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Nonexistence of Solutions in Nonconvex Multidimensional Variational Problems

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In the scalar n-dimensional situation, the extreme points in the set of certain gradient L^p-Young measures are studied. For $n = 1$, such Young measures must be composed from Diracs, while for $n \geq 2$ there are non-Dirac extreme points among them, for $n \geq 3$, some are even weakly* continuous. This is used to construct nontrivial examples of nonexistence of solutions of the minimization-type variational problem $\int_\Omega W(x, \nabla u)\, \mathrm{d}x$ with a Carathéodory (if $n \geq 2$) or even continuous (if $n \geq 3$) integrand W.

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1. Introduction

In this paper we want to study nonattainment due to material nonhomogeneity which may occur even in scalar variational problems. For this goal, we avoid the influence of any boundary conditions and consider the following scalar multidimensional variational problem

$$\text{Minimize } \int_\Omega W(x, \nabla u)\, \mathrm{d}x \qquad \text{for } \ u \in W^{1,p}(\Omega), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded simply connected Lipschitz domain, $W^{1,p}(\Omega)$ is the Sobolev space $\{u \in L^p(\Omega);\ \nabla u \in L^p(\Omega; \mathbb{R}^n)\}$, and $W : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function satisfying

$$\exists\ c_0,\ c_1 > 0,\ c_2 \qquad \forall (x, s) \in \Omega \times \mathbb{R}^n : \qquad c_0 + c_1|s|^p \leq W(x, s) \leq c_2(1 + |s|^p). \tag{1.2}$$

This condition ensures that the functional $u \mapsto \int_\Omega W(x, \nabla u)\, \mathrm{d}x$ is well defined on $W^{1,p}(\Omega)$ and coercive on the subspace

$$\tilde{W}^{1,p}(\Omega) := \left\{ u \in W^{1,p}(\Omega);\ \int_\Omega u(x)\, \mathrm{d}x = 0 \right\}. \tag{1.3}$$

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We are especially interested in the case when $W(x,\cdot)$ is nonconvex. Such problems often do not have any solution; we then speak about nonattainment while in the converse case we speak about attainment. For $n=1$ the attainment is quite typical; in fact, it holds even for Dirichlet boundary conditions and u vector-valued. For this sorts of results we refer to Aubert and Tahraoui [1, 2], Cellina and Colombo [7], Mariconda [14], Ornelas [16] and Raymond [18, 21]. This effect has also been known for a long time in context of optimal control theory, see Cesari [8]. However, contrary to the one-dimensional case, for $n \geq 2$ the Dirichlet boundary conditions may create very easily the nonattainment effect even if W is independent of x, though in special situations the attainment effect can occur even there. This spatially homogeneous case has been studied by Bauman and Phillips [5], Chipot [9], Cellina [6], Flores [10], Friesecke [11], Marcellini [13], Mascolo and Schianchi [15], and Raymond [19, 20]. To avoid rather trivial situations for nonattainment, we thus did not involve Dirichlet boundary conditions into (1.1) but allowed nonhomogeneous situations, i.e. W depends on x. For the relations with Dirichlet conditions in the homogeneous case we refer to [11, Theorem 1].

It seems that in our scalar case the attainment is closely related through the Bauer extremal principle [4] with the structure of extreme points of the set of admissible pairs (u,ν) for the so-called relaxed problem, as pointed out by Balder [3] in a more general context; here ν stands for a Young measure related with u by (2.3) below. We will address this question in Section 2, showing that attainment appears if all extreme points are trivial while existence of nontrivial "uniformly proper" (in the sense of (2.9) below) extreme points allows us to construct examples of (1.1) with non-attainment relying on the x-dependence of W. Thus, our aim is to construct the nontrivial extreme points in case $n \geq 2$. More precisely, in Section 3 we will do this for $n \geq 2$ by highly oscillatory manner, using Cantor sets, which result to W a mere Carathéodory function, while in Section 4 we will show it for $n \geq 3$ by vector fields with no nontrivial integration factors, which leads to W jointly continuous.

2. Extreme gradient Young measures

Since W does not depend on u but only on ∇u, we can replace $W^{1,p}(\Omega)$ in (1.1) by $\tilde{W}^{1,p}(\Omega)$. Then every solution of such modified problem solves (1.1) and conversely every solution to (1.1) can be shifted by a suitable constant to satisfy $\int_\Omega u\mathrm{d}x = 0$, obtaining thus a solution of the modified problem.

Then a natural relaxation of the variational problem (1.1) takes the form

$$\begin{cases} \text{Minimize} & \displaystyle\int_\Omega \int_{\mathbb{R}^n} W(x,s)\nu_x(\mathrm{d}s)\mathrm{d}x \\ \text{subject to} & \nabla u(x) = \displaystyle\int_{\mathbb{R}^n} s\,\nu_x(\mathrm{d}s) \quad \text{for a.a. } x\in\Omega, \\ & u\in \tilde{W}^{1,p}(\Omega), \quad \nu \in \mathcal{Y}^p(\Omega;\mathbb{R}^n), \end{cases} \tag{2.1}$$

where $\mathcal{Y}^p(\Omega;\mathbb{R}^n)$ is the set of the so-called L^p-Young measures defined by

$$\begin{aligned} \mathcal{Y}^p(\Omega;\mathbb{R}^n) := \big\{ \nu : x \mapsto \nu_x : \Omega \to \mathrm{rca}(\mathbb{R}^n) &\text{ weakly measurable;} \\ \nu_x \text{ is a probability measure for a.a. } &x\in\Omega, \\ \int_\Omega\int_{\mathbb{R}^n} |s|^p \nu_x(\mathrm{d}s)\mathrm{d}x < +\infty &\big\} \end{aligned} \tag{2.2}$$

where $\mathrm{rca}(\mathbb{R}^m) \cong C_0(\mathbb{R}^m)^*$ stands for the set of the Radon measures on $\mathbb{R}^m$.

Proposition 2.1 (Kinderlehrer-Pedregal [12]). *Let* (1.2) *be valid for* $p > 1$. *Then* $\inf(1.1) = \inf(2.1)$ *and* (2.1) *always has a solution.*

In fact, [12] deals with even a much more complicated vectorial situation. We refer also to monographs [17, Theorems 4.4 and 8.7] and [22, Proposition 5.2.6].

If (u,ν) is a solution to (2.1) and ν is a Dirac mass a.e. on Ω, then obviously $\nu_x = \delta_{\nabla u(x)}$ for a.a. $x \in \Omega$ with $\delta_s \in \mathrm{rca}(\mathbb{R}^n)$ denoting the Dirac mass supported at $s \in \mathbb{R}^n$. Then Proposition 2.1 implies that u solves the original problem (1.1).

The problem of nonattainment in (1.1) will be shown to have an intimate relation with the structure of extreme points of the convex set of the admissible pairs for (2.1), i.e. the set

$$D_{\mathrm{ad}} := \left\{ (u,\nu) \in \tilde{W}^{1,p}(\Omega) \times \mathcal{Y}^p(\Omega;\mathbb{R}^n);\ \nabla u(x) = \int_{\mathbb{R}^n} s\,\nu_x(\mathrm{d}s) \text{ for a.a. } x \in \Omega \right\}. \tag{2.3}$$

Note that, for any $(u,\nu) \in D_{\mathrm{ad}}$, the L^p-Young measure ν satisfies

$$\mathrm{curl}\Big(\int_{\mathbb{R}^n} s\,\nu_x(\mathrm{d}s)\Big) = 0. \tag{2.4}$$

Let us remind that $(u,\nu) \in D_{\mathrm{ad}}$ is called an extreme point if

$$\forall (u_1,\nu_1),(u_2,\nu_2) \in D_{\mathrm{ad}} :\ \frac{1}{2}(u_1,\nu_1) + \frac{1}{2}(u_2,\nu_2) = (u,\nu) \ \Rightarrow\ (u_1,\nu_1) = (u_2,\nu_2). \tag{2.5}$$

Note that, as $u \in \tilde{W}^{1,p}(\Omega)$ is determined uniquely by ν, the pair (u,ν) is an extreme point in D_{ad} if and only if ν is an extreme point in $\mathcal{G}^p(\Omega;\mathbb{R}^n) := \{\nu;\ \exists u:\ (u,\nu) \in D_{\mathrm{ad}}\}$, i.e. in the set of the so-called gradient L^p-Young measures whose underlying displacement u has the average $\int_\Omega u\mathrm{d}x$ zero. The interesting question is whether all extreme points of $\mathcal{G}^p(\Omega;\mathbb{R}^n)$ are composed from Dirac masses a.e. on Ω, i.e.

$$\begin{aligned}\Big(\nu_1,\nu_2 \in \mathcal{G}^p(\Omega;\mathbb{R}^n)\ \&\ \frac{\nu_1+\nu_2}{2} = \nu \ \Rightarrow\ \nu_1 = \nu_2\Big) \\ \Rightarrow\ \nu_x \text{ is a Dirac mass for a.a. } x \in \Omega.\end{aligned} \tag{2.6}$$

Proposition 2.2. *Let* (1.2) *be valid for* $p > 1$ *and let* (2.6) *hold. Then* (1.1) *has a solution.*

Sketch of the proof. By the coercivity (1.2), all minimizing sequences $\{u_k\}$ for (1.1) must eventually satisfy $\|\nabla u_k\|_{L^p(\Omega;\mathbb{R}^n)} \le R$ for R large enough, e.g. for $R > (\mathrm{meas}(\Omega)(c_2 - c_0)/c_1)^{1/p}$ with c_0, c_1 and c_2 from (1.2). Then the problem (2.1) has the same solution as the minimization problem of the affine continuous functional $(u,\nu) \mapsto \int_\Omega \int_{\mathbb{R}^n} W(x,s)\,\nu_x(\mathrm{d}s)\mathrm{d}x$ over the convex weakly* compact set

$$D_{\mathrm{ad}}^{\varrho} := \left\{ (u,\nu) \in D_{\mathrm{ad}};\ \left(\int_\Omega \int_{\mathbb{R}^n} |s|^p \nu_x(\mathrm{d}s)\mathrm{d}x\right)^{1/p} \le \varrho \right\} \tag{2.7}$$

provided $\varrho \geq R$. Using this fact for $\varrho = 2^{1/p}R$, we can see that there is at least one solution (u,ν) of this problem (and thus also of (2.1)) which is an extreme point in $D_{\rm ad}^{2^{1/p}R}$; this follows from Bauer's extremal principle [4] which says that a concave continuous functional on a convex compact set attains its minimum in an extreme point of this set. Yet, every solution to (2.1) must lie in $D_{\rm ad}^R$, so it is also true that $(u,\nu) \in D_{\rm ad}^R$. Then (u,ν) is an extreme point even in $D_{\rm ad}$ because $\frac{1}{2}(u_1,\nu_1) + \frac{1}{2}(u_2,\nu_2) = (u,\nu)$ for $(u_1,\nu_1) \in D_{\rm ad}$ and $(u_2,\nu_2) \in D_{\rm ad}$ implies automatically $(u_1,\nu_1) \in D_{\rm ad}^{2^{1/p}R}$ due to the estimate

$$\int_\Omega\int_{\mathbb{R}^n}|s|^p\nu_{1,x}(\mathrm{d}s)\mathrm{d}x \;=\; \int_\Omega\int_{\mathbb{R}^n}|s|^p(2\nu_x - \nu_{2,x})(\mathrm{d}s)\mathrm{d}x \;\leq\; 2\int_\Omega\int_{\mathbb{R}^n}|s|^p\nu_x(\mathrm{d}s)\mathrm{d}x \;\leq\; 2R^p,$$

and by symmetry also $(u_2,\nu_2) \in D_{\rm ad}^{2^{1/p}R}$. Then (2.6) implies that ν is a Dirac mass for a.a. $x \in \Omega$, so that necessarily $\nu_x = \delta_{\nabla u(x)}$ and u solves (1.1). □

Proposition 2.3. *Let $n = 1$. Then the condition (2.6) actually holds.*

Sketch of the proof. In case $n = 1$, the equality constraints in (2.3), i.e. $\nabla u(x) = \int_{\mathbb{R}^n} s\,\nu_x(\mathrm{d}s)$ and $\int_\Omega u(x)\mathrm{d}x = 0$, do not represent any actual restriction in the sense that $\mathcal{Y}^p(\Omega;\mathbb{R}) = \mathcal{G}^p(\Omega;\mathbb{R})$. Yet the extreme points in $\mathcal{Y}^p(\Omega;\mathbb{R})$ have just the structure $\nu = \{\delta_{y(x)}\}_{x\in\Omega}$ with $y \in L^p(\Omega)$, see [22, Proposition 3.2.11]. □

Therefore, Propositions 2.2 and 2.3 immediately implies the attainment in (1.1) for $n = 1$.

The converse implication, which may come into considerations for $n \geq 2$, i.e. whether failure of (2.6) implies nonattainment in (1.1), is not completely clear. Nevertheless, some "non-Dirac" extreme points of $\mathcal{G}^p(\Omega;\mathbb{R}^n)$ can give an immediate hint for a counter-example of attainment in (1.1) relying on a uniqueness (at least "locally", cf. (2.14) below) of the solution to (2.1) being just this extreme point. Let us first realize that any extreme point ν of $\mathcal{G}^p(\Omega;\mathbb{R}^n)$ can be decomposed as a pointwise convex combination of at most $n+1$ terms:

$$\text{for a.a. } x\in\Omega \;\; \exists\, m(x) \in \mathbb{N},\; m(x) \leq n+1, \;\; \exists\, a_i(x) \in [0,1], \;\; \exists\, y_i(x) \in \mathbb{R}^n :$$
$$\{y_i(x)\}_{i=1}^{m(x)} \text{ is an } m(x)\text{-dimensional simplex} \quad \& \quad \nu_x = \sum_{i=1}^{m(x)} a_i(x)\delta_{y_i(x)}, \tag{2.8}$$

otherwise one can easily construct mutually different $\nu_1,\nu_2 \in \mathcal{Y}^p(\Omega;\mathbb{R}^n)$ with the same barycenter as ν, and therefore $\nu_1,\nu_2 \in \mathcal{G}^p(\Omega;\mathbb{R}^n)$.

In view of (2.2), m can be considered measurable and, for all $i = 1,...,m(x)$, also (a_i, y_i) can be considered defined on a measurable set and itself to be measurable.

Proposition 2.4. *Let $(u,\nu) \in D_{\rm ad}$ be an extreme point of $D_{\rm ad}$ such that (2.8) holds with*

$$\exists \varepsilon > 0 \;\; \forall x\in\Omega \textit{ (a.a.) } \;\forall i = 1,...,m(x): \qquad a_i(x) \geq \varepsilon\,. \tag{2.9}$$

If $m(x) \geq 2$ for x from a non-zero measure set, then the problem (1.1) does not have any solution if the "energy density" W is taken to be

$$W(x,s) := \min_{i=1,\dots,m(x)} |s - y_i(x)|^p. \tag{2.10}$$

Proof. First, (u,ν) in question solves the relaxed problem (2.1) with W from (2.10) because obviously $\int_\Omega\int_{\mathbb{R}^n} W(x,s)\nu_x(\mathrm{d}s)\mathrm{d}x = \int_\Omega \sum_{i=1}^{m(x)} a_i(x)W(x,y_i(x))\mathrm{d}x = 0 \le \min(2.1)$ as $0\le W$ and $W(x,y_i(x))\equiv 0$. In particular, $\min(2.1)=0$.

We will prove that this solution is unique. Suppose that $(u_1,\nu_1)\in D_{\rm ad}$ is another solution to (2.1). As $\min(2.1)=0$, $\nu_{1,x}$ must be supported on the finite set $\{y_i(x)\}_{i=1}^{m(x)}$, i.e.

$$\exists\ a_{1,i}(x)\in[0,1]:\qquad \nu_{1,x}=\sum_{i=1}^{m(x)} a_{1,i}(x)\delta_{y_i(x)} \tag{2.11}$$

for a.a. $x\in\Omega$. As ν_1 is weakly measurable, $a_{1,i}$ can be assumed measurable, too. For a parameter $\alpha\in\mathbb{R}$, take (u_α,ν_α) given by

$$\begin{aligned} u_\alpha := \alpha u_1+(1-\alpha)u\ ,\qquad \nu_{\alpha,x} &:= \sum_{i=1}^{m(x)} a_{\alpha,i}(x)\delta_{y_i(x)}\ ,\\ a_{\alpha,i}(x) &:= \alpha a_{1,i}(x)+(1-\alpha)a_i(x)\ . \end{aligned} \tag{2.12}$$

Obviously, $L:=\{(u_\alpha,\nu_\alpha);\ \alpha\in\mathbb{R}\}$ is just the line going through (u,ν) and (u_1,ν_1). We then have

$$\int_\Omega u_\alpha \mathrm{d}x = \alpha\int_\Omega u_1\mathrm{d}x+(1-\alpha)\int_\Omega u\mathrm{d}x = 0 \tag{2.13}$$

and also, for a.a. $x\in\Omega$,

$$\begin{aligned}\int_{\mathbb{R}^n} s\,\nu_{\alpha,x}(\mathrm{d}s) &= \sum_{i=1}^{m(x)} a_{\alpha,i}(x)y_i(x) = \alpha\sum_{i=1}^{m(x)} a_{1,i}(x)y_i(x)+(1-\alpha)\sum_{i=1}^{m(x)} a_i(x)y_i(x)\\ &= \alpha\int_{\mathbb{R}^n} s\,\nu_{1,x}(\mathrm{d}s)+(1-\alpha)\int_{\mathbb{R}^n} s\,\nu_1(\mathrm{d}s) = \alpha\nabla u_1(x)+(1-\alpha)\nabla u(x)=\nabla u_\alpha(x).\end{aligned}$$

Hence we can see that (u_α,ν_α) will belong to $D_{\rm ad}$ provided $\nu_\alpha\in\mathcal{Y}^p(\Omega;\mathbb{R}^n)$. As always

$$\sum_{i=1}^{m(x)} a_{\alpha,i}(x) = \alpha\sum_{i=1}^{m(x)} a_{1,i}(x)+(1-\alpha)\sum_{i=1}^{m(x)} a_i(x) = \alpha+(1-\alpha)=1,$$

it suffices $a_{\alpha,i}(x)\in[0,1]$. This is valid for any $\alpha\in[-\varepsilon/(1-\varepsilon),1]$ because of (2.12) and of the assumptions $a_i(x)\in[\varepsilon,1]$ and $a_{1,i}(x)\in[0,1]$. However, this shows that $(u_1,\nu_1)=(u,\nu)$ otherwise $(u,\nu)=(u_0,\nu_0)$ would lie in the interior of the segment $D_{\rm ad}\cap L$ and could not thus be an extreme point in $D_{\rm ad}$.

Suppose for a moment that (1.1) has a solution, say $w\in W^{1,p}(\Omega)$. Without loss of generality we may suppose that $w\in\tilde{W}^{1,p}(\Omega)$. Then $(w,\delta_{\nabla w})$ solves (2.1). As we supposed $\{y_i(x)\}_{i=1}^{m(x)}$ not identically a singleton (recall that $m(x)\ge 2$ for x from a non-zero measure set, $\{y_i(x)\}_{i=1}^{m(x)}$ is an $m(x)$-dimensional simplex, and a_i are nonvanishing), ν from (2.8) must differ from $\delta_{\nabla w}$. Thus we can see that (2.1) would have two solutions, a contradiction. □

Hence, to construct examples of nonattainment in (1.1), we will seek extreme points in $D_{\rm ad}$ of the form (2.8) which are not identically Diracs and whose coefficients a_i are away from zero, as assumed in (2.9). Let us still remark that one can construct other examples than (2.10), not relying on uniqueness of the solution to (2.1); assuming $|y_i|$ bounded by r, it suffices to take a simply connected subdomain Ω_0 of Ω such that $\mathrm{meas}(\{x \in \Omega_0;\ m(x) \geq 2\}) > 0$, and then modify (2.10) as follows:

$$W(x,s) := \begin{cases} \min_{i=1,\dots,m(x)} |s - y_i(x)|^p & \text{for } x \in \Omega_0, \\ \max(|s|^p, r) & \text{for } x \in \Omega \setminus \Omega_0. \end{cases} \tag{2.14}$$

The modification of the proof of Proposition 2.4 is simple, and the solution (u,ν) to (2.1) is not unique because ν_x can be supported arbitrarily in the ball of the radius r if $x \in \Omega \setminus \Omega_0$.

3. The case $n \geq 2$

We will construct explicitly an extreme pair (u,ν) with ν satisfying the assumptions of Proposition 2.4. For this, we take n Cantor sets $K_1, K_2, \dots, K_n \subset \mathbb{R}$ having a positive Lebesgue measure and positioned in such a way that $K := K_1 \times K_2 \times \dots \times K_n \subset \Omega \subset \mathbb{R}^n$.

Proposition 3.1. *Let $y \in L^\infty(\Omega;\mathbb{R}^n)$ be arbitrary, $n \geq 2$. Then the pair (u,ν) with*

$$u(x) = 0\ , \quad \nu_x := \begin{cases} \frac{1}{2}\delta_{y(x)} + \frac{1}{2}\delta_{-y(x)} & \text{if } x \in K := K_1 \times K_2 \times \dots \times K_n, \\ \delta_0 & \text{if } x \in \Omega \setminus K, \end{cases} \tag{3.1}$$

forms an extreme point in $D_{\rm ad}$.

Proof. Take $(u_1,\nu_1) \in D_{\rm ad}$ and $(u_2,\nu_2) \in D_{\rm ad}$, and assume that

$$\frac{1}{2}u_1 + \frac{1}{2}u_2 = u \quad \text{and} \quad \frac{1}{2}\nu_1 + \frac{1}{2}\nu_2 = \nu. \tag{3.2}$$

The latter equality implies, in particular, that $\nu_{1,x} = \nu_{2,x} = \delta_0$ for $x \in \Omega \setminus K$, which is an open, connected (if $n \geq 2$ as supposed), and dense subset of $\mathbb{R}^n$. For $x \in K$, both $\nu_{1,x}$ and $\nu_{2,x}$ must be supported on the set $\{y(x), -y(x)\}$ which is supposed bounded in $\mathbb{R}^n$ uniformly with respect to x because $y \in L^\infty(\Omega;\mathbb{R}^n)$. Therefore, both u_1 and u_2 have a bounded gradient almost everywhere, and thus they are (even Lipschitz) continuous. As $\nabla u_1(x) = \nabla u_2(x) = 0$ for x ranging a dense, open, connected subset of Ω, both u_1 and u_2 must be constant everywhere on Ω. Since $u_1, u_2 \in \tilde{W}^{1,p}(\Omega)$ so that obviously $\int_\Omega (u_1 - u_2)\mathrm{d}x = 0$ we have $u_1 = 0 = u_2$.

Then $\nabla u_1 = 0 = \nabla u_2$ a.e. on Ω, which fixes the barycenter of $\nu_{1,x}$ and $\nu_{2,x}$. As both these measures are at most two-atomic, we got inevitably $\nu_{1,x} = \nu_x = \nu_{2,x}$ for a.a. $x \in \Omega$.

Altogether, we proved that $(u_1,\nu_1) = (u_2,\nu_2)$, which just means by the definition (2.3) that (u,ν) is indeed an extreme point in $D_{\rm ad}$. □

As mentioned, we will consider each K_1, K_2, ... K_n having a positive one-dimensional Lebesgue measure and the vector field y not vanishing on their product K. Then Proposition 3.1 gives an example of an extreme point $(u,\nu) \in D_{\rm ad}$ with ν not a Dirac mass a.e.

on Ω. More precisely, this ν satisfies the assumptions in Proposition 2.4 with $\varepsilon = 1/2$, so that there is the nonattainment in (1.1) provided the energy density W is defined as

$$W(x,s) := \begin{cases} \min(|s-y(x)|^p, |s+y(x)|^p) & \text{if } x \in K := K_1 \times K_2 \times \dots \times K_n, \\ |s|^p & \text{if } x \in \Omega \setminus K. \end{cases} \tag{3.3}$$

As y is measurable, W is a Carathéodory function. Moreover, as y is essentially bounded, (1.2) is satisfied, too.

4. The case $n \geq 3$

The construction from Section 3 yields an example of an extreme gradient Young measure $\nu \in \mathcal{G}^p(\Omega;\mathbb{R}^n)$ which is not a Dirac mass a.e. but $x \mapsto \nu_x$ oscillates widely. Now we will use a more sophisticated construction to show explicitly an extreme $\nu \in \mathcal{G}^p(\Omega;\mathbb{R}^n)$ such that ν_x is not a Dirac mass a.e. and even $x \mapsto \nu_x : \Omega \to \mathrm{rca}(\mathbb{R}^n)$ is weakly* continuous. We will consider only $n = 3$, the case $n \geq 4$ being then similar.

For this, we will use a vector field $y \in C^\infty(\bar{\Omega};\mathbb{R}^3)$ which does not have an integration factor, which means that there is no $\lambda \in L^\infty(\Omega)$ nontrivial (i.e. not identically zero) such that $\mathrm{curl}(\lambda y) = 0$ in the sense of distributions.

Lemma 4.1. *If $y \in C^\infty(\bar{\Omega};\mathbb{R}^3)$ satisfies $y \cdot \mathrm{curl}(y) \neq 0$ a.e. on Ω, then this field has no measurable nontrivial integration factor.*

Proof. Since $\mathrm{curl}(\lambda y) = \lambda\, \mathrm{curl}(y) + \nabla\lambda \times y$, it holds in the classical sense that

$$y \cdot \mathrm{curl}(\lambda y) = y \cdot (\lambda\ \mathrm{curl}(y)) + y \cdot (\nabla\lambda \times y) = \lambda(y \cdot \mathrm{curl}(y)) \tag{4.1}$$

at least for smooth λ's. This implies $y\cdot\mathrm{curl}(\lambda y) = \lambda(y\cdot\mathrm{curl}(y))$ in the sense of distributions even for any $\lambda \in L^\infty(\Omega)$ because one can make a limit passage with smooth λ's in the integral identity

$$\int_\Omega \lambda y \cdot \mathrm{curl}(yv)\mathrm{d}x = \int_\Omega \lambda(y \cdot \mathrm{curl}(y))v\mathrm{d}x \tag{4.2}$$

for any fixed $v \in C^\infty(\Omega)$ compactly supported. Thus, if $\mathrm{curl}(\lambda y) = 0$ in the sense of distributions, we get $\lambda(y\cdot\mathrm{curl}(y)) = 0$ in the sense of distributions. Assuming $y\cdot\mathrm{curl}(y) \neq 0$, one gets $\lambda = 0$ in the sense of distributions, hence in $L^\infty(\Omega)$, too. $\square$

Proposition 4.2. *Let $y \in C^\infty(\bar{\Omega};\mathbb{R}^3)$ and $y \cdot \mathrm{curl}(y) \neq 0$ hold a.e. on Ω. Then the pair (u,ν) with*

$$u(x) = 0\ , \quad \nu_x := \frac{1}{2}\delta_{y(x)} + \frac{1}{2}\delta_{-y(x)} \tag{4.3}$$

forms an extreme point in D_{ad}.

Proof. Take $(u_1,\nu_1) \in D_{\mathrm{ad}}$ and $(u_2,\nu_2) \in D_{\mathrm{ad}}$, and assume (3.2). This, together with the fact that both $\nu_{1,x}$ and $\nu_{2,x}$ are probability measures, implies that necessarily

$$\nu_{1,x} = \left(\frac{1}{2} + \varepsilon(x)\right)\delta_{y(x)} + \left(\frac{1}{2} - \varepsilon(x)\right)\delta_{-y(x)}\ , \tag{4.4a}$$

$$\nu_{2,x} = \left(\frac{1}{2} - \varepsilon(x)\right)\delta_{y(x)} + \left(\frac{1}{2} + \varepsilon(x)\right)\delta_{-y(x)} \tag{4.4b}$$

for some $\varepsilon \in L^\infty(\Omega)$ such that $\varepsilon(x) \in [-\frac{1}{2}, \frac{1}{2}]$. Moreover $(u_1, \nu_1) \in D_{\rm ad}$ implies $\nabla u_1(x) = \int_{\mathbb{R}^3} s\,\nu_{1,x}(\mathrm{d}s) =: y_1(x)$ so that the vector field y_1 is rotation free, i.e. $\mathrm{curl}(y_1) = 0$. However, from (4.4) it follows that $y_1 = 2\varepsilon y$, so that 2ε is a measurable integration factor for the vector field y. Yet, by Lemma 4.1, $\varepsilon = 0$ so that $\nu_1 = \nu$ and also $\nabla u_1 = y_1 = 2\varepsilon y = 0$. As $u_1 \in \tilde{W}^{1,p}(\Omega)$ so that $\int_\Omega u_1 \mathrm{d}x = 0$, we eventually obtain $u_1 = 0$. Altogether, we obtained $(u_1, \nu_1) = (u, \nu)$, and analogously $(u_2, \nu_2) = (u, \nu)$. Thus (u, ν) is an extreme point in $D_{\rm ad}$, as claimed. □

Let us remark that the field $y(x_1, x_2, x_3) = (0, x_1, 1)$ is an elementary example for a smooth y satisfying $y \cdot \mathrm{curl}(y) \neq 0$ as assumed in Proposition 4.2. We thus get an example for nonattainment in (1.1) provided W is defined as

$$W(x, s) := \min(|s - y(x)|^p, |s + y(x)|^p). \tag{4.5}$$

As y is now continuous on $\bar{\Omega}$, W is even jointly continuous and obviously (1.2) is again satisfied.

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