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Homographic Approximation for Some Nonlinear Parabolic Unilateral Problems

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We deal with nonlinear parabolic unilateral problems by means of the homographic approximation, introduced by C. M. Brauner and B. Nicolaenko in the linear elliptic case (see [7]). The interest in this kind of penalty method arises from the fact that, in contrast with the usual penalization (see [12], [16] and [10]), the homographic approximation is a "bounded penalty", which turns out to be convenient to have *a priori* estimates on the approximate solutions. We present two different situations in which the homographic approximation gives advantages to solve evolutionary unilateral problems. First, in a variational framework, we are interested in strong solutions to nonlinear parabolic variational inequalities; then, in a second case, we consider obstacle problems with L^1 data.

1. Introduction

Parabolic variational inequalites have been widely studied in the literature. In the case of regular obstacles, the classical penalty method of J. L. Lions [12] yields existence and uniqueness of weak solutions. Applying the same kind of approximation, F. Mignot and J. P. Puel obtained existence of minimal weak solutions if the obstacles are only measurable functions (see [14] and [16]).

The existence of a unique strong solution in the case of regular constraints has been previously proved by Brezis [1] and P. Charrier and G. M. Troianiello [8] for the linear case, and by F. Donati [10] for the nonlinear monotone case. In [8] the result is obtained by means of an elliptic regularization method and of dual estimates. The approach of [10] is based on a penalty method and yields a regularity result in terms of Lewy-Stampacchia inequality. In Section 3 we will extend the existence and uniqueness result of strong solutions to unilateral problems with obstacles admitting "downward jumps" with respect to time (see Remark 3.3). The proof, based on the homographic approximation, will also imply a dual estimate of Lewy-Stampacchia type.

Section 4 deals with the case of parabolic unilateral problems with L^1 data. Since the classical formulation of obstacle problem is not appropriate, it will be necessary to give a new definition of the solution. As for elliptic unilateral problems (see [3]), under some regularity assumptions on the obstacle, it is possible to give a definition of solution similar to that of entropy solution for parabolic equations with L^1 data (we refer to [15]), and to prove existence and uniqueness together with a Lewy-Stampacchia inequality. We will show that such solution also satisfies a minimality condition, given by (4.4) below. This new characterization of solution allows us to extend the existence and uniqueness results so as to include the case of obstacles of a more general type. For this purpose we will use,

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once more, the homographic approximation and, therefore, we will also prove an estimate of Lewy-Stampacchia type.

2. Notation and hypotheses

Throughout the paper, Ω is an open bounded set of $\mathbb{R}^N$, with $N \geq 2$, and Q is the cylinder $\Omega \times (0,T)$, $T > 0$.

We will denote by χ_E the characteristic function of a measurable set $E \subset \mathbb{R}^N$.

Let us consider the following nonlinear operator in divergence form

$$A(u) = -\operatorname{div}(a(x,t,Du)),$$

where $a : Q \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that, for almost every (x,t) in Q, and for every ξ, ξ_1 and ξ_2 in $\mathbb{R}^N$ with $\xi_1 \neq \xi_2$, one has

$$a(x,t,\xi)\cdot\xi \geq \alpha|\xi|^p, \tag{2.1}$$

$$|a(x,t,\xi)| \leq \beta(b(x,t) + |\xi|^{p-1}), \tag{2.2}$$

$$(a(x,t,\xi_1) - a(x,t,\xi_2))\cdot(\xi_1 - \xi_2) > 0, \tag{2.3}$$

where $\alpha, \beta > 0$, $1 < p < +\infty$ and b is a nonnegative function in $L^{p'}(Q)$.

These hypotheses are classical and assure that A is a coercive, continuous and pseudomonotone operator of Leray-Lions type, acting from $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual.

Define the reflexive Banach space

$$V = W_0^{1,p}(\Omega) \cap L^2(\Omega),$$

equipped with the norm $\|v\|_V = \|Dv\|_{L^p(\Omega)} + \|v\|_{L^2(\Omega)}$, and denote by V' the dual of V. Identifying $L^2(\Omega)$ and its dual, one has

$$V \subset L^2(\Omega) \subset V',$$

where the embeddings are continuous and dense. Therefore, from classical results (see [12]) it follows that, setting

$$\mathcal{W} = \{v \in L^p(0,T;V) : \ v_t \in L^{p'}(0,T;V')\},$$

one obtains

$$\mathcal{W} \subset C([0,T];L^2(\Omega)).$$

Denoting by $\langle\, , \rangle$ the duality pairing of V and V', we have that

$$\int_0^\tau \langle u_t, v\rangle = -\int_0^\tau \langle v_t, u\rangle + \int_\Omega u(\tau)v(\tau) - \int_\Omega u(0)v(0) \tag{2.4}$$

holds for all $\tau \in [0,T]$ and $u, v \in \mathcal{W}$. Moreover, if $\phi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz bounded function such that $\phi(0) = 0$, then

$$\int_0^\tau \langle u_t, \phi(u)\rangle = \int_\Omega \Phi(u(\tau)) - \int_\Omega \Phi(u(0)) \tag{2.5}$$

for all $u \in \mathcal{W}$, where

$$\Phi(s) = \int_0^s \phi(\sigma) d\sigma.$$

In the sequel we will use the truncation function $T_k : \mathbb{R} \to \mathbb{R}$, defined as the Lipschitz bounded function

$$T_k(s) = \max\{-k, \min\{k, s\}\}$$

for every positive real number k, and its primitive $\Theta_k : \mathbb{R} \to \mathbb{R}^+$

$$\Theta_k(s) = \int_0^s T_k(\sigma)\, d\sigma.$$

3. Variational framework: existence and uniqueness of the strong solution and proof of the Lewy-Stampacchia inequality

In this section we study existence of strong solutions for the evolutionary unilateral problem involving the nonlinear operator A in the case of right hand side f in $L^{p'}(0,T;V')$, initial datum u_0 in $L^2(\Omega)$ and obstacle function ψ in $L^p(0,T;V)$. In other words, defining the (nonempty) closed convex set

$$\mathcal{C}_\psi = \{v \in L^p(0,T;V) :\ v \geq \psi \text{ almost everywhere in } Q\}, \tag{3.1}$$

we look for a function $u \in \mathcal{C}_\psi \cap \mathcal{W}$ satisfying the initial condition $u(0) = u_0$ and the variational inequality

$$\int_0^T \langle u_t, v-u\rangle + \iint_Q a(x,t,Du)\cdot D(v-u) \geq \int_0^T \langle f, v-u\rangle, \tag{3.2}$$

for all $v \in \mathcal{C}_\psi$.

Let us first suppose that $\psi \in \mathcal{W}$, $\psi(0) \leq u_0$, and that the element h of $L^{p'}(0,T;V')$, defined by

$$h = \psi_t + A(\psi) - f,$$

can be decomposed as $h = h^+ - h^-$, where h^+ and h^- are nonnegative elements of the dual space $L^{p'}(0,T;V')$ (that is, $\psi_t + A(\psi) - f$ belongs to the ordered dual space of $L^p(0,T;V)$, which is well-know to be a proper subspace of $L^{p'}(0,T;V')$). Due to a result of [10] (proved by using classical penalty approximation), there exists a unique strong solution u of the parabolic unilateral problem for A with data f, u_0 and ψ. Moreover, the following inequality of Lewy-Stampacchia type holds

$$0 \leq u_t + A(u) - f \leq (\psi_t + A(\psi) - f)^+ \qquad \text{in } L^{p'}(0,T;V'). \tag{3.3}$$

From the above inequality (3.3) it follows that the regularity of the solution depends only on the positive part of h. This agrees with the nature of the problem. Indeed, the solutions of unilateral problems, with the obstacle placed below, intuitively may depart from the constraint only "jumping upwards". From here the interest in removing the assumption on the distribution $\psi_t + A(\psi) - f$ to belong to the ordered dual space of $L^p(0,T;V)$, and this is the actual content of the next theorem.

Theorem 3.1. *Let $f \in L^{p'}(0,T;V')$ and $u_0 \in L^2(\Omega)$. Suppose that $\psi \in L^p(0,T;V)$ and that there exists g in $L^{p'}(0,T;V')$ such that g is nonnegative and ψ is a (weak) subsolution of the nonlinear parabolic problem with right hand side $f+g$ and initial datum u_0, that is*

$$-\int_0^T \langle \varphi_t, \psi \rangle + \iint_Q a(x,t,D\psi)\cdot D\varphi \le \int_\Omega u_0\varphi(0) + \int_0^T \langle f+g, \varphi\rangle, \tag{3.4}$$

for all $\varphi \in \mathcal{W}$ such that $\varphi \ge 0$ and $\varphi(T)=0$.

Then there exists a unique function u such that $u \in \mathcal{C}_\psi \cap \mathcal{W}$, $u(0)=u_0$ and inequality (3.2) holds.

Moreover, the obstacle reaction associated with u, which is the nonnegative element of $L^{p'}(0,T;V')$ defined as $\mu = u_t + A(u) - f$, satisfies

$$0 \le \mu \le g \qquad \textit{in } L^{p'}(0,T;V'). \tag{3.5}$$

Proof. We begin by proving that there exists at most one strong solution. Let u and $\bar{u}$ be in $\mathcal{C}_\psi \cap \mathcal{W}$ such that $u(0)=\bar{u}(0)=u_0$ and satisfying inequality (3.2). Then

$$\int_0^T \langle u_t - \bar{u}_t, u-\bar{u}\rangle + \iint_Q (a(x,t,Du)-a(x,t,D\bar{u}))\cdot D(u-\bar{u}) \le 0,$$

where $\int_0^T \langle u_t - \bar{u}_t, u-\bar{u}\rangle = \frac{1}{2}\int_\Omega |u-\bar{u}|^2(T)$. From assumption (2.3) it then follows that $u=\bar{u}$.

To complete the proof of the theorem we proceed in two steps. First we assume that g belongs to $L^{p'}(Q)$ and, using the homographic approximation, we construct a strong solution u. To deal with the general case we then argue by approximating g. Inequality (3.5) will be a consequence of the penalty method.

First case: $g \in L^{p'}(Q)$, $g \ge 0$. Let $\lambda > 0$. We consider the following homographic approximation to the variational inequality (3.2):

$$\begin{cases} u_t^\lambda + A(u^\lambda) + g\frac{u^\lambda - \psi}{\lambda + |u^\lambda - \psi|} = f+g & \text{in } Q,\\ u^\lambda(x,0) = u_0(x) & \text{in } \Omega,\\ u^\lambda(x,t) = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{3.6}$$

From classical results (see [12]), equation (3.6) admits a unique solution $u^\lambda \in \mathcal{W}$. Notice that $u^\lambda \in C([0,T];L^2(\Omega))$, so that the initial datum makes sense.

Now, using assumption (3.4), we prove that $u^\lambda \ge \psi$. Take $\varphi \in \mathcal{W}$ such that $\varphi \ge 0$ and $\varphi(T)=0$. Multiply equation (3.6) by φ and integrate by parts. We have, from (3.4), that

$$\int_0^T \langle \varphi_t, \psi - u^\lambda\rangle + \iint_Q (a(x,t,Du^\lambda)-a(x,t,D\psi))\cdot D\varphi + \iint_Q g\frac{u^\lambda-\psi}{\lambda+|u^\lambda-\psi|}\varphi \ge 0. \tag{3.7}$$

Consider the time-regularization $z^{k,\nu}$ of the function $T_k(\psi-u^\lambda)^+ \in L^p(0,T;V)$, defined as the solution of the problem

$$\begin{cases} -z_t^{k,\nu} + \nu z^{k,\nu} = \nu T_k(\psi-u^\lambda)^+ & \text{in } Q, \\ z^{k,\nu}(x,T) = 0 & \text{in } \Omega, \\ z^{k,\nu}(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}$$

where $\nu > 0$. Explicitly, we have

$$z^{k,\nu}(x,t) = \nu \int_t^T T_k(\psi-u^\lambda)^+(s) \exp(\nu(t-s))\, ds,$$

and thus $z^{k,\nu}$ is a nonnegative sequence which belongs to $L^p(0,T;V) \cap C([0,T];L^2(\Omega))$ and converges to $T_k(\psi-u^\lambda)^+$ strongly in $L^p(0,T;V)$ as $\nu \to +\infty$ (see, for instance, [10]). Moreover, since $T_k(\psi-u^\lambda)^+$ belongs to $L^\infty(Q)$, and $z^{k,\nu}$ turns out to have the same regularity, then we get $z_t^{k,\nu} \in L^\infty(Q)$. Finally, we can see that

$$\iint_Q z_t^{k,\nu}(\psi-u^\lambda) \le 0, \tag{3.8}$$

for all $\nu > 0$; indeed, we have

$$\iint_Q z_t^{k,\nu}(\psi-u^\lambda) = \iint_Q z_t^{k,\nu}(\psi-u^\lambda)^+ - \iint_Q z_t^{k,\nu}(\psi-u^\lambda)^-,$$

where

$$\begin{aligned} \iint_Q z_t^{k,\nu}(\psi-u^\lambda)^+ &= \iint_Q z_t^{k,\nu} T_k(\psi-u^\lambda)^+ + \iint_Q z_t^{k,\nu}((\psi-u^\lambda)^+ - k)^+ = \\ &= \iint_Q z_t^{k,\nu}\left(-\frac{1}{\nu} z_t^{k,\nu} + z^{k,\nu}\right) + \nu \iint_Q (z^{k,\nu} - k)((\psi-u^\lambda)^+ - k)^+ = \\ &= -\frac{1}{\nu}\iint_Q |z_t^{k,\nu}|^2 - \frac{1}{2}\int_\Omega |z^{k,\nu}(0)|^2 - k\nu \iint_Q \exp(\nu(t-T))((\psi-u^\lambda)^+ - k)^+ \le 0, \end{aligned}$$

and

$$\iint_Q z_t^{k,\nu}(\psi-u^\lambda)^- = \nu \iint_Q (z^{k,\nu} - T_k(\psi-u^\lambda)^+)(\psi-u^\lambda)^- = \nu \iint_Q z^{k,\nu}(\psi-u^\lambda)^- \ge 0.$$

Let us take $\varphi = z^{k,\nu}$ in (3.7). Using (3.8) and taking the limit for $\nu \to +\infty$, we get

$$\iint_Q (a(x,t,D\psi) - a(x,t,Du^\lambda)) \cdot DT_k(\psi-u^\lambda)^+ \le 0,$$

so that from (2.3) it follows that $T_k(\psi - u^\lambda)^+ = 0$ for all $k > 0$, and thus $u^\lambda \geq \psi$.

Thus we can rewrite (3.6) as

$$u_t^\lambda + A(u^\lambda) = f + g\frac{\lambda}{\lambda + u^\lambda - \psi}, \tag{3.9}$$

and so we have that u^λ is monotone nonincreasing as $\lambda \to 0^+$; indeed, if $\lambda \leq \eta$, then u^λ is a subsolution of the problem with η so that

$$u_t^\lambda - u_t^\eta + A(u^\lambda) - A(u^\eta) \leq g\frac{\eta(u^\eta - u^\lambda)}{(\eta + u^\lambda - \psi)(\eta + u^\eta - \psi)},$$

and thus, multiplying the last expression by $(u^\lambda - u^\eta)^+$ and integrating on Q, we get that $u^\lambda \leq u^\eta$.

In view of the fact that the homographic term $(u^\lambda - \psi)/(\lambda + |u^\lambda - \psi|)$ is uniformly bounded in $L^\infty(Q)$ with respect to $\lambda > 0$ and that A is coercive, multiplying (3.6) by $u^\lambda \chi_{(0,\tau)}$ we obtain

$$\frac{1}{2}\int_\Omega |u^\lambda(\tau)|^2 - \frac{1}{2}\int_\Omega |u_0|^2 + \alpha \int_0^\tau \int_\Omega |Du^\lambda|^p \leq \int_0^\tau \langle f, u^\lambda\rangle + \int_0^\tau \int_\Omega g|u^\lambda| \leq c\left(\int_0^\tau \|u^\lambda\|_V^p\right)^{\frac{1}{p}}.$$

For $p \geq 2$, this implies, since $\|u^\lambda\|_{L^p(0,T;V)} \leq c\|u^\lambda\|_{L^p(0,T;W_0^{1,p}(\Omega))}$, that u^λ is bounded in $L^p(0,T;V)$. Let us consider the case $1 < p < 2$. We have that

$$\|u^\lambda(\tau)\|_{L^2(\Omega)}^2 + 2\alpha \int_0^\tau \|Du^\lambda\|_{L^p(\Omega)}^p \leq c + c\left(\int_0^\tau \|Du^\lambda\|_{L^p(\Omega)}^p\right)^{\frac{1}{p}} + c\left(\int_0^\tau \|u^\lambda\|_{L^2(\Omega)}^p\right)^{\frac{1}{p}}.$$

Hence, using Young's inequality,

$$\|u^\lambda(\tau)\|_{L^2(\Omega)}^2 + \alpha \int_0^\tau \|Du^\lambda\|_{L^p(\Omega)}^p \leq c + c\left(\int_0^\tau \|u^\lambda\|_{L^2(\Omega)}^p\right)^{\frac{2}{p}}. \tag{3.10}$$

It follows that

$$\|u^\lambda(\tau)\|_{L^2(\Omega)}^p \leq c + c\int_0^\tau \|u^\lambda\|_{L^2(\Omega)}^p$$

and then, by the Gronwall lemma, we deduce that $\|u^\lambda(\tau)\|_{L^2(\Omega)} \leq c$. This implies that u^λ is bounded in $L^p(0,T;L^2(\Omega))$ and, by estimate (3.10), in $L^p(0,T;W_0^{1,p}(\Omega))$. Finally, we conclude that $\|u^\lambda\|_{L^p(0,T;V)} \leq c$.

Now, from (2.2) it follows that $A(u^\lambda)$ is bounded in $L^{p'}(0,T;V')$, and therefore, going back to equation (3.6), we get

$$\|u_t^\lambda\|_{L^{p'}(0,T;V')} \leq c.$$

Consequently, there exists a function u such that u^λ converges to u weakly in $L^p(0,T;V)$ and u_t^λ converges to u_t weakly in $L^{p'}(0,T;V')$. By compact embedding theorems (see [17]) we then have that $u^\lambda \to u$ strongly in $L^p(Q)$. Thus $u \geq \psi$. We also deduce that $u \in C([0,T];L^2(\Omega))$, $u(0)=u_0$ and that

$$\int_0^T \langle u_t, u\rangle \leq \liminf_{\lambda\to 0^+} \int_0^T \langle u_t^\lambda, u^\lambda\rangle.$$

Let $v \in \mathcal{C}_\psi$ and multiply (3.9) by $v-u^\lambda$. Since $u^\lambda - v \leq \lambda + u^\lambda - \psi$ it follows that

$$\int_0^T \langle u_t^\lambda, v-u^\lambda\rangle + \iint_Q a(x,t,Du^\lambda)\cdot D(v-u^\lambda) \geq \int_0^T \langle f, v-u^\lambda\rangle - \lambda \iint_Q g. \tag{3.11}$$

From the above results we have that (3.11) yields (3.2), provided that

$$\limsup_{\lambda\to 0^+} \iint_Q a(x,t,Du^\lambda)\cdot D(v-u^\lambda) \leq \iint_Q a(x,t,Du)\cdot D(v-u). \tag{3.12}$$

To prove (3.12) we choose $v=u$ in (3.11) to find

$$\liminf_{\lambda\to 0^+} \iint_Q a(x,t,Du^\lambda)\cdot D(u-u^\lambda) \geq 0.$$

Thus, since A is pseudomonotone and u^λ converges to u weakly in $L^p(0,T;V)$, we have that (3.12) holds for all $v \in L^p(0,T;V)$ and that $A(u^\lambda)$ converges to $A(u)$ weakly in $L^{p'}(0,T;V')$ (see [12]).

Finally, equation (3.9) yields that

$$0 \leq u_t^\lambda + A(u^\lambda) - f \leq g \qquad \text{in } L^{p'}(0,T;V'),$$

hence, letting $\lambda \to 0^+$, we obtain that inequality (3.5) holds.

General case. Consider now $g \in L^{p'}(0,T;V')$, $g \geq 0$. It is shown in [10] that there exists $\{g_n\} \subset L^{p'}(Q)$, $g_n \geq 0$, such that $g_n \to g$ strongly in $L^{p'}(0,T;V')$ as $n \to +\infty$. Define

$$f_n = f - g_n + g.$$

Thus $f_n \in L^{p'}(0,T;V')$ and $f_n \to f$ strongly in $L^{p'}(0,T;V')$.

Since $f+g = f_n + g_n$, assumption (3.4) is automatically satisfied with right hand side $f_n + g_n$. From the above result, there exists $u^n \in \mathcal{C}_\psi \cap \mathcal{W}$ such that $u^n(0)=u_0$ and

$$\int_0^T \langle u_t^n, v-u^n\rangle + \iint_Q a(x,t,Du^n)\cdot D(v-u^n) \geq \int_0^T \langle f_n, v-u^n\rangle, \tag{3.13}$$

for all $v \in \mathcal{C}_\psi$. Moreover, $\mu_n = u^n_t + A(u^n) - f_n$, the obstacle reaction associated with u^n, is a nonnegative element of $L^{p'}(0,T;V')$ such that

$$0 \le \mu_n \le g_n \qquad \text{in } L^{p'}(0,T;V'). \tag{3.14}$$

This implies that $\|u^n_t + A(u^n)\|_{L^{p'}(0,T;V')} \le \|f_n\|_{L^{p'}(0,T;V')} + \|g_n\|_{L^{p'}(0,T;V')}$; thus, since $\{f_n\}$ and $\{g_n\}$ are convergent sequences, we have

$$\|u^n_t + A(u^n)\|_{L^{p'}(0,T;V')} \le c. \tag{3.15}$$

Taking $v = u^n + (\psi - u^n)\chi_{(0,\tau)} \in \mathcal{C}_\psi$ in (3.13), from estimate (3.15) and assumption (2.1), we get

$$\frac{1}{2}\int_\Omega |u^n(\tau)|^2 - \frac{1}{2}\int_\Omega |u_0|^2 + \alpha \int_0^\tau \int_\Omega |Du^n|^p \le \int_0^\tau \langle u^n_t, u^n\rangle + \int_0^\tau \int_\Omega a(x,t,Du^n)\cdot Du^n \le$$
$$\le \int_0^\tau \langle u^n_t, \psi\rangle + \int_0^\tau \int_\Omega a(x,t,Du^n)\cdot D\psi + \int_0^\tau \langle f_n, u^n - \psi\rangle \le c + c\left(\int_0^\tau \|u^n\|^p_V\right)^{\frac{1}{p}}.$$

Consequently, reasoning as before, we obtain

$$\|u^n\|_{L^p(0,T;V)} \le c.$$

Using (2.2) we also deduce that $A(u^n)$ is bounded in $L^{p'}(0,T;V')$. This implies, by (3.15), that

$$\|u^n_t\|_{L^{p'}(0,T;V')} \le c.$$

Therefore, there exists a function u such that, up to a subsequence, $u^n \to u$ weakly in $L^p(0,T;V)$, $u^n_t \to u_t$ weakly in $L^{p'}(0,T;V')$, $u^n \to u$ strongly in $L^p(Q)$. Moreover $u \in C([0,T];L^2(\Omega))$, $u(0) = u_0$ and

$$\int_0^T \langle u_t, u\rangle \le \liminf_{n\to+\infty} \int_0^T \langle u^n_t, u^n\rangle.$$

Choosing now $v = u$ in (3.13), we get

$$\liminf_{n\to+\infty} \iint_Q a(x,t,Du^n)\cdot D(u - u^n) \ge 0,$$

and since the Leray-Lions operator A is pseudomonotone, this yields

$$\limsup_{n\to+\infty} \iint_Q a(x,t,Du^n)\cdot D(v - u^n) \le \iint_Q a(x,t,Du)\cdot D(v-u),$$

for all v belonging to $L^p(0,T;V)$. Moreover, we have that $A(u^n)$ converges to $A(u)$ weakly in $L^{p'}(0,T;V')$.

Taking the limit in (3.13) and (3.14) we prove both the existence of a solution and inequality (3.5). $\square$

Remark 3.2. The solution of (3.2) is also the minimal function $u \in \mathcal{C}_\psi \cap \mathcal{W}$ such that $u(0) = u_0$ and

$$\int_0^T \langle u_t, \varphi \rangle + \iint_Q a(x,t,Du) \cdot D\varphi \geq \int_0^T \langle f, \varphi \rangle, \tag{3.16}$$

for all $\varphi \in L^p(0,T;V)$ with $\varphi \geq 0$. Indeed, if $\varphi \in L^p(0,T;V)$ and $\varphi \geq 0$, we can choose $v = u + \varphi \in \mathcal{C}_\psi$ in (3.2), so that (3.16) follows. Let now $\bar{u} \in \mathcal{C}_\psi \cap \mathcal{W}$ be such that $\bar{u}(0) = u_0$ and

$$\int_0^T \langle \bar{u}_t, \varphi \rangle + \iint_Q a(x,t,D\bar{u}) \cdot D\varphi \geq \int_0^T \langle f, \varphi \rangle,$$

for all $\varphi \in L^p(0,T;V)$ with $\varphi \geq 0$; taking $\varphi = (u - \bar{u})^+$ and choosing $v = u - (u - \bar{u})^+ \in \mathcal{C}_\psi$ in (3.2), we have that

$$\int_0^T \langle u_t - \bar{u}_t, (u-\bar{u})^+ \rangle + \iint_Q (a(x,t,Du) - a(x,t,D\bar{u})) \cdot D(u-\bar{u})^+ \leq 0,$$

which yields that $u \leq \bar{u}$.

Remark 3.3. If the obstacle ψ is such that $\psi \in \mathcal{W}$, $\psi(0) \leq u_0$, and that $h = \psi_t + A(\psi) - f$ belongs to the ordered dual space of $L^p(0,T;V)$, then hypothesis (3.4) is satisfied with $g = h^+$. Thus we have that there exists a unique function u such that $u \in \mathcal{C}_\psi \cap \mathcal{W}$, $u(0) = u_0$ and inequality (3.2) holds. Moreover, from estimate (3.5) we deduce the Lewy-Stampacchia inequality (3.3). This is the same result proved in [10]. We now give a simple example in which the obstacle is irregular in time (ψ_t does not belong to $L^{p'}(0,T;V')$) and condition (3.4) is satisfied, provided that ψ does not "jump up".

Consider the case where the operator A is the Laplacian, $f = 0$, and $u_0 = 0$. Define the obstacle function

$$\psi(x,t) = [t\chi_{(0,\tau)}(t) + c(1 - \chi_{(0,\tau)}(t))]w(x),$$

where $\tau \in (0,T)$ is fixed, c is a real constant, and $w \in H_0^1(\Omega)$ is the positive solution of the following elliptic problem

$$\begin{cases} -\Delta w = 1 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Note that, if $c \neq \tau$, ψ is not continuous in t. In particular, if $c < \tau$ then ψ "jumps down" and, if $c > \tau$ then ψ "jumps up".

Let us check that assumption (3.4) holds. Take φ in $L^2(0,T;H_0^1(\Omega))$ such that φ_t belongs to $L^2(0,T;H^{-1}(\Omega))$, $\varphi \geq 0$ and $\varphi(T)=0$. We have

$$-\int_0^T \langle \varphi_t, \psi\rangle + \iint_Q D\psi \cdot D\varphi = -\int_0^\tau t\langle \varphi_t, w\rangle - c\int_\tau^T \langle \varphi_t, w\rangle + \int_0^\tau\int_\Omega t\varphi + c\int_\tau^T\int_\Omega \varphi =$$

$$= \int_0^\tau\int_\Omega w\varphi - \tau\int_\Omega w\varphi(\tau) + c\int_\Omega w\varphi(\tau) + \int_0^\tau\int_\Omega t\varphi + c\int_\tau^T\int_\Omega \varphi =$$

$$= (c-\tau)\int_\Omega w\varphi(\tau) + \int_0^\tau\int_\Omega (t+w)\varphi + c\int_\tau^T\int_\Omega \varphi.$$

Thus, if $c \leq \tau$,

$$-\int_0^T \langle \varphi_t, \psi\rangle + \iint_Q D\psi \cdot D\varphi \leq \int_0^\tau\int_\Omega (t+w)\varphi + \tau\int_\tau^T\int_\Omega \varphi$$

and condition (3.4) is satisfied with

$$g(x,t) = \chi_{(0,\tau)}(t)(t+w(x)) + \tau(1-\chi_{(0,\tau)}(t)),$$

which is a nonnegative element of $L^\infty(0,T;L^2(\Omega))$.

Remark 3.4. Let us point out that hypothesis (3.4) implies that there exists $z \in \mathcal{C}_\psi \cap \mathcal{W}$. Indeed we can choose z as the solution of the parabolic problem

$$\begin{cases} z_t + A(z) = f + g & \text{in } Q, \\ z(x,0) = u_0 & \text{in } \Omega, \\ z(x,t) = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}$$

Assume now that $\mathcal{C}_\psi \cap \mathcal{W}$ is nonempty and consider the case where $A = -\Delta$. It is shown in [16] that there exists u such that: $u \in \mathcal{C}_\psi$ and

$$\int_0^T \langle v_t, v-u\rangle + \iint_Q Du \cdot D(v-u) + \frac{1}{2}\int_\Omega |v(0)-u_0|^2 \geq \int_0^T \langle f, v-u\rangle, \tag{3.17}$$

for all $v \in \mathcal{C}_\psi \cap \mathcal{W}$ (that is, u is a weak solution).

In [16] u is obtained as the nondecreasing limit, for $\varepsilon \to 0^+$, of the solution u^ε of the following penalized problem

$$\begin{cases} u^\varepsilon_t - \Delta u^\varepsilon - \frac{1}{\varepsilon}(u^\varepsilon - \psi)^- = f & \text{in } Q, \\ u^\varepsilon(x,0) = u_0 & \text{in } \Omega, \\ u^\varepsilon(x,t) = 0 & \text{on } \partial\Omega \times (0,T). \end{cases}$$

With this method u turns out to be the minimal weak solution and is characterized as the minimal function in $\mathcal{C}_\psi$ such that

$$-\int_0^T \langle \varphi_t, u\rangle + \iint_Q Du \cdot D\varphi \geq \int_0^T \langle f, \varphi\rangle + \int_\Omega u_0\varphi(0)$$

for all $\varphi \in \mathcal{W}$ with $\varphi \geq 0$ and $\varphi(T) = 0$.

On the other hand, the existence of weak solutions in the case of nonlinear parabolic operators of Leray-Lions type requires something more than the natural assumption that $\mathcal{C}_\psi \cap \mathcal{W}$ is nonempty. In [12, Chapter II n.9] it is given a "compatibility condition" on the convex $\mathcal{C}_\psi$ which assures existence and uniqueness of the weak solution. For instance, $\mathcal{C}_\psi$ satisfies the hypotheses of compatibility if $\psi = 0$ and $u_0 = 0$; for all $f \in L^{p'}(0, T; V')$ there exists then a unique function $u \in \mathcal{C}_0$ such that

$$\int_0^T \langle v_t, v-u\rangle + \iint_Q a(x,t,Du) \cdot D(v-u) + \frac{1}{2}\int_\Omega |v(0)|^2 \geq \int_0^T \langle f, v-u\rangle,$$

for all $v \in \mathcal{C}_0 \cap \mathcal{W}$. Notice that, in this case, Theorem 3.1 can be applied only if f belongs to the ordered dual space of $L^p(0, T; V)$.

Remark 3.5. It has been pointed out in [16] that if there exists a strong solution u of the parabolic unilateral problem involving a linear operator A then u is the unique weak solution.

The same situation happens in the nonlinear case so that, under the hypotheses of Theorem 3.1, we not only have uniqueness for (strong) solutions but also uniqueness for weak solutions. Indeed, let u be the strong solution of our problem; if there exists a weak solution $\bar{u} \in \mathcal{C}_\psi$ then, since $u \in \mathcal{C}_\psi \cap \mathcal{W}$ and $u(0) = u_0$, one has

$$\int_0^T \langle u_t, u-\bar{u}\rangle + \iint_Q a(x,t,D\bar{u}) \cdot D(u-\bar{u}) \geq \int_0^T \langle f, u-\bar{u}\rangle,$$

and, from inequality (3.2),

$$\int_0^T \langle u_t, \bar{u}-u\rangle + \iint_Q a(x,t,Du) \cdot D(\bar{u}-u) \geq \int_0^T \langle f, \bar{u}-u\rangle.$$

It follows that

$$\iint_Q (a(x,t,Du) - a(x,t,D\bar{u})) \cdot D(u-\bar{u}) \leq 0$$

and so $u = \bar{u}$.

4. L^1 data

Let us consider evolutionary unilateral problems with L^1 data. We take

$$f \in L^1(Q), \qquad u_0 \in L^1(\Omega), \qquad \psi^+ \in L^1(Q), \tag{4.1}$$

and define the (nonempty) closed convex set

$$\mathcal{K}_\psi = \{\varphi \in L^1(Q) : \varphi \geq \psi \text{ almost everywhere in } Q\}. \tag{4.2}$$

We recall that the study of parabolic equations with L^1 data has been carried out using the notion of entropy solution (see [15]), introduced in [2] for elliptic equations. Here, using the concept of entropy solution, we adapt to the case of data as in (4.1) the minimality condition which characterizes the solutions of parabolic unilateral problems in the variational framework (see Remark 3.2).

Set

$$\mathcal{Y} = \{\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap C([0,T];L^1(\Omega)) : \varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)\}, \tag{4.3}$$

and observe that $\mathcal{Y} \cap L^\infty(Q) \subset C([0,T];L^\gamma(\Omega))$ for all $1 \leq \gamma < +\infty$. Moreover, by virtue of Lemma 2.4 of [6], formula (2.5) holds for any $u \in \mathcal{Y}$ such that $u(0) \in L^2(\Omega)$, where now, and in what follows, $\langle\ ,\ \rangle$ denotes the duality between $W^{-1,p'}(\Omega) + L^1(\Omega)$ and $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We will say that u is the solution of the parabolic unilateral problem with data (4.1) if u is the minimal function in $\mathcal{K}_\psi \cap C([0,T];L^1(\Omega))$ such that $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$ for all $k > 0$, and

$$\int_\Omega \Theta_k(\varphi(0) - u_0)^+ + \int_0^T \langle \varphi_t, T_k(\varphi - u)^+ \rangle + \iint_Q a(x,t,Du) \cdot DT_k(\varphi - u)^+ \geq$$
$$\geq \iint_Q f T_k(\varphi - u)^+ + \int_\Omega \Theta_k(\varphi - u)^+(T) \tag{4.4}$$

for all $k > 0$ and $\varphi \in \mathcal{Y} \cap L^\infty(Q)$.

We have the following result.

Theorem 4.1. *Let $f \in L^1(Q)$ and $u_0 \in L^2(\Omega)$. Suppose that $T_k(\psi) \in L^p(0,T;W^{1,p}(\Omega))$, $T_k(\psi - \varphi)^+ \in L^p(0,T;W_0^{1,p}(\Omega))$ for all $k > 0$ and $\varphi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, and that there exists a nonnegative function $g \in L^1(Q)$ such that*

$$\int_0^T \langle \varphi_t, T_k(\psi - \varphi)^+ \rangle + \iint_Q a(x,t,D\psi) \cdot DT_k(\psi - \varphi)^+ \leq$$
$$\leq \iint_Q (f+g) T_k(\psi - \varphi)^+ + \int_\Omega \Theta_k(u_0 - \varphi(0))^+, \tag{4.5}$$

for all $k > 0$ *and* $\varphi \in \mathcal{Y} \cap L^\infty(Q)$.

Then there exists a unique minimal function $u \in \mathcal{K}_\psi \cap C([0,T]; L^1(\Omega))$ *such that* $T_k(u)$ *belongs to* $L^p(0,T; W_0^{1,p}(\Omega))$ *for all* $k > 0$, *and* (4.4) *holds.*

Moreover, the obstacle reaction associated with u, *that is* $\mu = u_t + A(u) - f$, *is a nonnegative element of* $L^1(Q)$ *such that*

$$0 \le \mu \le g \qquad \textit{almost everywhere in } Q. \tag{4.6}$$

Before giving the proof of Theorem 4.1 we recall that the entropy solution of a nonlinear parabolic Dirichlet problem with right hand side $f \in L^1(Q)$ and initial datum $u_0 \in L^1(\Omega)$ is a function u such that $T_k(u) \in L^p(0,T; W_0^{1,p}(\Omega))$ for all $k > 0$ and satisfying the following inequality

$$\int_\Omega \Theta_k(u-\varphi)(T) + \int_0^T \langle \varphi_t, T_k(u-\varphi)\rangle + \iint_Q a(x,t,Du)\cdot DT_k(u-\varphi) \le$$
$$\le \iint_Q f T_k(u-\varphi) + \int_\Omega \Theta_k(u_0 - \varphi(0)),$$

for all $k > 0$ and $\varphi \in \mathcal{Y} \cap L^\infty(Q)$. Existence and uniqueness of the entropy solution have been proved in [15].

Proof of Theorem 4.1. The proof is divided into various steps.

The homographic approximation. We first consider the family of nonlinear parabolic problems

$$\begin{cases} u_t^\lambda + A(u^\lambda) + g\frac{u^\lambda - \psi}{\lambda + |u^\lambda - \psi|} = f + g & \text{in } Q, \\ u^\lambda(x,0) = u_0(x) & \text{in } \Omega, \\ u^\lambda(x,t) = 0 & \text{on } \partial\Omega \times (0,T), \end{cases} \tag{4.7}$$

depending on the parameter $\lambda > 0$.

Let us prove that there exists a unique entropy solution of (4.7). Take $f_n = T_n(f)$, $g_n = T_n(g)$, $u_0^n = T_n(u_0)$ and consider the approximate equations

$$\begin{cases} u_t^{\lambda,n} + A(u^{\lambda,n}) + g_n\frac{u^{\lambda,n} - \psi}{\lambda + |u^{\lambda,n} - \psi|} = f_n + g_n & \text{in } Q, \\ u^{\lambda,n}(x,0) = u_0^n & \text{in } \Omega, \\ u^{\lambda,n}(x,t) = 0 & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{4.8}$$

Due to calssical results (see [12]) problem (4.8) admits a unique solution $u^{\lambda,n} \in \mathcal{W}$; further, Theorem 7.1 of [13], Chapter III, implies that $u^{\lambda,n} \in L^\infty(Q)$. Let us point out that, since the equation gives $u_t^{\lambda,n} \in L^{p'}(0,T; W^{-1,p'}(\Omega))$, $u^{\lambda,n}$ actually belongs to $\mathcal{Y} \cap L^\infty(Q)$. One can also see, using the fact that the sequence $\{f_n + g_n(1 - \frac{u^{\lambda,n}-\psi}{\lambda + |u^{\lambda,n} - \psi|})\}$ is bounded in $L^1(Q)$ and arguing as in [5], that

$$\|T_k(u^{\lambda,n})\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \le ck,$$

and

$$\|u^{\lambda,n}\|_{L^\infty(0,T;L^1(\Omega))} \leq c,$$

for all $k > 0$, $\lambda > 0$ and $n \in \mathbb{N}$. Then, applying the well-known Gagliardo-Nirenberg embedding result (see Proposition 3.1 of [9]), it follows that

$$\operatorname{meas}(\{(x,t) \in Q : |Du^{\lambda,n}| > k\})k^{\frac{p(N+1)}{N}} \leq \iint_Q |T_k(u^{\lambda,n})|^{\frac{p(N+1)}{N}} \leq$$

$$\leq c\|T_k(u^{\lambda,n})\|^{\frac{p}{N}}_{L^\infty(0,T;L^1(\Omega))} \iint_Q |DT_k(u^{\lambda,n})|^p \leq ck,$$

and thus

$$\operatorname{meas}(\{(x,t) \in Q : |u^{\lambda,n}| > k\}) \leq ck^{-p_1}, \qquad p_1 = p\frac{N+1}{N} - 1, \tag{4.9}$$

for all $k > 0$, $\lambda > 0$ and $n \in \mathbb{N}$. Moreover, reasoning as in the elliptic case (see [2]), we deduce that

$$\operatorname{meas}(\{(x,t) \in Q : |Du^{\lambda,n}| > k\}) \leq ck^{-p_2}, \qquad p_2 = p - \frac{N}{N+1}, \tag{4.10}$$

for all $k > 0$, $\lambda > 0$ and $n \in \mathbb{N}$. From estimate (4.9), equation (4.8) and standard compactness results, one can prove that there exists u^λ such that (up to a subsequence) $u^{\lambda,n}$ converges to u^λ almost everywhere in Q. Consequently, using (4.10), (4.8) and Theorem 3.3 of [4], it follows that $Du^{\lambda,n}$ converges to Du^λ almost everywhere in Q, where Du^λ denotes the measurable function defined almost everywhere in Q by $Du^\lambda\chi_{\{|u^\lambda|<k\}} = DT_k(u^\lambda)$ for every $k > 0$. Taking now $n, m \in \mathbb{N}$ and $\tau \in (0, T]$, and multiplying (4.8) by $T_1(u^{\lambda,n} - u^{\lambda,m})\chi_{(0,\tau)}$, we get

$$\int_\Omega \Theta_1(u^{\lambda,n} - u^{\lambda,m})(\tau) \leq$$

$$\leq \int_\Omega \Theta_1(u_0^n - u_0^m) + \iint_Q |f_n - f_m| + \iint_Q |g_n - g_m| + 2\iint_Q g|T_1(u^{\lambda,n} - u^{\lambda,m})|.$$

Thus, using the fact that $u^{\lambda,n}$ converges to u^λ almost everywhere in Q and following the proof of [15], we deduce that $u^\lambda \in C([0,T]; L^1(\Omega))$ and $u^{\lambda,n}$ converges to u^λ strongly in $C([0,T]; L^1(\Omega))$. Finally, observing that the sequence $\{f_n + g_n(1 - \frac{u^{\lambda,n}-\psi}{\lambda+|u^{\lambda,n}-\psi|})\}$ strongly converges to the function $f + g(1 - \frac{u^\lambda-\psi}{\lambda+|u^\lambda-\psi|})$ in $L^1(Q)$ as $n \to +\infty$, and arguing again as in [15], one can check that u^λ is an entropy solution of equation (4.7). Further, using assumption (2.3) and the monotonicity of the homographic map $s \to \frac{s}{\lambda+|s|}$, one can easly see that if v is another entropy solution of (4.7) then $v = u^\lambda$.

Let us check that $u^\lambda \geq \psi$. Choosing $\varphi = u^{\lambda,n}$ in (4.5) and multiplying equation (4.8) by

$T_k(\psi - u^{\lambda,n})^+$, we get

$$\iint_Q (a(x,t,D\psi) - a(x,t,Du^{\lambda,n})) \cdot DT_k(\psi - u^{\lambda,n})^+ \leq$$

$$\leq \iint_Q (f - f_n) T_k(\psi - u^{\lambda,n})^+ + \int_\Omega \Theta_k(u_0 - u_0^n)^+.$$

It then follows from Fatou's lemma and hypothesis (2.3) that $T_k(\psi - u^\lambda)^+ = 0$ for all $k > 0$, and this implies $u^\lambda \geq \psi$. Consequently, equation (4.7) can be rewritten as

$$u_t^\lambda + A(u^\lambda) = f + g\frac{\lambda}{\lambda + u^\lambda - \psi}. \tag{4.11}$$

The solution of the obstacle problem. Consider the entropy solution u^λ of equation (4.11), given by the previous step. We begin by proving that u^λ is monotone nonincreasing as $\lambda \to 0^+$. Let $\lambda \leq \eta$, and let $u^{\lambda,n}$ and $u^{\eta,n}$ be the corresponding solutions of (4.8); then

$$\int_\Omega \Theta_k(u^{\lambda,n} - u^{\eta,n})^+(T) + \iint_Q (a(x,t,Du^{\lambda,n}) - a(x,t,Du^{\eta,n})) \cdot DT_k(u^{\lambda,n} - u^{\eta,n})^+ \leq$$

$$\leq \iint_Q g_n \left(\frac{u^{\eta,n} - \psi}{\eta + |u^{\eta,n} - \psi|} - \frac{u^{\lambda,n} - \psi}{\lambda + |u^{\lambda,n} - \psi|} \right) T_k(u^{\lambda,n} - u^{\eta,n})^+$$

and thus

$$\iint_Q (a(x,t,Du^\lambda) - a(x,t,Du^\eta)) \cdot DT_k(u^\lambda - u^\eta)^+ \leq$$

$$\leq \iint_Q g \left(\frac{u^\eta - \psi}{\eta + u^\eta - \psi} - \frac{u^\lambda - \psi}{\lambda + u^\lambda - \psi} \right) T_k(u^\lambda - u^\eta)^+ \leq \iint_Q g \frac{(u^\lambda - \psi)(\lambda - \eta)k}{(\lambda + u^\lambda - \psi)(\eta + u^\eta - \psi)}.$$

Then

$$\iint_Q (a(x,t,Du^\lambda) - a(x,t,Du^\eta)) \cdot DT_k(u^\lambda - u^\eta)^+ \leq 0,$$

which implies, from assumption (2.3), that $u^\lambda \leq u^\eta$.

We have then proved that there exists a function u such that $u \geq \psi$ and the solution u^λ of (4.7) converges to u, as λ goes to zero. Our goal is now to show that such a function u satisfies inequality (4.4) and the minimality property.

As a consequence of the previous step we have both the following estimates

$$\|T_k(u^\lambda)\|^p_{L^p(0,T;W_0^{1,p}(\Omega))} \leq ck,$$

and

$$\|u^\lambda\|_{L^\infty(0,T;L^1(\Omega))} \leq c,$$

for all $k > 0$ and $\lambda > 0$. Therefore $T_k(u)$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ for all $k > 0$, $u \in L^1(Q)$ and, thanks again to the Gagliardo-Nirenberg inequality, we obtain

$$\text{meas}(\{(x,t) \in Q : |u^\lambda| > k\}) \leq ck^{-p_1}, \qquad p_1 = p\frac{N+1}{N} - 1,$$

and

$$\text{meas}(\{(x,t) \in Q : |Du^\lambda| > k\}) \leq ck^{-p_2}, \qquad p_2 = p - \frac{N}{N+1},$$

for all $k > 0$ and $\lambda > 0$.

Let us take $\lambda, \eta > 0$ and $\tau \in (0,T]$. Multiplying (4.8) by $T_1(u^{\lambda,n} - u^{\eta,n})\chi_{(0,\tau)}$, and letting $n \to +\infty$, we get

$$\int_\Omega \Theta_1(u^\lambda - u^\eta)(\tau) \leq 2 \iint_Q g|T_1(u^\lambda - u^\eta)|.$$

Then, since $u^\lambda \to u$ almost everywhere in Q, we are able to establish, arguing as in [15], that u^λ is a Cauchy sequence in $C([0,T];L^1(\Omega))$, and thus $u \in C([0,T];L^1(\Omega))$ and u^λ converges to u strongly in $C([0,T];L^1(\Omega))$.

Moreover, using the fact that $\{f + g - g\frac{u^\lambda - \psi}{\lambda + |u^\lambda - \psi|}\}$ is uniformly bounded in $L^1(Q)$, it is possible to apply, once more, Theorem 3.3 of [4] to obtain the almost everywhere convergence of Du^λ to Du, where Du denotes the measurable function defined almost everywhere in Q by $Du\chi_{\{|u|<k\}} = DT_k(u)$ for every $k > 0$.

To prove (4.4) we multiply (4.8) by $T_k(u^{\lambda,n} - \varphi)^+$, with φ as in the statement of the theorem, and we get, as $n \to +\infty$,

$$\int_\Omega \Theta_k(\varphi(0) - u_0)^+ + \int_0^T \langle \varphi_t, T_k(\varphi - u^\lambda)^+\rangle + \iint_Q a(x,t,Du^\lambda) \cdot DT_k(\varphi - u^\lambda)^+ \geq$$
$$\geq \iint_Q fT_k(\varphi - u^\lambda)^+ + \int_\Omega \Theta_k(\varphi - u^\lambda)^+(T).$$

The claim then follows taking the limit as $\lambda \to 0^+$.

We now prove the minimality of u. Let $\bar{u} \in \mathcal{K}_\psi \cap C([0,T];L^1(\Omega))$ be such that $T_k(\bar{u})$ belongs to $L^p(0,T;W_0^{1,p}(\Omega))$ for all $k > 0$, and such that (4.4) holds. Setting $\varphi = u^{\lambda,n}$ in (4.4) and multiplying equation (4.8) by $T_k(u^{\lambda,n} - \bar{u})^+$, we obtain

$$\int_\Omega \Theta_k(u^{\lambda,n} - \bar{u})^+(T) + \iint_Q (a(x,t,Du^{\lambda,n}) - a(x,t,D\bar{u})) \cdot DT_k(u^{\lambda,n} - \bar{u})^+ \leq$$
$$\leq \iint_Q \left\{g_n\left(1 - \frac{u^{\lambda,n} - \psi}{\lambda + |u^{\lambda,n} - \psi|}\right) + f_n - f\right\} T_k(u^{\lambda,n} - \bar{u})^+ + \int_\Omega \Theta_k(u_0^n - u_0)^+.$$

Again, by Fatou's lemma, we have

$$\iint_Q (a(x,t,Du^\lambda) - a(x,t,D\bar{u})) \cdot DT_k(u^\lambda - \bar{u})^+ \leq \iint_Q g\frac{\lambda}{\lambda + u^\lambda - \psi}T_k(u^\lambda - \bar{u})^+.$$

Hence, since $u^\lambda \geq \psi$ and $\bar{u} \geq \psi$,

$$\iint_Q (a(x,t,Du^\lambda) - a(x,t,D\bar{u})) \cdot DT_k(u^\lambda - \bar{u})^+ \leq \lambda \iint_Q g,$$

so that it follows, again from Fatou's lemma and the coercivity of A, that $u \leq \bar{u}$.

Inequality (4.6)**.** The constructive approximation of the solution u given in the previous step allows us to verify without difficulty inequality (4.6). Let us set

$$\mu_\lambda = g\frac{\lambda}{\lambda + u^\lambda - \psi}. \tag{4.12}$$

Since $0 \leq \mu_\lambda \leq g$, there exists a function $\widehat{\mu} \in L^1(Q)$ such that, up to a subsequence, $\mu_\lambda \to \widehat{\mu}$ weakly in $L^1(Q)$ and $0 \leq \widehat{\mu} \leq g$ almost everywhere in Q. Thanks now to the penalized equation (4.7), we get

$$-\iint_Q u^\lambda \varphi_t + \iint_Q a(x,t,Du^\lambda) \cdot D\varphi - \iint_Q f\varphi = \iint_Q \varphi\mu_\lambda,$$

for every $\varphi \in \mathcal{D}(Q)$; this yields, as $\lambda \to 0^+$, that the distribution $\mu = u_t + A(u) - f$ coincides with $\widehat{\mu}$.

The proof of Theorem 4.1 is now completed. □

Remark 4.2. Observe that if $p > 2 - \frac{1}{N+1}$ then $p_2 = p - \frac{N}{N+1}$ is greater than 1. Consequently, in this case, $u \in L^q(0,T;W_0^{1,q}(\Omega))$ for any $1 \leq q < p_2$. We deduce, in particular, that $Du \in L^1(Q)$.

In the following proposition we will suppose, in addition to assumption (4.5), that $\psi^+ \in L^\infty(\Omega)$, and we will prove that the minimal function $u \in \mathcal{K}_\psi \cap C([0,T];L^1(\Omega))$ such that $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$ for all $k > 0$, and (4.4) holds, is also characterized as the unique solution of an "entropy variational inequality" (see formulation (4.13) below).

Proposition 4.3. *Assume that the obstacle function ψ satisfies assumption* (4.5) *and that $\psi^+ \in L^\infty(Q)$. Then, $\mathcal{K}_\psi \cap \mathcal{Y} \cap L^\infty(Q)$ is nonempty and the solution u of the parabolic unilateral problem with data* (4.1) *is the unique function such that $u \in \mathcal{K}_\psi \cap C([0,T];L^1(\Omega))$, $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$ for all $k > 0$, and*

$$\int_\Omega \Theta_k(\varphi(0) - u_0) + \int_0^T \langle \varphi_t, T_k(\varphi - u)\rangle + \iint_Q a(x,t,Du) \cdot DT_k(\varphi - u) \geq$$
$$\geq \iint_Q fT_k(\varphi - u) + \int_\Omega \Theta_k(\varphi - u)(T) \tag{4.13}$$

for all $k > 0$ and $\varphi \in \mathcal{K}_\psi \cap \mathcal{Y} \cap L^\infty(Q)$.

Remark 4.4. Let us point out that if $\mathcal{K}_\psi \cap \mathcal{Y} \cap L^\infty(Q)$ is nonempty then $\psi^+ \in L^\infty(Q)$. From Proposition 4.3 it thus follows that, under assumption (4.5), $\mathcal{K}_\psi \cap \mathcal{Y} \cap L^\infty(Q)$ is nonempty if, and only if, $\psi^+ \in L^\infty(Q)$.

Proof of Proposition 4.3. We first observe that if $\mathcal{K}_\psi \cap \mathcal{Y} \cap L^\infty(Q)$ is nonempty (and thus inequality (4.13) makes sense) and ψ satisfies assumption (4.5), then the solution u of the obstacle problem is also a solution of the entropy variational inequality (4.13). Indeed, multiplying (4.8) by $T_k(\varphi - u^{\lambda,n})$, with $\varphi \in \mathcal{K}_\psi \cap \mathcal{Y} \cap L^\infty(Q)$, and letting $n \to +\infty$, we have

$$\begin{aligned}
&\int_\Omega \Theta_k(u^\lambda - \varphi)(T) + \int_0^T \langle \varphi_t, T_k(u^\lambda - \varphi)\rangle + \iint_Q a(x,t,Du^\lambda)\cdot DT_k(u^\lambda - \varphi) \leq \\
&\leq \int_\Omega \Theta_k(u_0 - \varphi(0)) + \iint_Q fT_k(u^\lambda - \varphi) + \iint_Q g\frac{\lambda}{\lambda + u^\lambda - \psi}T_k(u^\lambda - \varphi) \leq \\
&\leq \int_\Omega \Theta_k(u_0 - \varphi(0)) + \iint_Q fT_k(u^\lambda - \varphi) + \lambda \iint_Q g,
\end{aligned}$$

which yields, taking the limit as $\lambda \to 0^+$, inequality (4.13).

We now check that if assumption (4.5) is satisfied and $\psi^+ \in L^\infty(Q)$, then $\mathcal{K}_\psi \cap \mathcal{Y} \cap L^\infty(Q)$ is nonempty. Approximate T_j by a sequence of regular functions T_j^ε : for $j > \varepsilon$, define $T_j^\varepsilon \in C^2(\mathbb{R},\mathbb{R})$ as follows

$$\begin{cases}
(T_j^\varepsilon)'(s) = 0 & \text{if } |s| \geq j, \\
(T_j^\varepsilon)'(s) = 1 & \text{if } |s| \leq j - \varepsilon, \\
|T_j^\varepsilon(s)| \leq |s| & \text{for all } s \in \mathbb{R}, \\
\leq (T_j^\varepsilon)'(s) \leq 1 & \text{for all } s \in \mathbb{R}.
\end{cases}$$

Let us choose $j > \varepsilon + \|\psi^+\|_\infty$, so that $T_j^\varepsilon(u^\lambda) \geq T_j^\varepsilon(\psi) \geq \psi$ almost everywhere in Q. Multiplying equation (4.8) by $(T_j^\varepsilon)'(u^{\lambda,n})\varphi$, with $\varphi \in \mathcal{D}(Q)$, and by passage to the limit $(n \to +\infty)$, we get

$$\begin{aligned}
&(T_j^\varepsilon(u^\lambda))_t - \operatorname{div}\left((T_j^\varepsilon)'(u^\lambda)a(x,t,Du^\lambda)\right) + (T_j^\varepsilon)''(u^\lambda)a(x,t,Du^\lambda)\cdot Du^\lambda = \\
&\qquad = \left(f + g\frac{\lambda}{\lambda + u^\lambda - \psi}\right)(T_j^\varepsilon)'(u^\lambda) \quad \text{in } \mathcal{D}'(Q), \qquad (4.14)
\end{aligned}$$

which implies that $(T_j^\varepsilon(u^\lambda))_t$ belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$. As a consequence $T_j^\varepsilon(u^\lambda)$ is in $\mathcal{K}_\psi \cap \mathcal{Y} \cap L^\infty(Q)$. This yields that $\mathcal{K}_\psi \cap \mathcal{Y} \cap L^\infty(Q)$ is nonempty.

To prove uniqueness of the solution of inequality (4.13), we show that if $\bar{u}$ belongs to $\mathcal{K}_\psi \cap C([0,T];L^1(\Omega))$, $T_k(\bar{u}) \in L^p(0,T;W_0^{1,p}(\Omega))$ for all $k > 0$, and

$$\begin{aligned}
&\int_\Omega \Theta_k(\varphi(0) - u_0) + \int_0^T \langle \varphi_t, T_k(\varphi - \bar{u})\rangle + \iint_Q a(x,t,D\bar{u})\cdot DT_k(\varphi - \bar{u}) \geq \\
&\qquad \geq \iint_Q fT_k(\varphi - \bar{u}) + \int_\Omega \Theta_k(\varphi - \bar{u})(T), \qquad (4.15)
\end{aligned}$$

for all $k > 0$ and $\varphi \in \mathcal{K}_\psi \cap \mathcal{Y} \cap L^\infty(Q)$, then $\bar{u} = u$. To this end we will essentially use the same method as for the proof of uniqueness of entropy solutions for parabolic equations with L^1 data given by A. Prignet in [15].

We first observe that (4.14) yields

$$\int_0^T \langle (T_j^\varepsilon(u^\lambda))_t, \varphi \rangle + \iint_Q (T_j^\varepsilon)'(u^\lambda) a(x,t,Du^\lambda) \cdot D\varphi + \\ + \iint_Q (T_j^\varepsilon)''(u^\lambda) a(x,t,Du^\lambda) \cdot Du^\lambda \varphi = \iint_Q \left(f + g \frac{\lambda}{\lambda + u^\lambda - \psi} \right) (T_j^\varepsilon)'(u^\lambda)\varphi,$$

for all $\varphi \in L^p(0,T; W_0^{1,p}(\Omega)) \cap L^\infty(Q)$, and thus for $\varphi = T_k(T_j^\varepsilon(u^\lambda) - \bar{u})$. Then, since

$$(T_j^\varepsilon)'(u^\lambda) T_k(T_j^\varepsilon(u^\lambda) - \bar{u}) \le \\ \le (T_j^\varepsilon)'(u^\lambda) T_k(T_j^\varepsilon(u^\lambda) - \psi) \le (T_j^\varepsilon)'(u^\lambda)(T_j^\varepsilon(u^\lambda) - \psi) \le (T_j^\varepsilon(u^\lambda) - \psi)\chi_{\{|u^\lambda| \le j\}},$$

we obtain

$$\int_0^T \langle (T_j^\varepsilon(u^\lambda))_t, T_k(T_j^\varepsilon(u^\lambda) - \bar{u}) \rangle + \iint_Q (T_j^\varepsilon)'(u^\lambda) a(x,t,Du^\lambda) \cdot DT_k(T_j^\varepsilon(u^\lambda) - \bar{u}) + \\ + \iint_Q (T_j^\varepsilon)''(u^\lambda) a(x,t,Du^\lambda) \cdot Du^\lambda T_k(T_j^\varepsilon(u^\lambda) - \bar{u}) \le \\ \le \iint_Q f(T_j^\varepsilon)'(u^\lambda) T_k(T_j^\varepsilon(u^\lambda) - \bar{u}) + \lambda \iint_{\{|u^\lambda| \le j\}} g \frac{T_j^\varepsilon(u^\lambda) - \psi}{\lambda + u^\lambda - \psi}. \quad (4.16)$$

Now, taking $\varphi = T_j^\varepsilon(u^\lambda) \in \mathcal{K}_\psi \cap \mathcal{Y} \cap L^\infty(Q)$ as a test function in (4.15), we also have

$$\int_\Omega \Theta_k(T_j^\varepsilon(u_0) - u_0) + \int_0^T \langle (T_j^\varepsilon(u^\lambda))_t, T_k(T_j^\varepsilon(u^\lambda) - \bar{u}) \rangle + \iint_Q a(x,t,D\bar{u}) \cdot DT_k(T_j^\varepsilon(u^\lambda) - \bar{u}) \\ \ge \iint_Q fT_k(T_j^\varepsilon(u^\lambda) - \bar{u}) + \int_\Omega \Theta_k(T_j^\varepsilon(u^\lambda) - \bar{u})(T), \quad (4.17)$$

which implies, from (4.16), that

$$\iint_Q (a(x,t,DT_j^\varepsilon(u^\lambda)) - a(x,t,D\bar{u})) \cdot DT_k(T_j^\varepsilon(u^\lambda) - \bar{u}) +$$
$$+ \int_\Omega \Theta_k(T_j^\varepsilon(u^\lambda) - \bar{u})(T) + \iint_Q f(1 - (T_j^\varepsilon)'(u^\lambda))T_k(T_j^\varepsilon(u^\lambda) - \bar{u}) \leq$$
$$\leq \int_\Omega \Theta_k(T_j^\varepsilon(u_0) - u_0) + k \iint_Q |(T_j^\varepsilon)''(u^\lambda)| a(x,t,Du^\lambda) \cdot Du^\lambda + \lambda \iint_{\{|u^\lambda| \leq j\}} g \frac{T_j^\varepsilon(u^\lambda) - \psi}{\lambda + u^\lambda - \psi}. \tag{4.18}$$

Using the fact that $\lim_{\varepsilon \to 0^+} \lambda \iint_{\{|u^\lambda| \leq j\}} g \frac{T_j^\varepsilon(u^\lambda) - \psi}{\lambda + u^\lambda - \psi} \leq \lambda \iint_Q g$, and following the same calculations of [15], we can take the limit in (4.18) as $\varepsilon \to 0^+$, then $\lambda \to 0^+$ and then $j \to +\infty$, and we obtain

$$\int_\Omega \Theta_k(u - \bar{u})(T) + \iint_Q (a(x,t,Du) - a(x,t,D\bar{u})) \cdot DT_k(u - \bar{u}) \leq 0, \quad \text{for all } k > 0.$$

From assumption (2.3), it then follows that $u = \bar{u}$. □

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