analytical Theory of Biological Populations.*

In 1936, again with Louis I. Dublin, Lotka published the book *Length of life*. He published a revised version of *Length of life* again with Louis I. Dublin, who was at that time second vice-president of the Metropolitan Insurance Company, but also with Mortimer Spiegelman, an assistant statistician in the same institution.

## 2. The Two First Contributions to "Demographic Analysis" in 1907 and 1911

In 1907, Lotka published a very short article in *Science*, reacting to a paper presented by C. E. Woodruff to the American Association for the Advancement of Science on the relation between birth rates and deaths rates. Lotka proposes a "mathematical expression" of this relation when "the .general conditions in the community are constant" and if there is a negligible effect of migration.

If, wrote Lotka, $N_t$ represents the number of individuals in the community at time $t$; and coefficient $c(a)$ the proportion of the $N_t$ individuals of age $a$, the number of individuals aged between $a$ and $a+da$ is given by :

$$N_t c(a)da.$$

Lotka noted $B(t\text{-}a)$ the total birth rate at time ($t$-$a$) and $p(a)$ the probability for an individual of surviving from his birth to the age $a$. It was possible to write:

$$N_t c(a)da = B_{(t-a)} p(a)da$$

and consequently :

$$c(a) = \frac{B_{(t-a)}}{N_t} p(a)$$

The hypothesis of constant conditions in the community results in a geometric progression of $N$ and $B$ over time, at the rate $r$:

$$B_{(t-a)} = B_t e^{-ra} da$$

And $c(a)$ may be given by:

$$c(a) = \frac{B_t}{N_t} e^{-ra} p(a)$$

Lotka noted $b$ the birth rate per head and obtained a relation between age structure, birth rate and probability of survival of an individual from his birth to the age $a$:

$$c(a) = be^{-ra} p(a) \quad [1]$$

As, by definition, the sum of $c(a)da$ from age 0 to the age limit $\infty$ equals 1, Lotka could specify the relation between the birth rate per head $b$, the natural increase per head $r$ and, consequently, the mortality rate per head $d$ which is the difference between $r$ and $b$:

$$\frac{1}{b} = \int_0^\infty e^{-ra} p(a)da \quad [2]$$

Four years later, jointly with F. R. Sharpe, a mathematician and professor at Cornell University, Lotka [1911] published an article in which the two authors introduced the concept of “stable age-distribution of a population”:

> “It seems therefore that there must be a limiting "stable" type about which the actual distribution varies, and towards which it tends to return if through any agency disturbed therefrom”.

Sharpe and Lotka demonstrated that the “stable” distribution was also the “fixed” distribution, corresponding to the equation [1]. After a displacement from the fixed distribution, a population will return to this fixed distribution, if mortality and fertility remain constant.

### 3. Another Measure of the Natural Rate of Increase of a Population

What is the level of fertility necessary to ensure the reproduction of a population? It is possible to answer this question considering the net reproduction rate, an indicator introduced in 1880 by Richard Böckh, from the Statistical Office in Berlin, and popularised by Robert R. Kuczynski [1928].

To measure the fertility of a population the rawest indicator is the *birth rate*, which is the number of births divided by the size of the population. But this indicator is very sensitive to the proportion in the population of women in childbearing age. The *general fertility rate*, the number of births per 1000 women aged 15 to 50 years, is a better measure of the fertility indicator but it is "calculated without regard to the specific age composition of the women in child-bearing age" as pointed out by Kuczynski. A third indicator of fertility is the *total fertility*[1] obtained by summing the *specific fertility rates* by the age of mothers. This indicator gives the number of children per woman, when mortality is not affecting women in child-bearing ages. To know the number of daughters per woman, multiplying total fertility by the proportion of female births among total births gives a fourth measurement of fertility: the *gross reproduction rate*.

The measurement of fertility is one thing, the measurement of replacement is another one. Robert K. Kuczyniski clearly presented this issue in his book *The Balance of Births and Deaths*:

> "The pertinent question is not: is there an excess of births over deaths? but rather: are natality and mortality such that a generation which would be permanently subject to them would during its lifetime, that is until it has died out, produce sufficient children to replace that generation?"[2]

To appreciate reproduction in that way, a fifth indicator may be computed: the *net reproduction rate.* From a life table, it is possible to determine, from an initial 1000 female births, how many women are still alive at age 15-16, 16-17, 17-18, etc. The sum of the female fertility rate for each year of age multiplied by the number of women at each age in the stationary population associated with the life table gives the net reproduction rate of the population considered. If this indicator equals 1, the population remains constant, if it is less than 1 the population will decrease, and if it is more than 1, it will increase.

Lotka proposed an alternative indicator to measure the natural increase of a population. He introduced first the "*mean length T of one generation*". If $R_0$ is the "ratio of total births in two successive generations", i.e. the net reproduction rate, the "standardized" or "stable natural rate of increase" $r$ is given by:

$$(1+r)^T = R_0$$

and

$$r = \sqrt[T]{R_0} - 1$$

The "true" rate of increase is different from the "actual" rate of increase (birth rate minus death rate); it reflects the "biological condition" of a population, leaving aside the particularities of the age structure[3]. This "true" rate of natural increase growth is the rate of growth of the stable population associated with the constant levels of fertility and mortality by age.

The French statistician who introduced Lotka's demographic work in France, Raoul Huson (1931), compared the respective advantages of the net reproduction rate and the natural rate of increase. He considered the first indicator as "a characteristic of an isolated generation and not of a global population". The net reproduction rate indicates how many daughters have been born to a generation of women, taking into account the mortality of these women from their birth to the maximum child-bearing age. Husson criticized this indicator for ignoring time, which is a crucial factor with respect to reproduction. For instance, the speed of reproduction of generations is rapid when the female age at birth is low and slow when it is high. Lotka's natural rate of increase takes time explicitly into account, and consequently the speed of population change. Pierre Depoid (1941), also a French statistician at the *Statistique générale de la France*, in his publication on net reproduction in Europe, compared several indicators and concluded that the best one was the rate of natural increase, suggested by Lotka.

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1 Called now "total fertility rate" (TFR).

2 Robert R. Kuczynski, *The Balance of Births and Deaths.*, Volume I, p. 41-42.

3 Effect of past age-selective immigration for instance.

## 4. “Malthusian” and “Stable” Populations

In his *Théorie analytique des associations biologiques*, Lotka (1939) expressed the number $N(t)$ of individuals living at the instant $t$ as a function of the series of births in the past and of the probabilities for individuals born at an instant $t-a$ of being still alive at the instant $t$:

$$N(t) = \int_0^{\omega} B(t-a)p(a)da$$

If the age distribution of a population is constant, the proportion $c(a,t)$ of people of age $a$ at the instant $t$ is independent of time :

$$c(a,t) = c(a)$$

The birth rate $b$ is given by:

$$b = c(0) = constant$$

As neither the age structure nor mortality vary through time; the death rate $d$ is constant and consequently the rate of growth $r$ is also constant:

$$N_t = N_0 e^{rt}$$

In that case:

$$B_t = B_0 e^{rt}$$

and

$$D_t = D_0 e^{rt}$$

The “numbers in the population, annual births and annual deaths are expressed by the same Malthusian law, the law of compound interest” and this population satisfying the equation [1] and [2] presented above, is named “Malthusian population” and “its characteristics (age distribution, rates of growth, birth, death, etc) will be Malthusian Characteristics”[4].

When the population is stationary (*i.e.* the growth rate is equal to zero), the relation between birth rate, death rate and life expectancy ($e_0$) is given by:

$$b = d = \frac{1}{e_0}$$

The “Malthusian populations” become “Stable populations” when the fertility by age is constant. In that case, if $m(a)$ is the frequency of childbearing at age $a$, the following equation is verified:

$$1 = \int_0^{\infty} e^{-ra} p(a)m(a)da$$

The stable state is the “final state” attained, when fertility and mortality do not change over a very long period.

## 5. Epistemology of Population Dynamics

As explained by Sharon E. Kingsland in *Modeling Nature, Episodes in the History of Population Ecology,* in his book *Elements of Physical Biology* published in 1925 Lotka

[4] The translation by David and Rossert [1998] of Lotka’ *Théorie analytique* is used for each quotation from this text.

(1925) regards the entire world as a system. It is an application, writes Kingsland, of physical principles to biological systems. His "clear exposition of the systems approach […] ha[d] some influence on ecosystem ecology". In the first part of his *Théorie analytique des associations biologiques*, Lotka (1934) still refers to a systemic approach when he asserts that the living species are "in mutual relation one with another":

> "On cannot insist too strongly on the fact that the evolution of a system comprising a certain number of biological species ought to be conceived as a whole. It is not this species or that one which evolves, but the system itself as an entire system." [5]

In a paper presented at the International Population Conference of Paris, in 1927, Lotka estimated that it was necessary to reconcile "empirical" and "rational or formal" methods in quantitative demography. In this text, he mentioned "three fundamental types" of populations: stationary, Malthusian and logistic, and considered the relations between theoretical and actual populations. For him, "the Malthusian type has been almost been achieved in several cases". On the other hand, he considered that the decline of the birth rate in the developed countries justified a special attention to the logistic model.

From an epistemological point of view, the distinction made by Lotka between "Demographic Statistics" and "Demographic Analysis" is worthy of special attention. With observational data that are collected, it is useful, even important, thinks Lotka, to analyse the relations between variables. He gives the example of the numerical relation between density and death rate. The study of such a relation is "of great importance". But it is impossible with observational data to escape from "an empirical framework". This kind of investigation is contained in the field of Demographic Statistics. Another completely different way to consider population issues is to focus on the "necessary relations […] between the quantities serving to describe the state and changes in the state of population". This approach falls within the province of Demographic Analysis. This discipline focuses on the "linkages" between demographic quantities. What is the utility of this approach? These populations are theoretical. The necessary relations may be demonstrated after making strong assumptions. Reality is much more complex. Lotka is aware of the complexity of the real world but he considers that demographic analysis helps to a better understanding of relations between demographic quantities:

> "The conditions that present themselves in an actual population are always excessively complicated. Whoever has failed to grasp clearly the necessary relations among the characteristics of a theoretical population subject to simple hypotheses, will certainly be unable to manage in the much more complicated relations that exist in a real population."[6].

His grounding in physics led Lotka to compare demographic analysis and classical rational physics. In this science, it is necessary to assimilate difficult concepts such as entropy, space-time, quanta, etc. This is not the case for demography, whose concepts are "ideas familiar from our daily lives: the size of a population, births, ages, the number of descendants in the first generation, the second, etc." In the case of classic rational physics, writes Lotka, "it is possible to formulate functional relations": the period of a simple pendulum may be expressed as a function of its length and of the force of gravity. A single value may be attributed to this period. It is not the same in demographic analysis. For instance, the relation between weight and size, in a population, is a "probabilistic relation" and not a "simple functional relation". He concludes that:

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[5] Alfred J. Lotka, *Analytical Theory of Biological Populations*, p. 3.

[6] *Ibid*, p. 50.

"[t]hese two circumstances, the large number of variables, and the probabilistic nature of the relations that link them, lend a unique character to demographic analysis."[7]

## 6. Other Demographic Work

In the field of demography, the scientific work of Lotka is extensive and various. For example, he published papers on malaria epidemiology (with F. R. Sharpe), on population forecasts, on the size of American families in the eighteenth century, on sterility in American marriages, on the spreads of generations, on orphanhood, on the extinction of families, on longevity, on the construction of Life Tables, on the law of urban concentration, etc. Two important books resulted from his collaboration with Louis I. Dublin: *The Length of Life* and *The money value of a man.*
Dublin and Lotka began their book *Length of life* by an interesting comment on the collective character of personal events:

> "Human life is a very personal affair. It is your life and mine and that of your neighbor. Each life is a separate ands distinct entity; yet there is a common stamp upon all."

The two authors were interested in the "impersonal aspect of longevity"; they were not concerned by individual lives but by "life in aggregate". Dublin and Lotka analysed long-term mortality trends as consequences of scientific and industrial progress and considered the geographic variations of longevity in the United States. They focused also on the biological factors affecting longevity and presented some applications of the life tables to economic problems.
The linkage between demographic analysis and economic issues is much stronger in the book *The money value of a man* published in 1930 by Dublin and Lotka. They showed with great simplicity how to obtain an economic measure of the value of a human life:

> "Every individual who insures himself for the protection of the members of his family has in mind providing them, in the event of his death, with a sum of money that shall, as nearly as possible, take the place of his contribution to them while living. Human life in this sense my be equated to a sum of money."

As the authors said themselves, the "aim and purpose of this book [was] from first to last essentially and fundamentally practical", that is to say, they wanted to specify the economic loss of a wage-earner, often a bread-winner for a family during a part of his life, and to consider the possibility of securing this loss. In their book, Dublin and Lotka presented tables of the "economic value of a man" by age, depending on his class of earnings, in the case of standard mortality (mortality prevailing in the United States as a whole). They constructed another set of tables for people whose "mortality [was] above normal" (with an excess of mortality of 40%, 75% and 125%). The developments of this book were, of course, of great interest for the professionals of the Life Insurance Company or others insurance companies, but they show also that the construction of life tables may lead to practical applications.

## 7. An Application of the Theoretical Populations

In the mid 1950s, the United Nations developed model life tables to prepare population projections for countries where mortality by age and sex was unknown. The model life tables were constructed through regression analysis, measuring the relation between the mortality

[7] *Ibid.*, p. 51.

rate at a particular age and the mortality rate at the following age. With the estimations obtained, it was possible to construct theoretical life tables, associated with different levels of infant mortality, the health situation of a country being defined precisely through infant mortality. The construction of model life tables led to intense scientific debate on how to estimate the coefficients between successive mortality rates; other methods for constructing model life tables have since been proposed.

Once model life tables have been constructed, it is possible to associate different levels of mortality, theoretical populations and particularly stable populations. The demographers Ansley J. Coale and Paul Demeny proposed in their *Regional Model Life Tables and Stable Populations* (first edition in 1966 and second in 1983) various series of model life tables and they computed stable populations for different levels of mortality and different growth rates. First they calculated four families of model life tables, on a "regional" basis: "East" model life tables (using tables for eastern countries), "North" model life tables (Norway, Sweeden, Iceland), "South" model life tables (Spain, Italy, Portugal) and "West" model life tables (other countries). Coale and Demeny presented a series of life tables, for each region and for a life expectancy ranging from 20 to 80 years. Using Lotka's formulas (in particular the equation [1]) they also constructed series of stable populations, for different rates of population growth and various demographic indices.

These tables could be used, and have been used, to estimate the demographic parameters of developing countries with very limited statistical data.

## 8. "Quasi-stable" and "Nearly stable" Populations

One of the demographers who paid much attention to Lotka's work was Jean Bourgeois-Pichat, who worked for some time at the Population Division of the United Nations, in New York, and was director of the National institute for demographic studies (INED) in the 1960s. In his publications The *Concept of Stable Population* [United Nations, 1968] and *La dynamique des populations* [1994], he recalled that Lotka's contribution to demographic analysis lacked practical applications until the construction of model life tables launched by the Population Division of the United Nations and the Office of Population Research (OPR) in Princeton. Bourgeois-Pichat [1973] also presented new developments of the demographic analysis initiated by Lotka:

> "Theoretical considerations have played their part. The concept of stable population evolved before the last war by Alfred J. Lotka has been debated afresh and two new concepts have emerged –those of "quasi-stable" and "nearly stable" populations. The former are empirical populations, that is population arrived at by maintaining the fertility constant and varying the mortality rate over the area defined by the sample tables just referred to. It has been observed that these populations differ little from stable populations.
>
> Nearly stable populations are populations that are invariable in their composition by age. It has been shown that, at any given moment, the relations existing between the various characteristics of these populations are the same as those in the case of stable populations."[8]

For researchers such as Bourgeois-Pichat who investigated the relations central to demographic analysis as defined by Lotka, the aim was to apply the theoretical results to the actual situations of countries with deficient demographic data.

[8] Jean Bourgeois-Pichat, *Main Trends in Demography*, p. 60.

## 9. Conclusion

Lotka is well-known for his contribution to mathematical demography. The references to the fundamental articles he wrote on this subject and to the *Théorie analytique des associations biologiques* are frequent in the literature, even though his book published in 1939 for the second part, was very difficult to find, until its translation in English in 1998.
Employed as a statistician at the Metropolitan Life Insurance Company until his retirement, Lotka was also a major demographer and he was recognized as such by the Academic community. This is testified by his election as President of the Population Association of America (PAA) in 1938-1939. His contribution to various subjects of population science is also important (he wrote on population growth, on the length of life, on the value of human life, etc.).
As regards the epistemology of demography and more generally the application of mathematical reasoning to human and social issues, Lotka was concerned as much by the rigour of the demonstrations as by the necessary recourse to approximation in the social sciences.

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