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Wellposedness in the Calculus of Variations

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We consider the stability of solutions of variational problems with respect to perturbations of the integrand, raised by Ulam in [21]. In this paper we prove some results concerning Ulam's problem by using the theory of wellposedness. We consider the notion of wellposedness introduced in [23] and we deal with perturbations of the integrands related to variational convergence. Moreover some criteria to obtain variational convergence of sequences of non-convex integrals are given.

Keywords: calculus of variations, non convex integrals, wellposedness, variational convergence

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1. Introduction

Wellposedness of a problem means, roughly speaking, existence and uniqueness of its solution and also the continuous dependence of the solution on problem's data. We are interested in wellposedness of global minimization problems of functionals of the Calculus of Variations. There are only few wellposedness results in the classical theory, see for example Chapter VIII in [10]. There Tikhonov wellposedness of the Lagrange problem is characterized by means of regularity of the value function. Moreover, Hadamard wellposedness is shown to be linked to topologies on the space of integrands coming from epi-convergence or Mosco-convergence, in the case of convex functionals.

In this paper we study the stability problem of solutions of variational problems with respect to perturbations of the integrand (with fixed boundary data) arisen by Ulam in [21], using wellposedness theory, in particular the notion of wellposedness by perturbation introduced in [23]. This problem of Ulam has been considered by Bobylev [5] and Sychev [20], for local minimizers. Bobylev gives an example of an integral such that, for small perturbation of its integrand in the uniform metric, existence of minimizers fails. Both Bobylev and Sychev show that by imposing further regularity conditions on the integrand, stability results are obtained.

In this paper we consider a one-dimensional integral functional with a unique minimizer. For perturbations of the integrand which do not involve the derivative we show that their variational convergence, which is weaker than uniform convergence, suffices to obtain wellposedness. We are interested in the strong convergence in the Sobolev space of any asymptotically minimizing sequence to the minimizer of the unperturbed problem (in particular, of any minimizing sequence). We remark that by using epi-convergence only the weak convergence of minimizers is achieved (see e.g. [8]). Moreover we do not require existence of minimizers of perturbed problems.

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Recent results of A. D. Ioffe and A. J. Zaslavski [14] show that variational problems are generically wellposed. More precisely, these results imply that for perturbations of the integrands with respect to a topology coming from the uniform convergence modulo given growth, there is a generic subset G in the space of the integrands such that for each $f \in G$, the corresponding integral functional is wellposed with respect to changes of the integrands in the same topological space.

In this paper we prove that Tikhonov wellposed problems are wellposed with respect to perturbations of the integrand related to variational convergence. Hence the new integrands are not necessarily close to the original one in the uniform norm. Consider, for example, the linear perturbations of the quadratic functional

$$I(u,p_n) = \frac{1}{2}\int_0^1 \dot{u}^2(t)dt + \int_0^1 p_n(t)u(t)dt \quad u \in H_0^1([0,1]),$$

where

$$p_n(t) = \chi_{[0,\frac{1}{n^\beta}]}(t)n^\alpha \text{ in } [0,1], \text{ with } \beta < \alpha < \frac{3}{2}\beta$$

and χ denotes the characteristic function. In this example we are able to show that we have stability under the perturbation p_n of the quadratic functional, even if $\|p_n\|_\infty$ is divergent (see Section 5).

We emphasize that we do not impose any strict convexity assumption on the integrands and that we consider a class of perturbations of the original integrand larger than the one considered in [5] and [20]. Moreover, we establish criteria to obtain variational convergence of integral functionals which are not convex; we point out that so far only characterizations of epi-convergence of convex integral functionals are known (see [2], [6], [8], [10], [16]).

The paper is organized as follows. In section 2 we recall the relevant definitions and notations. In section 3 we present criteria for Tikhonov wellposedness of an integral functional; there the integrands are not necessarily strictly convex. In section 4 we prove wellposedness by perturbations of global minimization problems in the Calculus of Variations . Moreover we detect a notion of convergence in the space of the integrands which implies the variational convergence of integral functionals. In the final section 5 we consider examples; for linear perturbation of the quadratic functional we obtain a complete characterization of wellposedness by perturbation.

2. Definitions and preliminaries

Let $q > 1$, $N \geq 1$ and let $|\cdot|$ be any norm in $\mathbf{R}^N$. Throughout this paper we denote by $W^{1,q}([a,b],\mathbf{R}^N)$ the space of functions $u : [a,b] \to \mathbf{R}^N$, such that both u and $\dot{u}$ belong to L^q. Moreover we write

$$\|u\|_{1,q} = \|u\|_q + \|\dot{u}\|_q \text{ for all } u \in W^{1,q}([a,b],\mathbf{R}^N).$$

Let q' denote the conjugate exponent of q, i.e. $\frac{1}{q} + \frac{1}{q'} = 1$.

If $f = f(t,s,v) : [a,b] \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ we write f_t, f_s, f_v for the partial derivatives and we say that f is a Carathéodory function if:

(1) $f(\cdot,s,v)$ is measurable for every $(s,v) \in \mathbf{R}^{2N}$,

(2) $f(t, \cdot, \cdot)$ is continuous for a.e. $t \in [a, b]$.

In the general setting we consider the problem (X, J) of globally minimizing the proper extended real valued functional

$$J : X \longrightarrow (-\infty, +\infty]$$

defined on the metric space X. Following [23], the given problem is embedded in a family of minimization problems parametrized by the elements of a space P which may be endowed with either a topology or a convergence structure The unperturbed problem (X, J) corresponds to a given parameter value p_0. Given a subset L of P containing p_0 we denote by

$$I : X \times L \longrightarrow (-\infty, +\infty] \tag{2.1}$$

the embedding and define the corresponding value function as

$$V(p) = \inf\{I(x, p) : x \in X\}, \quad p \in L. \tag{2.2}$$

Definition 2.1. We say that $v_n \in X$ is an asymptotically minimizing sequence corresponding to the sequence $p_n \to p_0$ in P if $V(p_n) > -\infty$ and

$$I(v_n, p_n) - V(p_n) \longrightarrow 0 \text{ as } n \to +\infty. \tag{2.3}$$

Definition 2.2. The global minimization problem (X, J) is wellposed by perturbations (with respect to the embedding defined in (2.1)) if $V(p) > -\infty$ for all $p \in L$ and there exists a unique minimum point $u_0 = \operatorname{argmin}(X, I(\cdot, p_0))$ such that for all sequences p_n converging to p_0, every corresponding asymptotically minimizing sequence v_n is convergent to u_0.

(This definition was introduced in [23]). This notion of wellposedness is stronger than the more classical notion of Tikhonov wellposedness and is related to Hadamard wellposedness. We recall these definitions.

Definition 2.3. The problem (X, J) is Tikhonov wellposed iff has a unique global minimum point towards which every minimizing sequence converges.

The concept of Tikhonov wellposedness can be extended to minimum problems without uniqueness of the optimal solution as in the following definition.

Definition 2.4. The problem (X, J) is Tikhonov wellposed in the extended sense iff the $\operatorname{argmin}(X, J)$ is nonempty and every minimizing sequence has a subsequence which converges to some optimal solution.

Hadamard wellposedness deals with the continuous dependence of the unique solution from problem's data. There are many forms of Hadamard wellposedness for the problem (X, J). We are interested in considering a convergence for a sequence of functionals and studying the convergence of every asymptotically minimizing sequence. We introduce now a suitable notion of convergence which implies Hadamard wellposedness under minimal assumptions.

Definition 2.5. Let $J, J_n : X \to (-\infty, +\infty]$. We say that J_n variationally converges to J, $J_n \stackrel{VAR}{\to} J$, iff

(1) $x_n \to x$ implies $\liminf_n J_n(x_n) \geq J(x)$,
(2) for every $x \in X$ there exists x_n such that $\limsup_n J_n(x_n) \leq J(x)$

(For a reference see Section 2, Chapter IV in [10]).

Observe that variational convergence is not a Kuratowski convergence (see [13]) since the limit is not necessarily unique.

We consider also the natural extension of the definition of wellposedness by perturbations given by relaxing the uniqueness requirement for the solution of the unperturbed problem.

Definition 2.6. The global minimization problem (X, J) is wellposed in the extended sense (with respect to the embedding defined in (2.1)) if $V(p) > -\infty$ for all $p \in L$, $\operatorname{argmin}(X, I(\cdot, p_0))$ is nonempty and, for every sequence p_n converging to p_0, every corresponding asymptotically minimizing sequence has a subsequence converging to some optimal solution of the unperturbed problem.

3. Tikhonov Wellposedness

In this paragraph we present criteria concerning Tikhonov wellposedness of the minimization problem for functionals of the Calculus of Variations

$$F(u) = \int_a^b f(t, u(t), \dot{u}(t))dt, \tag{3.1}$$

where $f = f(t, s, v) : [a, b] \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ is a Carathéodory function.

It is well known that if $f(t, \cdot, \cdot)$ is strictly convex and verifies suitable growth conditions, then the minimization problem $(W^{1,q}([a, b]), F)$ is wellposed in the sense of Tikhonov (see [22]). Here we want to obtain a similar result under a milder convexity assumption. In order to do this we need the following definition of strict convexity at a point introduced in [19].

Definition 3.1. Let U be a closed convex subset in $\mathbf{R}^N$. Fix $v_0 \in U$. We say that a continuous function $L : U \to \mathbf{R}$ is strictly convex at v_0 with respect to the set U if $\sum_{i=1}^r c_i L(v_i) > L(v_0)$ for any $v_i \in U$, $v_i \neq v_0$, and $c_i \geq 0$, where $i = 1, ..r$, $r \in \mathbf{N}$, $\sum_{i=1}^r c_i = 1$ with $\sum_{i=1}^r c_i v_i = v_0$.

It is easy to find examples of convex superlinear functions which are strictly convex on U but not elsewhere.

3.1. Lipschitz functions

We start with some existence results concerning minimizers of integral functionals for which we refer to [15]. Given $k > 0$, consider the integral functional (3.1) defined on

$$C_k = \left\{u \in C^{0,1}([a, b]) : u(a) = A, u(b) = B, \|u\|_{C^{0,1}} \leq k\right\} \tag{3.2}$$

endowed with the $W^{1,q}([a, b])$-norm, where $A, B \in \mathbf{R}^N$ and

$$\|u\|_{C^{0,1}} = \|u\|_\infty + \sup_{\{x,y \in [a,b] \,|\, x \neq y\}} \frac{|u(x) - u(y)|}{|x - y|}.$$

There are several existence results for the problem (3.1), (3.2), which are essentially based on the direct method due to Tonelli [4] (see also [7] and [11]).

In general the minimizers are not Lipschitz continuous. We are interested in obtaining an a priori estimate of the gradient of minima in order to weaken the strict convexity assumption in Corollary 1 of [19] by considering only a local convexity condition.

We need also the following result which follows immediately from Theorem 11.10 in [15], generalized to vector valued functions; in this case we consider the function $\bar{u}(x) = A + \frac{B-A}{b-a}(x-a)$.

Proposition 3.2. *Let $f \in C^1((a,b) \times \mathbf{R}^N \times \mathbf{R}^N)$, let f, f_s, $f_v \in C^0([a,b] \times \mathbf{R}^N \times \mathbf{R}^N)$ and assume $f(t,s,\cdot)$ convex. Then for all $k \geq \|\bar{u}\|_{C^{0,1}}$ the minimum of* (3.1) *on C_k exists.*

The above Proposition allows us to give a criterion for the Tikhonov wellposedness of (3.1), namely we have:

Proposition 3.3. *Let F be as in* (3.1)*, let f be as in Proposition* 3.2*. Assume $f(t,\cdot,\cdot)$ is in C^2 and satisfies the following assumptions:*

(1) *For $i = 1,2$ there exist constants $\gamma_i > 1$ and functions $\phi_i \in L^{r_i}((a,b))$, where $r_1 \geq 1$ and $r_2 > 1$ such that for some suitable exponent q and a positive constant δ, $\gamma_2 < q\frac{r_2-1}{r_2}$ and*

$$\delta|v|^q + \phi_2(t)|s|^{\gamma_2} + \phi_1(t) \leq f(t,s,v) \text{ for a. e. } t,\ s \in \mathbf{R}^N,\ v \in \mathbf{R}^N; \tag{3.3}$$

(2) *$f(t,s,\cdot)$ is convex for a. e. t and every $s \in \mathbf{R}^N$.*

Fix $k \geq \|\bar{u}\|_{C^{0,1}}$ and consider the global minimization problem (C_k, F). If for a.e. $t \in [a,b]$ and for all $s \in \mathbf{R}^N$, $f(t,s,\cdot)$ is strictly convex in the closed ball B_k of radius k, then the problem (C_k,F) is Tikhonov wellposed in the extended sense with respect to the $W^{1,q}([a,b])$-norm.

Moreover, if $N = 1$ and $f(t,\cdot,\cdot)$ is convex for a. e. t, then (C_k,F) is Tikhonov wellposed.

Proof. (Step 1) First we prove that every minimizing sequence u_n, i.e. $F(u_n) \to \inf F(C_k)$, has a weakly convergent subsequence in $W^{1,q}([a,b])$. To do this it suffices to show that $F(u_n)$ is minorized by a coercive function of the norm of u_n in $W^{1,q}([a,b])$. Set $\alpha = \left(\frac{(r_2-1)q}{r_2\gamma_2}\right)'$. By (3.3) we have

$$\begin{aligned}
F(u_n) &\geq \delta \int_a^b |\dot{u}_n(t)|^q dt - \left(\int_a^b |\phi_2(t)|^{r_2} dt\right)^{\frac{1}{r_2}} \left(\int_a^b |u_n(t)|^{\gamma_2 \frac{r_2}{r_2-1}} dt\right)^{\frac{r_2-1}{r_2}} + \int_a^b (\phi_1(t))dt \\
&\geq \delta \int_a^b |\dot{u}_n(t)|^q dt - \|\phi_2\|_{r_2} \left(\int_a^b |u_n(t)|^q dt\right)^{\frac{\gamma_2}{q}} (b-a)^{\frac{1}{\alpha}} + \int_a^b (\phi_1(t))dt \\
&\geq \delta \int_a^b |\dot{u}_n(t)|^q dt - \left[M + N\left(\int_a^b |\dot{u}_n(t)|^q dt\right)^{\frac{\gamma_2}{q}}\right] + \int_a^b (\phi_1(t))dt
\end{aligned}$$

where M, N are suitable constants depending on a, b, A and γ_2. Since $\gamma_2 < q$ the claim follows now from the coercivity of the last term. Thus, a subsequence, which we also denote by u_n, weakly converges to u in $W^{1,q}([a,b])$. Since $u_n \in C_k$ and the subsequence converges uniformly to u, we have that $u \in C_k$ as well. Since F is lower semicontinuous on C_k with respect to uniform convergence, it follows that u belongs to $\operatorname{argmin}(C_k, F)$.

(Step 2) Now we prove the strong convergence of some subsequence of u_n to u in $W^{1,q}([a,b])$. Observe that f and u_n satisfy the hypothesis of Corollary 1 of [19]. Therefore u_n strongly converges to u in $W^{1,1}([a,b])$. To see that we have actually strong convergence in $W^{1,q}([a,b])$ note that

$$|\dot{u}_n(t) - \dot{u}(t)|^q \leq (2k)^q, \qquad \text{for a.e. } t.$$

By the Lebesgue convergence theorem a subsequence of $\dot{u}_n$ converges to $\dot{u}$ in L^q, which proves the result.

Let $N = 1$. Then since f is convex, it follows from an easy generalization of the proof of Theorem 11.9 of [15] that the $\operatorname{argmin}(F)$ is a singleton. We observe that in the proof of this theorem the global strict convexity of f with respect to the last variable can be weakened by local strict convexity in $[-k,k]$. As a matter of fact what is required is the positivity of $\frac{\partial^2}{\partial v^2} f(t,s,v)$ for almost every $t \in [a,b]$, for all $s, v \in [-k,k] \times [-k,k]$, that the hessian matrix of $f(t,\cdot,\cdot)$ is positive semidefinite on an open set containing $[-k,k]\times[-k,k]$ and finally that $\frac{\partial}{\partial s} f(t,\cdot,v)$ is non decreasing in $[-k,k]$, for every v. □

Remark 3.4. We observe that $\gamma_2 < q\frac{r_2-1}{r_2}$ if $r_2 > q'$.

Remark 3.5. The strict convexity of the integrand f with respect to the last variable at the point $v = \dot{u}(t)$ is a necessary condition for the strong convergence of the sequence u_n if we have a suitable growth condition on f, see Theorem 3 of [19].

3.2. The autonomous case

Now we present a result concerning Tikhonov wellposedness of the integral functional

$$G(u) = \int_a^b g(u(t), \dot{u}(t))dt \tag{3.4}$$

defined on the space

$$W = \left\{ u \in W^{1,q}([a,b], \mathbf{R}^N) : u(a) = A, u(b) = B \right\}$$

where $A, B \in \mathbf{R}^N$, endowed with the $W^{1,q}([a,b], \mathbf{R}^N)$-norm. Before proving Tikhonov wellposedness of (W, G) we need a lemma, which provides us some a priori estimates for the minimum point and its derivative. This result derives from Lemma VIII.11 in [10].

Lemma 3.6. *Let G be as in (3.4) with $g, g_v : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ continuous and satisfying*

(1) *$g(s,\cdot)$ is convex, for all $s \in \mathbf{R}^N$;*

(2) *there exists a strictly increasing function $\theta : [0,+\infty) \to \mathbf{R}$ such that $\frac{\theta(|v|)}{|v|} \to \infty$ as $|v| \to \infty$ such that*

$$g(s,v) \geq \theta(|v|) \text{for every } s \in \mathbf{R}^N,\ v \in \mathbf{R}^N; \tag{3.5}$$

(3) *g is locally bounded: for all $r \geq 0$ there exists $M_r \geq 0$ such that*

$$|g(s,v)| \leq M_r \text{ if } |s| + |v| \leq r; \tag{3.6}$$

(4) *g_v is locally bounded: for all $r \geq 0$ there exists $P_r \geq 0$ such that*

$$|g_v(s,v)| \leq P_r \text{ if } |s| + |v| \leq r. \tag{3.7}$$

Let $H = G(\bar{u})$, where $\bar{u}(t) = A + \frac{B-A}{(b-a)}(t-a)$ and consider d such that $\theta(d) \geq \left(\frac{2H}{(b-a)}\right) - \theta(0)$. Then for every $u \in \operatorname{argmin}(W, G)$ the following statements hold:

(1) *Let k be such that $\theta(z) \geq z$ for $z \geq k$ and $A_1 = |A| + k(b-a) + H$, then $\|u\|_\infty \leq A_1$.*

(2) *If R is such that $\theta(h) \geq h\,(1 + M_{A_1+1} + M_{A_1+2d} + 2dP_{A_1+2d})$ whenever $h \geq R$ then $|\dot{u}(t)| \leq R$ for a.e. $t \in [a,b]$.*

Proof. See the proof of Lemma VIII.11 in [10]: there the smoothness of the minimizers is not needed since the required change of variable leading to (37) follows from Corollary 6 of [18]. □

Theorem 3.7. *Let G be defined as in* (3.4). *Consider the global minimization problem (W, G). Under the same assumptions of Lemma* 3.6, *if furthermore, for every s, $g(s,\cdot)$ is strictly convex in the ball of radius R in $\mathbf{R}^N$ we have that the problem (W, G) is Tikhonov wellposed in the extended sense with respect to the $W^{1,1}$- norm.*

Proof. Let u_n be a minimizing sequence. By (3.5) we have that u_n is weakly convergent to some u_0, up to a subsequence, in $W^{1,1}$. Since G is lower semicontinuous with respect to the weak topology in $W^{1,1}$ (Corollario 4.1, [11]), $u_0 \in \operatorname{argmin}(W, G)$ and satisfies $G(u_n) \to G(u_0)$. Hence we can apply Corollary 1 in [19] and we obtain that $u_n \to u_0$ in $W^{1,1}([a,b], \mathbf{R}^N)$. □

4. Wellposedness by perturbations

Given $f = f(t,s,v) : [a,b] \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$, we consider the following embedding

$$I(u,p) = \int_a^b f(t, u(t), \dot{u}(t))dt + \int_a^b p(t, u(t))dt. \tag{4.1}$$

The space of parameters P is the space of all functions $p : [a,b] \times \mathbf{R}^N \to \mathbf{R}$, measurable in the first variable and verifying the following assumptions:

(1) For every bounded subset $B \subset \mathbf{R}^N$ and for every $p \in P$ there exists $k_p \in L^1$ such that

$$|p(t,s_1) - p(t,s_2)| \leq k_p(t)|s_1 - s_2| \text{ for all } s_1, s_2 \in B. \tag{4.2}$$

Moreover, for all $B \subset \mathbf{R}^N$ there exists $M_B \geq 0$ such that

$$\int_a^b |k_p(t)|dt \leq M_B \text{ for all } p \in P. \tag{4.3}$$

(2) There exist $\psi_i \in L^{r_i}((a,b))$, $r_i \geq 1$, for $i = 0,1,2,3$, $\gamma_1 > 1$ and $\gamma_2 > 1$ such that for every $p \in P$

$$p(t,s) \leq \psi_0(t) + \psi_1(t)|s|^{\gamma_1} \tag{4.4}$$

and

$$p(t,s) \geq -\psi_2(t)|s|^{\gamma_2} + \psi_3(t). \tag{4.5}$$

We consider on the space P the following convergence of sequences:

$$p_n \to p_o \text{ iff } I(\cdot, p_n) \stackrel{VAR}{\to} I(\cdot, p_o) \tag{4.6}$$

where the variational convergence is meant with respect to the weak convergence in the Sobolev space $W_0^{1,q}([a,b], \mathbf{R}^N)$ (we remark that the convergence structure on P does not fulfill all axioms of [13]).

In this setting we obtain the wellposedness of our problem with respect to the embedding (4.1).

Theorem 4.1. *Let P be the space of parameters defined as above. Let $q \geq 2$. Consider the embedding $I : W_0^{1,q}([a,b], \mathbf{R}^N) \times P \to (-\infty, +\infty]$ defined in* (4.1) *and fix $p_0 \in P$. Assume that the integrand f is a Carathéodory function, convex with respect to the last variable and verifies the following growth conditions*

$$\delta |v|^q + \phi_2(t)|s|^{\gamma_2} + \phi_3(t) \leq f(t,s,v) \textit{ for a. e. } t,\ s \in \mathbf{R}^N,\ v \in \mathbf{R}^N \tag{4.7}$$

and

$$f(t,0,0) \leq \phi_0(t) \textit{ for a.e. } t \in [a,b]. \tag{4.8}$$

Here δ is a positive constant, the functions $\phi_i \in L^{r_i}((a,b))$ for $i = 0,2,3$, with $r_i = 1$, for $i = 0,3$ and $r_2 > 1$; γ_1, γ_2 are such that $\gamma_1 > 1$ and $1 < \gamma_2 < q\frac{r_2-1}{r_2}$. Furthermore assume that $I(\cdot, p_0)$ is Tikhonov wellposed with respect to the strong topology of $W_0^{1,q}([a,b])$.

Then the problem of minimizing $I(\cdot, p_0)$ is wellposed by perturbations with respect to the embedding (4.1).

Proof. (Step 1) Consider a sequence p_n converging to p_0 in P and v_n a corresponding asymptotically minimizing sequence. We prove that v_n is weakly convergent in $W_0^{1,q}([a,b], \mathbf{R}^N)$. To do this it suffices to verify that the sequence $I(\cdot, p_n)$ is equicoercive with respect to the weak topology and $V(p_n)$ is bounded. Set $\alpha = \left(\frac{(r_2-1)q}{r_2\gamma_2}\right)'$, by (4.5) and (4.7) we have

$$\begin{aligned}
I(u, p_n) &\geq \delta \int_a^b |\dot{u}(t)|^q dt - \left(\int_a^b |\phi_2(t) + \psi_2(t)|^{r_2} dt\right)^{\frac{1}{r_2}} \left(\int_a^b |u(t)|^{\gamma_2 \frac{r_2}{r_2-1}} dt\right)^{\frac{r_2-1}{r_2}} \\
&\quad + \int_a^b (\phi_3(t) + \psi_3(t)) dt \\
&\geq \delta \int_a^b |\dot{u}(t)|^q dt - \|\phi_2 + \psi_2\|_{r_2} \left(\int_a^b |u(t)|^q dt\right)^{\frac{\gamma_2}{q}} (b-a)^{\frac{1}{\alpha}} \\
&\quad + \int_a^b (\phi_3(t) + \psi_3(t)) dt \\
&\geq \delta \int_a^b |\dot{u}(t)|^q dt - \|\phi_2 + \psi_2\|_{r_2} c_p^{\gamma_2} \left(\int_a^b |\dot{u}(t)|^q dt\right)^{\frac{\gamma_2}{q}} (b-a)^{\frac{1}{\alpha}} \qquad (4.9) \\
&\quad + \int_a^b (\phi_3(t) + \psi_3(t)) dt
\end{aligned}$$

for each $u \in W_0^{1,q}([a,b],\mathbf{R}^N)$ and c_p is the Poincaré constant. Since $\gamma_2 < q$ the last term of (4.9) is coercive. Now we prove that $V(p_n)$ is bounded; actually by (4.8) and (4.4) we have

$$V(p_n) \leq \int_a^b f(t,0,0)dt + \int_a^b p_n(t,0)dt \leq \int_a^b (\phi_0 + \psi_0)(t)dt.$$

Thus a subsequence, which we also denote by v_n, is weakly convergent to u_0 in $W_0^{1,q}([a,b], \mathbf{R}^N)$. As proved in Theorem IV.5 in [10], the properties of the variational convergence yields that

$$u_0 \in \operatorname{argmin}(W_0^{1,q}([a,b],\mathbf{R}^N), I(\cdot,p_0)),$$

but, by Tikhonov wellposedness, $\operatorname{argmin}(W_0^{1,q}([a,b],\mathbf{R}^N), I(\cdot,p_0))$ is a singleton and so, for every subsequence of v_n, some further subsequence weakly converges to u_0. This yields the claim.

(Step 2) Now we prove the strong convergence of v_n in $W_0^{1,q}([a,b],\mathbf{R}^N)$. By the characterization of Tikhonov wellposedness in Theorem I.12 of [10], there exists a forcing function c such that, writing $k_n = k_{p_n}$, we have

$$\begin{aligned}
I(v_n,p_n) - V(p_n) &\geq c\left(\|\dot{v_n} - \dot{u}_0\|_q\right) + \int_a^b \left[p_0(t,u_0(t)) - p_0(t,v_n(t))\right] dt \\
&\quad + \int_a^b \left[p_n(t,v_n(t)) - p_n(t,u_0(t))\right] dt \\
&\geq c\left(\|\dot{v_n} - \dot{u}_0\|_q\right) - \int_a^b k_n(t)|v_n(t) - u_0(t)|dt \\
&\quad - \int_a^b k_0(t)|v_n(t) - u_0(t)|dt \qquad (4.10)\\
&\geq c\left(\|\dot{v_n} - \dot{u}_0\|_q\right) - (const.)\|v_n - u_0\|_\infty.
\end{aligned}$$

In (4.10) we use Rellich's theorem to obtain uniform boundedness of the sequence v_n and then the assumption (4.2). Finally, by the forcing property of c, the proof is completed by passing to the lim sup as $n \to +\infty$. □

4.1. Variational convergence

In Theorem 4.1 we have considered a particular convergence mode on the space of parameters P, defined by (4.6). Since the variational convergence of a sequence of non-convex integral functionals is not so far characterized, now we show explicit conditions on the integrands to obtain the variational convergence of sequences of integrals with respect to the weak topology in a Sobolev space.

More precisely, we first show a criterion concerning Mosco convergence of sequences of non convex integrals (4.1). Then we improve this result considering sequences of integral functionals whose integrands have no uniform bounds.

Write $f^- = \max(0,-f)$.

Definition 4.2. A function $f : [a,b]\times\mathbf{R}^N\times\mathbf{R}^N \to \mathbf{R}$ is said to satisfy the inf-compactness property if for every sequence u_n strongly converging in L^1 and every sequence v_n weakly converging in L^1 such that

$$\int_a^b f(t, u_n(t), v_n(t))dt \leq const. < +\infty$$

the functions $f^-(t, u_n(t), v_n(t))$ are equintegrable.

Proposition 4.3. *Let I be as in (4.1), with $f : [a,b] \times \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R}$ a Carathéodory function, convex with respect to the last variable and satisfying the inf-compactness property. Consider the sequence $p_n \in P$ such that $p_n(t,s)$ converges to $p_0(t,s)$ for almost every $t \in [a,b]$ and for all $s \in \mathbf{R}^N$.*

Then the sequence $I(\cdot, p_n)$ Mosco converges to $I(\cdot, p_0)$ in $W_0^{1,q}([a,b], \mathbf{R}^N)$.

Proof. We prove that the sequence $I(\cdot, p_n)$ pointwise converges to $I(\cdot, p_0)$; indeed by assumptions (4.4) and (4.5) we have

$$|p_n(t, u(t))| \leq \max\{\psi_0(t) + \psi_1(t)|u(t)|^{\gamma_1}, \psi_2(t)|u(t)|^{\gamma_2} - \psi_3(t)\}$$

for all $u \in W^{1,q}$, and so the claim follows by the Lebesgue convergence theorem. Now, since F given by (3.1) is weakly lower semicontinuous in $W^{1,1}([a,b], \mathbf{R}^N)$ (see Theorem 3.9 in [1]), it suffices to verify that for every sequence u_n weakly convergent to $\bar{u}$ in $W_0^{1,q}([a,b], \mathbf{R}^N)$ we have

$$\liminf_n \int_a^b p_n(t, u_n(t))dt \geq \int_a^b p_0(t, \bar{u}(t))dt.$$

We have

$$\begin{aligned}
&\int_a^b p_n(t, u_n(t))dt - \int_a^b p_0(t, \bar{u}(t))dt \\
&= \int_a^b [p_n(t, u_n(t)) - p_n(t, \bar{u}(t))]\, dt + \int_a^b [p_n(t, \bar{u}(t)) - p_0(t, \bar{u}(t))]\, dt \\
&\geq -\int_a^b k_n(t)|u_n(t) - \bar{u}(t)|dt + \int_a^b [p_n(t, \bar{u}(t)) - p_0(t, \bar{u}(t))]\, dt,
\end{aligned}$$

where $k_n = k_{p_n}$. Now taking the limit as $n \to \infty$, we deduce that the last term converges to zero because u_n uniformly converges to $\bar{u}$ and the integral functionals with integrands p_n are pointwise convergent. This yields the result. $\square$

Now we deal with variational convergence of a sequence of non-convex integral functionals (4.1) under milder assumptions on the integrand.

Proposition 4.4. *Consider the sequence of Carathéodory functions $p_n : [a,b] \times \mathbf{R}^N \to \mathbf{R}$ such that*

(1) *there exist $\psi_i \in L^{r_i}((a,b))$, with $r_i \geq 1$, for $i = 1, 2$ and $\gamma > 1$ such that*

$$p_n(t,s) \geq -\psi_2(t)|s|^\gamma + \psi_1(t) \text{ for all } n = 1, 2, 3, .., \text{ and } s \in \mathbf{R}^N; \tag{4.11}$$

(2) *for every sequence* $y_n \to y_0$

$$\liminf_n p_n(t, y_n) \geq p_0(t, y_0) \text{ for a.e. } t \in [a, b]. \tag{4.12}$$

Consider also the corresponding sequence of functionals $I(\cdot, p_n)$*, defined in* (4.1)*. Let* f *be as in Proposition* 4.3*.*

Then for every sequence v_n *weakly convergent to* v_0 *in* $W_0^{1,q}$*, we have*

$$\liminf_n I(v_n, p_n) \geq I(v_0, p_0).$$

Proof. First of all, we observe that the functional F in (3.1) is weakly lower semicontinuous in $W_0^{1,q}([a, b], \mathbf{R}^N)$ because of the properties of the integrand f. Hence we prove that for every sequence v_n, weakly convergent to v_0, we have

$$\liminf_n \int_a^b p_n(t, v_n(t))dt \geq \int_a^b p_0(t, v_0(t))dt. \tag{4.13}$$

By assumption (4.11), Fatou's Lemma yields:

$$\liminf_n \int_a^b p_n(t, v_n(t))dt \geq \int_a^b \liminf_n p_n(t, v_n(t))dt$$

and this concludes the proof by (4.12) since v_n is uniformly convergent to v_0. □

Proposition 4.4 is only a partial result; indeed, to obtain the Variational convergence it is necessary to check also condition (2) in Definition 2.5.

In the following we consider the class of autonomous integral functionals

$$G(u) = \int_a^b g(u(t), \dot{u}(t))dt, \quad u \in W_0^{1,q}([a, b]). \tag{4.14}$$

Proposition 4.5. *Let* $q > 1$ *and* G_0, G_n *be defined in* (4.14) *with respective integrands* $g_0, g_n : \mathbf{R}^N \times \mathbf{R}^N \to \mathbf{R} \cup \{+\infty\}$ *Borel functions such that:*

(1) $g_0(y, \cdot)$, $g_n(y, \cdot)$ *are lower semicontinuous for all* $y \in \mathbf{R}^N$ *and* $n = 1, 2, 3, ..$

(2) *There exist a positive constant* γ*, real numbers* M *and* N*, with* $M > 0$ *and a non negative function* $\psi : \mathbf{R} \to [0, +\infty)$ *such that* $\psi(y) \leq \beta|y|^q$*, with* $\beta \geq 0$ *and*

$$g_n(y, z) \geq \gamma|z|^q - \psi(y) \text{ and } g_n(y, 0) \leq M|y|^q + N. \tag{4.15}$$

for every $y \in \mathbf{R}^N$*,* $z \in \mathbf{R}^N$ *and all* $n \in \mathbf{N}$*.*

(3) *There exist* $k_0 \in L^1$ *and* α*,* $\delta > 0$ *such that for all* $n \in \mathbf{N}$ *and for all* $(y, z) \in \mathbf{R}^{2N}$ *there exists a sequence of measurable functions* $\phi_n : [a, b] \to (0, 1]$ *such that:*

$$\phi_n(t) g_n\left(y, \frac{z}{\phi_n(t)}\right) \leq k_0(t) + \alpha|y|^q + \delta|z|^q. \tag{4.16}$$

(4) *For every* $(y, z) \in \mathbf{R}^{2N}$ *there exists a sequence of real number* $\tau_n = \tau_n(y, z)$, $\tau_n \leq 1$, $\tau_n(y, z) \to 1$ *such that*

$$\limsup_n g_n\left(y, \frac{z}{\tau_n(y, z)}\right) \leq g_0(y, z). \tag{4.17}$$

Then for every $\bar{u} \in W_0^{1,q}([a, b], \mathbf{R}^N)$ *there exists* $u_n \in W_0^{1,q}([a, b], \mathbf{R}^N)$ *such that*

$$\limsup_n G_n(u_n) \leq G_0(\bar{u}). \tag{4.18}$$

Proof. Let $\bar{u} \in W_0^{1,q}([a, b], \mathbf{R}^N)$. We can assume that $g_0(\bar{u}(s), \dot{\bar{u}}(s)) < +\infty$ almost everywhere, otherwise the conclusion is trivially true. We consider now $\mathcal{A} = \{s \in [a, b] \mid \dot{\bar{u}}(s) \neq 0\}$. If $\mathcal{A}$ has measure zero, then $\bar{u}$ is the zero constant and the conclusion is achieved by taking $u_n = 0$. Otherwise, for every $s \in \mathcal{A}$, we consider the sequence of functions

$$f_n(s, v) = \begin{cases} g_n(\bar{u}(s), \frac{\dot{\bar{u}}(s)}{v})v & \text{if } 0 < v \leq 1 \\ +\infty & \text{if } v \leq 0 \text{ or } v > 1. \end{cases}$$

The lower semicontinuity of $f_n(s, \cdot)$ follows easily from the lower semicontinuity of g_n with respect to the second variable and from the growth condition in (4.15) . Moreover we consider the sequence of functions

$$h_n(s, v) = \max\{0, f_n(s, v) - g_0(\bar{u}(s), \dot{\bar{u}}(s))\} \tag{4.19}$$

for all $s \in \mathcal{A}$ and $v \in \mathbf{R}$; hence $h_n(s, \cdot)$ is lower semicontinuous and non negative. Now by the smooth variational principle Theorem 3.3 of [17]. Fix a sequence ε_n converging to zero, then there exists a sequence of functions $\xi_n = \xi_n(s) \in \mathbf{R}$ such that $|1 - \xi_n(s)| \leq \varepsilon_n$ and the function

$$v \mapsto h_n(s, v) + |v - \xi_n(s)|^2$$

has a unique minimizer $v_n = v_n(s)$ in $Dom(h_n(s, \cdot))$. Thus $0 < v_n(s) \leq 1$ for almost every s in $\mathcal{A}$.

We observe that v_n and ξ_n are measurable by the proof of Theorem 3.3 in [17] (as limits of measurable functions). Now we prove that $h_n(\cdot, v_n(\cdot))$ pointwise converges to zero in $\mathcal{A}$. As a matter of fact, by (4.17), there exists $\tau_n = \tau_n(\bar{u}(s), \dot{\bar{u}}(s))$ such that:

$$\begin{aligned} h_n(s, v_n(s)) &\leq h_n(s, v_n(s)) + |v_n(s) - \xi_n(s)|^2 \\ &\leq h_n(s, \tau_n) + |\tau_n - \xi_n(s)|^2 \\ &\leq \max\left\{0, g_n\left(\bar{u}(s), \frac{\dot{\bar{u}}(s)}{\tau_n}\right)\tau_n - g_0(\bar{u}(s), \dot{\bar{u}}(s))\right\} + |\tau_n - \xi_n(s)|^2, \end{aligned} \tag{4.20}$$

for almost every $s \in \mathcal{A}$. Taking the limit as $n \to \infty$, we see that $h_n(\cdot, v_n(\cdot))$ pointwise converges to zero in $\mathcal{A}$. Moreover from (4.16) and (4.20) we obtain that

$$\begin{aligned} h_n(s, v_n(s)) &\leq h_n(s, \phi_n(s)) + |\phi_n(s) - \xi_n(s)|^2 \\ &\leq \max\left\{0, g_n\left(\bar{u}(s), \frac{\dot{\bar{u}}(s)}{\phi_n(s)}\right)\phi_n(s) - g_0(\bar{u}(s), \dot{\bar{u}}(s))\right\} + |\phi_n(s) - \xi_n(s)|^2 \\ &= \max\{0, k_0(s) + \alpha|\bar{u}(s)|^q + \delta|\dot{\bar{u}}(s)|^q - g_0(\bar{u}(s), \dot{\bar{u}}(s))\} + const. \end{aligned} \tag{4.21}$$

So the claim follows from the Lebesgue convergence theorem, i.e. $h_n(\cdot, v_n(\cdot))$ converges to zero in L^1. Arguing as before, it is proved that

$$|v_n(s) - \xi_n(s)|^2 \to 0 \quad \text{and} \quad |v_n(s)|^2 \text{ is bounded by a summable function.}$$

Thus v_n converges to one in L^2.

Now we consider two different cases.

First case: we assume that the set $\{s \in [a, b] \mid \dot{\bar{u}}(s) = 0\}$ has measure zero. Hence

$$\limsup_n \int_a^b \left[g_n\left(\bar{u}(s), \frac{\dot{\bar{u}}(s)}{v_n(s)}\right) v_n(s) - g_0(\bar{u}(s), \dot{\bar{u}}(s)) \right] \leq \lim_n \int_a^b \max\{0, g_n(s, v_n(s)) - g_0(\bar{u}(s), \dot{\bar{u}}(s))\} = 0 \quad (4.22)$$

because $0 < v_n(s) \leq 1$ almost everywhere. Hence we define the sequence $w_n(s) = a + \int_a^s v_n(t)dt$ such that w_n is strictly increasing in $[a, b]$, $w_n(b) \leq b$ and $w_n(b) \to b$, as $n \to +\infty$. Thus the inverse function w_n^{-1} exists and its derivative is given by

$$\frac{d}{dt} w_n^{-1}(t) = \frac{1}{\dot{w}_n(w_n^{-1}(t))} \quad \text{for a. e. } t \in [a, w_n(b)]$$

by Corollary 4 in [18]. Moreover, by Theorem 1 of [3], provided only that one of the integrals exists, the change of variables $t = w_n(s)$ can be used, yielding

$$\int_a^{w_n(b)} g_n\left(\bar{u}(w_n^{-1}(t)), \frac{\dot{\bar{u}}(w_n^{-1}(t))}{\dot{w}_n(w_n^{-1}(t))}\right) dt = \int_a^b g_n\left(\bar{u}(s), \frac{\dot{\bar{u}}(s)}{\dot{w}_n(s)}\right) \dot{w}_n(s) ds.$$

As a matter of fact the function $f(s) = g_n\left(\bar{u}(s), \frac{\dot{\bar{u}}(s)}{\dot{w}_n(s)}\right) \dot{w}_n(s)$ is integrable because $h_n(\cdot, v_n(\cdot))$ is dominated by a summable function and by the growth condition in (4.15). Now we observe that by Corollary 4 in [18] the derivative of $\bar{u}(w_n^{-1})$ is given by

$$\frac{d}{ds}(\bar{u}(w_n^{-1}))(s) = \frac{\dot{\bar{u}}(w_n^{-1}(s))}{\dot{w}_n(w_n^{-1}(s))}$$

since w_n^{-1} is increasing and $\bar{u}$ is absolutely continuous. Hence

$$\limsup_n \int_a^b g_n\left(\bar{u}(s), \frac{\dot{\bar{u}}(s)}{\dot{w}_n(s)}\right) \dot{w}_n(s) ds = \limsup_n \int_a^{w_n(b)} g_n\left(\bar{u}(w_n^{-1}(t)), \frac{d}{dt}(\bar{u}(w_n^{-1}))(t)\right) dt.$$

Now we check that the derivative $\frac{d}{dt}(\bar{u}(w_n^{-1}))$ is in L^q. Therefore, in order to apply Theorem 1 in [3], we first prove that the function $\frac{|\dot{\bar{u}}(s)|^q}{|\dot{w}_n(s)|^{q-1}}$ is integrable. By (4.15), since

$\dot{w}_n(s)$ is a minimum point for $h_n(s,\cdot)+|\cdot-\xi_n(s)|^2$, we have

$$\begin{aligned}
&\gamma\int_a^b \frac{|\dot{\bar{u}}(t)|^q}{|\dot{w}_n(t)|^{q-1}}dt\\
&\leq \int_a^b g_n\left(\bar{u}(s),\frac{\dot{\bar{u}}(s)}{\dot{w}_n(s)}\right)\dot{w}_n(s)ds+\int_a^b \psi(\bar{u}(s))\dot{w}_n(s)ds\\
&\leq \int_a^b [h_n(s,\dot{w}_n(s))+g_0(\bar{u}(s),\dot{\bar{u}}(s))]ds+\int_a^b |\dot{w}_n(s)-\xi_n(s)|^2ds\\
&\quad-\int_a^b |\dot{w}_n(s)-\xi_n(s)|^2ds+\int_a^b \psi(\bar{u}(s))\dot{w}_n(s)ds.
\end{aligned}$$

The claim is proved because, by (4.21), the integrands on the right hand side are bounded by a summable function and since $0<\dot{w}_n\leq 1$. Thus again by Theorem 1 in [3] the change of variables $s=w_n(t)$ is admissible in the following integral and we obtain:

$$\int_a^{w_n(b)} \frac{|\dot{\bar{u}}(w_n^{-1}(s))|^q}{|\dot{w}_n(w_n^{-1}(s))|^q}ds=\int_a^b \frac{|\dot{\bar{u}}(t)|^q}{|\dot{w}_n(t)|^{q-1}}dt. \tag{4.23}$$

Now we consider the sequence of absolutely continuous functions $u_n(x)=\int_a^x \dot{u}_n(s)ds$, where

$$\dot{u}_n(s)=\begin{cases}\frac{d}{ds}(\bar{u}(w_n^{-1}))(s) & \text{in } [a,w_n(b))\\ 0 & \text{in } [w_n(b),b].\end{cases}$$

It is easy to check that $u_n(x)=\bar{u}(w_n^{-1}(x))$ in $[a,w_n(b))$ and is zero elsewhere. Thus (4.22) may be written as

$$\begin{aligned}
&\limsup_n \int_a^b g_n(u_n(t),\dot{u}_n(t))dt\\
&\leq \limsup_n\left[\int_a^{w_n(b)} g_n\left(\bar{u}(w_n^{-1}(t)),\frac{d}{dt}(\bar{u}(w_n^{-1}))(t)\right)dt\right]+\limsup_n\left[\int_{w_n(b)}^b g_n(0,0)dt\right]\\
&\leq \int_a^b g_0(\bar{u}(s),\dot{\bar{u}}(s))ds,
\end{aligned}$$

where the last inequality is due to (4.22) and the fact that, by the Lebesgue convergence theorem and (4.15)

$$\limsup_n\left[\int_{w_n(b)}^b g_n(0,0)dt\right]=0.$$

Hence the sequence $u_n\in W_0^{1,q}([a,b],\mathbf{R}^N)$ and satisfies (4.18).

Second case: assume that the set $\{s\in[a,b]\mid \dot{\bar{u}}(s)=0\}$ has positive measure. Integrating $h_n(\cdot,v_n(\cdot))$ on $\mathcal{A}$ we have

$$\limsup_n \int_{\mathcal{A}}\left[g_n\left(\bar{u}(s),\frac{\dot{\bar{u}}(s)}{v_n(s)}\right)v_n(s)-g_0(\bar{u}(s),\dot{\bar{u}}(s))\right]\leq 0. \tag{4.24}$$

Now consider the sequence of absolutely continuous functions w_n such that

$$\dot{w}_n(s) = \begin{cases} v_n(s) & \text{in } \mathcal{A} \\ r_n & \text{elsewhere} \end{cases}$$

where r_n is a real number such that

$$\int_a^b \dot{w}_n(s)ds = b - a.$$

Hence $r_n \geq 1$ and r_n converges to one; indeed

$$r_n = \frac{b - a - \int_{\mathcal{A}} v_n(s)ds}{meas\left(\{s \mid \dot{\bar{u}}(s) = 0\}\right)}.$$

Therefore we obtain

$$\begin{aligned}
&\limsup_n \int_a^b g_n\left(\bar{u}(s), \frac{\dot{\bar{u}}(s)}{\dot{w}_n(s)}\right) \dot{w}_n(s)ds \\
&\leq \limsup_n \left[\int_{\mathcal{A}} g_n\left(\bar{u}(s), \frac{\dot{\bar{u}}(s)}{\dot{w}_n(s)}\right) \dot{w}_n(s)ds\right] + \limsup_n \left[r_n \int_{\{\dot{\bar{u}}=0\}} g_n(\bar{u}(s), 0)ds\right] \\
&\leq \limsup_n \left[\int_{\mathcal{A}} g_n\left(\bar{u}(s), \frac{\dot{\bar{u}}(s)}{\dot{w}_n(s)}\right) \dot{w}_n(s)ds\right] + \int_{\{\dot{\bar{u}}=0\}} g_0(\bar{u}(s), 0)ds \\
&\leq \int_{\mathcal{A}} g_0(\bar{u}(s), \dot{\bar{u}}(s)) + \int_{\{\dot{\bar{u}}=0\}} g_0(\bar{u}(s), 0)ds,
\end{aligned}$$

where the last inequality follows from (4.24), and by Fatou's Lemma. Finally, arguing as in the first case, by Theorem 1 of [3] we have

$$\begin{aligned}
\int_a^b g_n\left(\bar{u}(s), \frac{\dot{\bar{u}}(s)}{\dot{w}_n(s)}\right) \dot{w}_n(s)ds &= \int_a^b g_n\left(\bar{u}(w_n^{-1}(t)), \frac{\dot{\bar{u}}(w_n^{-1}(t))}{\dot{w}_n(w_n^{-1}(t))}\right) dt \\
&= \int_a^b g_n\left(\bar{u}(w_n^{-1}(t)), \frac{d}{dt}(\bar{u}(w_n^{-1}))(t)\right) dt.
\end{aligned}$$

We observe that the integrability of the function $f(s) = g_n\left(\bar{u}(s), \frac{\dot{\bar{u}}(s)}{\dot{w}_n(s)}\right) \dot{w}_n(s)$ is obtained as before on $\mathcal{A}$, while outside $\mathcal{A}$ the function f is equal to $g_n(\bar{u}(s), 0)r_n$, which is equibounded. Thus we consider the sequence of absolutely continuous functions $u_n(s) = \int_a^s \frac{d}{dv}(\bar{u}(w_n^{-1}))(v)dv$. By (4.23) the first derivative of u_n is in L^q and $u_n(s) = (\bar{u}(w_n^{-1}))(s)$. because if $s \in [a, b]$, Theorem 1 of [3], with $A = [a, w_n^{-1}(s)]$, $f = \frac{d}{dv}(\bar{u}(w_n^{-1}))$, yields

$$\int_a^{w_n^{-1}(s)} \dot{\bar{u}}(x)dx = \int_a^s \frac{\dot{\bar{u}}(w_n^{-1}(t))}{\dot{w}_n(w_n^{-1}(t))}(v)dv.$$

Hence we have found a sequence of functions u_n in $W_0^{1,q}([a, b], \mathbf{R}^N)$ which satisfy (4.18). This ends the proof. $\square$

Remark 4.6. The convergence mode defined in (4.17) holds if the sequence g_n pointwise converges to g, but it does not imply pointwise convergence. As an example the sequence of functions:

$$g_n(z) = \begin{cases} z^n - 1 & \text{if } z \in [0,1]; \\ 0 & \text{otherwise} \end{cases}$$

fulfills (4.17) with $g_0 = 0$ and $\tau_n = 1 - \frac{1}{n}$, but it is not pointwise convergent to zero. Another example is the sequence of convex functions

$$g_n(z) = \begin{cases} z^2 & \text{if n is even;} \\ 2(z-1)^2 + 2 & \text{if n is odd} \end{cases}$$

which does not converge pointwise nor is epi-convergent, but for every sequence τ_n converging to one we have

$$\limsup_n g_n\left(\frac{z}{\tau_n}\right) = epi - \limsup_n g_n(z) = 2(z-1)^2 + 2.$$

Remark 4.7. The assumption (4.16) in the previous proposition is satisfied in particular when $\phi_n = 1$ considering the sequence of functions $g_n = g_n(y, z)$, convex with respect to the second variable and equicoercive, i.e.

$$A + \gamma|z|^q - \beta|y|^q \leq g_n(y, z) \leq \alpha|y|^q + \delta|z|^q + k_0.$$

In this case the sequence is locally equibounded and so the epi-convergence of the sequence is equivalent to the pointwise convergence. The same is true for the the corresponding integral functionals.

As a consequence of the previous results we obtain the following

Corollary 4.8. *Let $p_n : \mathbf{R}^N \to \mathbf{R}$ satisfy the growth condition (4.11) and $f : \mathbf{R}^N \times \mathbf{R}^N$ be such that $f + p_n$ verifies (4.15) and (4.16). If p_n is both pointwise and epi-convergent to p_0 then the sequence of non-convex autonomous integrals*

$$\int_a^b f(u(t), \dot{u}(t))dt + \int_a^b p_n(u(t))dt \tag{4.25}$$

is variational convergent in $W_0^{1,q}([a,b], \mathbf{R}^N)$ with respect to the weak topology.

Remark 4.9. A result of this kind is, for example, contained in [12], where epi-convergence of the integrand is proved to imply Mosco convergence of the corresponding integral functionals in L^p space, but only convex integrals are considered there.

5. Examples

We want to exhibit an example characterizing completely the weakest convergence on the space of parameters which guarantees wellposedness.

Example 5.1. We consider linear perturbations of the quadratic functional

$$u \mapsto \int_0^1 \dot{u}^2(t)dt.$$

More precisely consider the space of parameters P defined as follows

$$P = \left\{(t,s) \mapsto p(t)s \ : p \in L^1([0,1]) \text{ and } s \in \mathbf{R}\right\}. \tag{5.1}$$

Fix $p_0 \in L^1$. We study the wellposedness by perturbations of the minimization problem defined by

$$I(u,p_0) = \frac{1}{2}\int_0^1 \dot{u}^2(t)dt + \int_0^1 p_0(t)u(t)dt, \qquad u \in H_0^1([0,1]). \tag{5.2}$$

It is well known that there exists a unique minimizer w of (5.2) which satisfies the Euler-Lagrange equation

$$\frac{d}{dt}\dot{w}(t) = p_0(t), \qquad w \in H_0^1([0,1]). \tag{5.3}$$

Now we prove that the following convergence of sequences

$$p_n \to p_o \text{ iff } \quad p_n \in L^1 \quad \text{and} \quad \delta(p_n,p_0) \to 0$$

where

$$\delta(p_n,p_0) =$$

$$= \int_0^1 \left[\int_0^t (p_n(s)-p_0(s))ds\right]^2 dt - \left[\int_0^1 \int_0^t (p_n(s)-p_0(s))dsdt\right]^2 \to 0 \tag{5.4}$$

as $n \to +\infty$, is the weakest one guaranteeing wellposedness by perturbations of this minimization problem.

To clarify we observe that (5.4) is equivalent to strong convergence in $H_0^1([0,1])$ of $\operatorname{argmin}(H_0^1([0,1]), I(\cdot,p_n))$ to $\operatorname{argmin}(H_0^1([0,1]), I(\cdot,p_0))$. As a matter of fact when $u_n = \operatorname{argmin}(H_0^1([0,1]), I(\cdot,p_n))$ then from (5.3)

$$\dot{u_n}(x) = -\int_0^1 \int_0^y p_n(t)dtdy + \int_0^x p_n(t)dt \quad \text{for } n = 0,1,2,3,.. \tag{5.5}$$

and it is easy to see that by (5.5)

$$\begin{aligned}
\|\dot{u}_n - \dot{u}_0\|_2^2 &= \int_0^1 \left[-\int_0^1 \int_0^y (p_n(t)-p_0(t))dtdy + \int_0^x (p_n(t)-p_0(t))dt\right]^2 dx \\
&= \int_0^1 \left[\int_0^t (p_n(s)-p_0(s))ds\right]^2 dt - \left[\int_0^1 \int_0^t (p_n(s)-p_0(s))dsdt\right]^2.
\end{aligned}$$

We remark that the convergence in (5.4) comes from a topology (see [9]). Let us check the uniqueness of the limit. Let p_0, q_0 be such that $\delta(p_n,p_0)$ and $\delta(p_n,q_0)$ converge to zero.

Denote by u_0 the unique solution of $\frac{d}{dt}\dot{u} = p_0$ in $H_0^1([0,1]$ and by v_0 the unique solution of $\frac{d}{dt}\dot{u} = q_0$. Then $\dot{u}_n$ converges both to $\dot{u}_0$ and to $\dot{v}_0$ in L^2. Thus $\dot{u}_0 = \dot{v}_0$ almost everywhere and so $p_0 = q_0$, thanks to the Euler-Lagrange equation.

Denote by v_n any asymptotically minimizing sequence. Then we can write, using the Euler-Lagrange equation

$$\begin{aligned} I(v_n, p_n) - V(p_n) &= \frac{1}{2}\int_0^1 \left(\dot{v_n} - \dot{u_n}\right)^2(t)dt + \int_0^1 \dot{u_n}(\dot{v_n} - \dot{u_n})dt \\ &\quad + \frac{1}{2}\int_0^1 \dot{u_n}^2 + \int_0^1 p_n(t)(v_n(t) - u_n(t))dt \\ &= \frac{1}{2}\int_0^1 \left(\dot{v_n} - \dot{u_n}\right)^2(t)dt, \end{aligned} \tag{5.6}$$

so the asymptotically minimizing sequence has the same asymptotic character of the sequence u_n.

Thus we have obtained that δ is the weakest convergence structure on the space of parameters guaranting wellposedness. This convergence is a necessary condition to wellposedness being equivalent to the strong convergence of $u_n = \text{argmin}(H_0^1([0,1]), I(\cdot, p_n))$ to $u_0 = \text{argmin}(H_0^1([0,1]), I(\cdot, p_0))$ and it is also a sufficient condition.

When (5.4) does not hold we have a Tikhonov wellposed functional which is illposed with respect to linear perturbations.

Example 5.2. Let us consider

$$p_n(t) = \chi_{[0, \frac{1}{n^\beta}]}(t) n^\alpha \text{ in } [0,1], \text{ with } \alpha > \frac{3}{2}\beta, \tag{5.7}$$

where we have denoted by χ the characteristic function. Let $p_0 = 0$. In this case we have

$$\delta(p_n, 0) = \int_0^{\frac{1}{n^\beta}} n^{2\alpha} t^2 dt + \int_{\frac{1}{n^\beta}}^1 n^{2(\alpha-\beta)} dt - \left(\int_0^{\frac{1}{n^\beta}} n^\alpha t dt + \int_{\frac{1}{n^\beta}}^1 n^{\alpha-\beta}\right)^2 \tag{5.8}$$

$$= n^{2\alpha - 3\beta}\left(\frac{1}{3} - \frac{1}{4} n^{-\beta}\right) \to +\infty \quad \text{as } n \to +\infty. \tag{5.9}$$

So $\delta(p_n, 0)$ does not converge, hence the problem is illposed.

The following example deals with perturbations involving the derivative.

Example 5.3. Let us consider the embedding $I : P \times H_0^1([-1,1]) \to \mathbf{R}$, defined as

$$I(u,p) = \int_{-1}^1 p(t)\dot{u}^2(t)dt,$$

where

$$P = \{p \in L^\infty([-1,1]) | \quad p(t) > 0 \quad \text{a.e.}\}.$$

We claim that if we endow the parameter space P with L^q-convergence (any $q > 1$) wellposedness of this minimization problem fails. Consider the sequence

$$p_n(t) = \frac{1}{n^5} \text{ for } |t| < \frac{1}{n} \text{ and } p_n(t) = 1 \text{ otherwise}$$

which converges to 1 in L^q and is uniformly bounded. The asymptotically minimizing sequence v_n given by

$$\dot{v}_n(t) = \frac{t|t||1 - p_n(t)|}{\sqrt{p_n(t)}}$$

does not converge to $0 = \operatorname{argmin}(I(\cdot, 1))$ in $H_0^1([-1, -1])$.

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