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# Turnpike Theorem for Convex Infinite Dimensional Discrete-Time Control Systems

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In this work we study the structure of "approximate" solutions for an infinite dimensional discrete-time optimal control problem determined by a convex function $v : K \times K \to R^1$, where $K$ is a convex closed bounded subset of a Banach space. We show that for a generic function $v$ there exists $y_v \in K$ such that each "approximate" optimal solution $\{x_i\}_{i=0}^n \subset K$ is a contained in a small neighborhood of $y_v$ for all $i \in \{N, \dots, n-N\}$, where $N$ is a constant which depends on the neighborhood and does not depend on $n$.

*Keywords:* Turnpike property, Banach space, convex function, generic function

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## 1. Introduction

Let $X$ be a Banach space, $||\cdot||$ be the norm on $X$, and let $K \subset X$ be a closed convex bounded set. Denote by $\mathfrak{A}$ the set of all bounded convex functions $v : K \times K \to R^1$ which satisfy the following assumption:

**Assumption 1.1 (uniform continuity).** For each $\epsilon > 0$ there exists $\delta > 0$ such that for each $x_1, x_2, y_1, y_2 \in K$ satisfying $||x_i - y_i|| \le \delta$, $i = 1, 2$ the relation $|v(x_1, x_2) - v(y_1, y_2)| \le \epsilon$ holds.

We consider the metric space $\mathfrak{A}$ with the metric

$$\rho(u, v) = \sup\{|v(x, y) - u(x, y)| : x, y \in K\}, \quad u, v \in \mathfrak{A}.$$

Evidently the metric space $\mathfrak{A}$ is complete.

In this paper we investigate the structure of "approximate" solutions of optimization problems

$$\sum_{i=0}^{n-1} v(x_i, x_{i+1}) \to \min, \tag{P}$$

$$\{x_i\}_{i=0}^n \subset K, \quad x_0 = y, \; x_n = z$$

where $v \in \mathfrak{A}$, $y, z \in K$ and an integer $n \ge 1$.

The interest in these discrete-time optimal problems stems from the study of various optimization problems which can be reduced to this framework, e.g., continuous-time control

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systems which are represented by ordinary differential equations whose cost integrand contains a discounting factor (see Leizarowitz [5]), the infinite-horizon control problem of minimizing $\int_0^T L(z, z')dt$ as $T \to \infty$ (see Leizarowitz [6]) and the analysis of a long slender bar of a polymeric material under tension in Leizarowitz and Mizel [7] and Zaslavski [16-18]. Similar optimization problems are also considered in mathematical economics (see Makarov and Rubinov [8], Rubinov [11] and survey [12]).

In this paper we establish the existence of a set $\mathfrak{F} \subset \mathfrak{A}$ which is a countable intersection of open everywhere dense sets in $\mathfrak{A}$ and such that for each $v \in \mathfrak{F}$ the following property holds:

There is $y_v \in K$ such that for all large enough $n$ and each $y, z \in K$ an "approximate" solution $\{x_i\}_{i=0}^n$ of problem (P) is contained in a small neighborhood of $y_v$ for all $i \in \{N, \dots, n-N\}$ where $N$ is a constant which depends on the neighborhood and does not depend on $n$.

This phenomenon which is called the turnpike property is well known in mathematical economics. The term was first coined by Samuelson in 1958 (see [13]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated by Radner [10], McKenzie [9], Makarov and Rubinov [8] and others for optimal trajectories of a von Neumann-Gale model and for optimal trajectories of continuous-time convex autonomous systems (see Carlson, Haurie and Leizarowitz [3, Chap. 4,6]). Asymptotic turnpike property for optimal control problems with infinite time horizon were studied by Carlson [1], Carlson, Haurie and Jabrane [2] and Zaslavski [14]. A related weak version of the turnpike property was studied in Zaslavski [15] with a nonconvex function $v : K \times K \to R^1$ and a compact metric space $K$.

In almost all studies of discrete time control systems the turnpike property was considered for a single cost function $v$ and a space of states $K$ which was a compact convex set in a finite dimensional space. In these studies the compactness of $K$ playes an important role. Perhaps the methods use there can be extended but only to obtain the turnpike property for a weakly compact set $K$ in an infinite dimensional Banach space. Specifically for the optimization problems considered in this paper if a function $v$ has the turnpike property then as we will see later, its "turnpike" $y_v$ is a unique solution of the following optimization problem

$$v(x, x) \to \min, \quad x \in K.$$

The existence of solution of this problem is guaranteed only if $K$ satisfies some compactness assumptions. To obtain the uniqueness of the solution we need additional assumptions on $v$ such as its strict convexity.

In the present paper, instead of considering the turnpike property for a single cost function $v$, we investigate it for the space of all such functions equipped with some natural metric, and show that this property holds for most of these functions. This allows us to establish the turnpike property without compactness assumption on the space of states and assumptions on functions themselves.

For each $v \in \mathfrak{A}$, each integers $m_1, m_2 > m_1$ and each $y_1, y_2 \in K$ we define

$$\sigma(v, m_1, m_2) = \inf\{\sum_{i=m_1}^{m_2-1} v(z_i, z_{i+1}) : \{z_i\}_{i=m_1}^{m_2} \subset K\},$$

$$\sigma(v, m_1, m_2, y_1, y_2) = \inf\{\sum_{i=m_1}^{m_2-1} v(z_i, z_{i+1}) : \{z_i\}_{i=m_1}^{m_2} \subset K,\ z_{m_1} = y_1, z_{m_2} = y_2\},$$

and the minimal growth rate

$$\mu(v) = \inf\{\liminf_{N\to\infty} N^{-1} \sum_{i=0}^{N-1} v(z_i, z_{i+1}) : \{z_i\}_{i=0}^{\infty} \subset K\}.$$

In this paper we will establish the existence of a set $\mathfrak{F} \subset \mathfrak{A}$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{A}$ and such that the following theorems are valid.

**Theorem 1.2.** *Let $v \in \mathfrak{F}$. Then there exists a unique $y_v \in K$ such that $v(y_v, y_v) = \mu(v)$ and the following assertion holds:*

*for each $\epsilon > 0$ there exist a neighborhood $\mathfrak{U}$ of $v$ in $\mathfrak{A}$ and $\delta > 0$ such that for each $u \in \mathfrak{U}$ and each $y \in K$ satisfying $u(y, y) \leq \mu(u) + \delta$ the relation $||y - y_v|| \leq \epsilon$ holds.*

**Theorem 1.3.** *Let $w \in \mathfrak{F}$ and $\epsilon > 0$. Then there exist $\delta \in (0, \epsilon)$, a neighborhood $\mathfrak{U}$ of $w$ in $\mathfrak{A}$ and an integer $N \geq 1$ such that for each $u \in \mathfrak{U}$, each integer $n \geq 2N$ and each sequence $\{x_i\}_{i=0}^{n} \subset K$ satisfying*

$$\sum_{i=0}^{n-1} u(x_i, x_{i+1}) \leq \sigma(u, 0, n, x_0, x_n) + \delta$$

*there exist $\tau_1 \in \{0, \dots N\}$ and $\tau_2 \in \{n - N, \dots n\}$ such that*

$$||x_t - y_w|| \leq \epsilon, \quad t = \tau_1, \dots \tau_2.$$

*Moreover, if $||x_0 - y_w|| \leq \delta$ then $\tau_1 = 0$, and if $||y_w - x_n|| \leq \delta$ then $\tau_2 = n$.*

## 2. Preliminary results

Set

$$D_0 = \sup\{||x|| : x \in K\}. \tag{2.1}$$

For each bounded function $u : K \times K \to R^1$ we set

$$||u|| = \sup\{|u(x, y)| : x, y \in K\}. \tag{2.2}$$

**Proposition 2.1.** *Let $v \in \mathfrak{A}$. Then*

$$\mu(v) = \inf\{v(z, z) : z \in K\}.$$

**Proof.** Clearly

$$\inf\{v(x,x) : x \in K\} \geq \mu(v) \tag{2.3}$$

and for each integer $m \geq 1$

$$\sigma(v,0,m) \leq m\mu(v). \tag{2.4}$$

Let $\epsilon > 0$. Then there exists $\delta \in (0,\epsilon)$ such that for each $x_1, x_2, y_1, y_2 \in K$ satisfying $||x_i - y_i|| \leq \delta$, $i = 1, 2$ the following relation holds

$$|v(x_1,x_2) - v(y_1,y_2)| \leq \epsilon. \tag{2.5}$$

Fix an integer $m \geq 1$ for which

$$8m^{-1}(D_0 + 1) \leq \delta. \tag{2.6}$$

There exists $\{y_i\}_{i=0}^m \subset K$ such that

$$\sum_{i=0}^{m-1} v(y_i, y_{i+1}) \leq \sigma(v,0,m) + \delta.$$

Define

$$z_0 = m^{-1}\sum_{i=0}^{m-1} y_i, \quad z_1 = m^{-1}\sum_{i=1}^{m} y_i.$$

It is easy to see that

$$||z_0 - z_1|| \leq 2m^{-1}D_0 < \delta, \tag{2.7}$$

$$v(z_0, z_1) \leq m^{-1}\sum_{i=0}^{m-1} v(y_i, y_{i+1}) \leq m^{-1}[\sigma(v,0,m) + \delta]. \tag{2.8}$$

By (2.7) and the definition of $\delta$ (see (2.5))

$$|v(z_0,z_0) - v(z_0,z_1)| \leq \epsilon.$$

Together with (2.8) and (2.4) this implies that

$$v(z_0,z_0) \leq 2\epsilon + m^{-1}\sigma(v,0,m) \leq \mu(v) + 2\epsilon.$$

Since $\epsilon$ is an arbitrary positive number we conclude that $\inf\{v(z,z) : z \in K\} \leq \mu(v)$. This completes the proof of the proposition. □

**Proposition 2.2.** *Let $v \in \mathfrak{A}$, $\epsilon \in (0,1)$. Then there exist $\delta \in (0,\epsilon)$, $u \in \mathfrak{A}$ and $z_0 \in K$ such that*

$$0 \leq u(x,y) - v(x,y) \leq \epsilon, \quad x, y \in K, \quad \mu(v) + \delta \geq v(z_0,z_0) \tag{2.9}$$

*and for each $y \in K$ satisfying $u(y,y) \leq \mu(u) + \delta$ the relation $||y - z_0|| \leq \epsilon$ holds.*

**Proof.** Choose numbers $\delta, \gamma > 0$ such that

$$\gamma(8D_0 + 4) \leq \epsilon, \quad \delta < 8^{-1}\gamma\epsilon. \tag{2.10}$$

By Proposition 2.1 there exists $z_0 \in K$ such that $v(z_0, z_0) < \mu(v) + \delta$. Define $u : K \times K \to R^1$ as

$$u(x, y) = v(x, y) + \gamma(||x - z_0|| + ||y - z_0||), \quad x, y \in K. \tag{2.11}$$

It is easy to see that $u \in \mathfrak{A}$ and (2.9) is valid.

Assume that $y \in K$ and

$$u(y, y) \leq \mu(u) + \delta. \tag{2.12}$$

It follows from Proposition 2.1, (2.11), (2.12), (2.10) and the definition of $z_0$ that

$$\mu(v) \leq \mu(u) \leq u(z_0, z_0) = v(z_0, z_0) \leq \mu(v) + \delta,$$

$$2\gamma||y - z_0|| + v(y, y) = u(y, y) \leq \mu(u) + \delta$$
$$\leq \mu(v) + 2\delta \leq v(y, y) + 2\delta, \quad ||y - z_0|| \leq \delta\gamma^{-1} < \epsilon.$$

This completes the proof of the proposition. □

**Proposition 2.3.** *There exists a set $\mathfrak{F}_0$ which is a countable intersection of open everywhere dense subsets of $\mathfrak{A}$ and such that for each $v \in \mathfrak{F}_0$ the following assertions hold:*

(i) *there exists a unique $y_v \in K$ such that $v(y_v, y_v) = \mu(v)$.*
(ii) *for each $\epsilon > 0$ there exist a neighborhood $\mathfrak{U}$ of $v$ in $\mathfrak{A}$ and $\delta > 0$ such that for each $u \in \mathfrak{U}$ and each $y \in K$ satisfying $u(y, y) \leq \mu(u) + \delta$ the relation $||y - y_v|| \leq \epsilon$ holds.*

**Proof.** Let $w \in \mathfrak{A}$ and $i \geq 1$ be an integer. By Proposition 2.2 there exist $\delta(w, i) \in (0, 4^{-i})$, $u^{(w,i)} \in \mathfrak{A}$ and $z(w, i) \in K$ such that

$$0 \leq u^{(w,i)}(x, y) - w(x, y) \leq 4^{-i}, \quad x, y \in K, \tag{2.13}$$

$$w(z(w, i), z(w, i)) \leq \mu(w) + \delta(w, i)$$

and for each $z \in K$ satisfying

$$u^{(w,i)}(z, z) \leq \mu(u^{(w,i)}) + \delta(w, i)$$

the relation $||z - z(w, i)|| \leq 4^{-i}$ holds.

Set

$$\mathfrak{U}(w, i) = \{u \in \mathfrak{A} : \rho(u, u^{(w,i)}) < 8^{-1}\delta(w, i)\}. \tag{2.14}$$

It is easy to verify that the following property holds:

(a) for each $u \in \mathfrak{U}(w, i)$ and for each $z \in K$ satisfying

$$u(z, z) \leq \mu(u) + 8^{-1}\delta(w, i)$$

the relation $||z - z(w, i)|| \leq 4^{-i}$ is valid.

Define

$$\mathfrak{F}_0 = \cap_{q=1}^{\infty} \cup \{\mathfrak{U}(w, i) : w \in \mathfrak{A}, \quad i = q, q+1, \dots\}.$$

Evidently $\mathfrak{F}_0$ is a countable intersection of open everywhere dense subsets of $\mathfrak{A}$.

Assume that $v \in \mathfrak{F}_0$. We will show that assertions (i) and (ii) are valid. There exists a sequence $\{x_j\}_{j=1}^{\infty} \subset K$ such that

$$\lim_{j \to \infty} v(x_j, x_j) = \mu(v). \tag{2.15}$$

It follows from the definition of $\mathfrak{F}_0$ and property (a) that $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence. Therefore there exists $\lim_{j \to \infty} x_j$ and

$$v(\lim_{j \to \infty} x_j, \lim_{j \to \infty} x_j) = \mu(v).$$

Since any sequence $\{x_j\}_{j=1}^{\infty} \subset K$ satisfying (2.15), converges in $K$, we conclude that there exists a unique $y_v \in K$ such that $v(y_v, y_v) = \mu(v)$.

Let $\epsilon > 0$. Choose an integer $q \geq 1$ such that

$$4^{-q} < 8^{-1}\epsilon. \tag{2.16}$$

By the definition of $\mathfrak{F}_0$ there exists $w \in \mathfrak{A}$ and an integer $i \geq q$ such that $v \in \mathfrak{U}(w, i)$. Property (a) implies that

$$||y_v - z(w, i)|| \leq 4^{-i}. \tag{2.17}$$

It follows from (2.17), property (a) and (2.16) that for each $u \in \mathfrak{U}(w, i)$ and each $y \in K$ satisfying

$$u(y, y) \leq \mu(u) + 8^{-1}\delta(w, i)$$

the relation $||y - y_v|| \leq \epsilon$ holds. This completes the proof of the proposition. □

**Remark 2.4.** The statement (i) of Proposition 2.3 is similar to a result in [4].

## 3. Proof of Theorems 1.2 and 1.3

Let the set $\mathfrak{F}_0$ be as guaranteed in Proposition 2.3. For each $w \in \mathfrak{F}_0$ there exists a unique $y_w \in K$ such that

$$w(y_w, y_w) = \mu(w). \tag{3.1}$$

Let $v \in \mathfrak{F}_0$, $\gamma \in (0, 1)$. Define

$$v_\gamma(x, y) = v(x, y) + \gamma(||x - y_v|| + ||y - y_v||), \quad x, y \in K. \tag{3.2}$$

Clearly $v_\gamma \in \mathfrak{A}$.

**Lemma 3.1.** *Let $\epsilon \in (0,1)$. Then there exists an integer $n \geq 1$ such that for each sequence $\{x_i\}_{i=0}^n \subset K$ which satisfies*

$$\sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \leq \sigma(v_\gamma, 0, n, x_0, x_n) + 4 \tag{3.3}$$

*there is $j \in \{0, \dots n-1\}$ such that*

$$||x_j - y_v||, \quad ||x_{j+1} - y_v|| \leq \epsilon. \tag{3.4}$$

**Proof.** Choose an integer

$$n > (\epsilon\gamma)^{-1}(5 + 4(||v_\gamma|| + ||v||)). \tag{3.5}$$

It is easy to verify that

$$n\mu(v) \leq \sigma(v, 0, n) + 2||v||. \tag{3.6}$$

Assume that $\{x_i\}_{i=0}^n \subset K$ satisfies (3.3). Define $\{y_i\}_{i=0}^n \subset K$ as

$$y_i = x_i, \quad i = 0, n, \quad y_i = y_v, \quad i = 1, \dots n-1. \tag{3.7}$$

It follows from (3.2), (3.3), (3.7), (3.1), (3.6) and (3.5) that

$$\sigma(v, 0, n) + \gamma \sum_{i=0}^{n-1} (||x_i - y_v|| + ||x_{i+1} - y_v||) \leq \sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1})$$

$$\leq \sum_{i=0}^{n-1} v_\gamma(y_i, y_{i+1}) + 4 \leq 4||v_\gamma|| + 4 + nv(y_v, y_v)$$

$$= 4 + 4||v_\gamma|| + n\mu(v) \leq 4 + 4||v_\gamma|| + \sigma(v, 0, n) + 2||v||,$$

$$\inf\{||x_i - y_v|| + ||x_{i+1} - y_v|| : i = 0, \dots n-1\}$$

$$\leq (n\gamma)^{-1}(4 + 4||v_\gamma|| + 2||v||) < \epsilon.$$

This completes the proof of the lemma. □

Lemma 3.1 implies

**Lemma 3.2.** *Let $\epsilon \in (0,1)$. Then there exist a neighborhood $\mathfrak{U}$ of $v_\gamma$ in $\mathfrak{A}$ and an integer $n \geq 1$ such that for each $u \in \mathfrak{U}$ and each sequence $\{x_i\}_{i=0}^n \subset K$ which satisfies*

$$\sum_{i=0}^{n-1} u(x_i, x_{i+1}) \leq \sigma(u, 0, n, x_0, x_n) + 3$$

*there is $j \in \{0, \dots n-1\}$ such that $||x_j - y_v||, ||x_{j+1} - y_v|| \leq \epsilon$.*

**Lemma 3.3.** *Let $\epsilon \in (0,1)$. Then there exists $\delta \in (0,\epsilon)$ such that for each integer $n \geq 1$ and each sequence $\{x_i\}_{i=0}^n \subset K$ which satisfies*

$$||x_i - y_v|| \leq \delta, \quad i = 0, n, \quad \sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \leq \sigma(v_\gamma, 0, n, x_0, x_n) + \delta, \tag{3.8}$$

*the following relation holds*

$$||x_i - y_v|| \leq \epsilon, \quad i = 0, \dots n. \tag{3.9}$$

**Proof.** It is easy to see that for each integer $m \geq 1$

$$\sigma(v, 0, m, y_v, y_v) = m\mu(v). \tag{3.10}$$

Fix

$$\delta_0 \in (0, 8^{-1}\gamma\epsilon). \tag{3.11}$$

There exists $\delta \in (0, 2^{-1}\delta_0)$ such that for each $x_1, x_2, y_1, y_2 \in K$ satisfying $||x_i - y_i|| \leq \delta$, $i = 1, 2$ the following relation holds

$$|v(x_1, x_2) - v(y_1, y_2)| \leq 64^{-1}\delta_0. \tag{3.12}$$

Assume that an integer $n \geq 1$ and a sequence $\{x_i\}_{i=0}^n \subset K$ satisfies (3.8). We will show that (3.9) is valid.

Let us assume the converse. Then $n \geq 2$ and there exists $j \in \{1, \dots, n-1\}$ such that

$$||x_j - y_v|| > \epsilon. \tag{3.13}$$

Define

$$z_i = x_i, \quad i = 0, n, \quad z_i = y_v, \quad i = 1, \dots n-1, \tag{3.14}$$

$$h_i = y_v, i = 0, n, \;\; h_i = x_i, \;\; i = 1, \dots, n-1.$$

It follows from (3.2), (3.13), (3.8), (3.14) and (3.1) that

$$\gamma\epsilon + \sum_{i=0}^{n-1} v(x_i, x_{i+1}) \leq \sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \leq \sum_{i=0}^{n-1} v_\gamma(z_i, z_{i+1}) + \delta \tag{3.15}$$

$$= \delta + v_\gamma(x_0, y_v) + v_\gamma(y_v, x_n) + (n-2)\mu(v).$$

By (3.8), (3.1) and the definition of $\delta$ (see (3.12))

$$|v(x_i, y_v) - \mu(v)|, \quad |v(y_v, x_i) - \mu(v)| \leq 64^{-1}\delta_0, \quad i = 0, n, \tag{3.16}$$

$$|v(y_v, x_1) - v(x_0, x_1)| \leq 64^{-1}\delta_0, \quad |v(x_{n-1}, x_n) - v(x_{n-1}, y_v)| \leq 64^{-1}\delta_0.$$

(3.16), (3.8), (3.15), (3.10), (3.14) and (3.2) imply that

$$\gamma\epsilon + \sum_{i=0}^{n-1} v(h_i, h_{i+1}) \leq \gamma\epsilon + \sum_{i=0}^{n-1} v(x_i, x_{i+1}) + 32^{-1}\delta_0$$

$$\leq 32^{-1}\delta_0 + \delta + (n-2)\mu(v) + v_\gamma(x_0, y_v) + v_\gamma(y_v, x_n)$$
$$\leq 32^{-1}\delta_0 + \delta + \mu(v)n + 32^{-1}\delta_0 + 2\gamma\delta \leq \mu(v)n + 16^{-1}\delta_0$$
$$+3\delta = \sigma(v, 0, n, y_v, y_v) + 16^{-1}\delta_0 + 3\delta.$$

Together with (3.14) this implies that $\gamma\epsilon \leq 4\delta_0$. This is contradictory to (3.11). The obtained contradiction proves the lemma. □

**Lemma 3.4.** *Let $\epsilon \in (0,1)$. Then there exist $\delta \in (0,\epsilon)$, a neighborhood $\mathfrak{U}$ of $v_\gamma$ in $\mathfrak{A}$ and an integer $N \geq 1$ such that for each $u \in \mathfrak{U}$, each integer $n \geq 2N$ and each sequence $\{x_i\}_{i=0}^n \subset K$ satisfying*

$$\sum_{i=0}^{n-1} u(x_i, x_{i+1}) \leq \sigma(u, 0, n, x_0, x_n) + \delta \tag{3.17}$$

*there exist $\tau_1 \in \{0, \dots, N\}$, $\tau_2 \in \{-N+n, n\}$ such that*

$$||x_i - y_v|| \leq \epsilon, \quad t = \tau_1, \dots \tau_2, \tag{3.18}$$

*and moreover if $||x_0 - y_v|| \leq \delta$ then $\tau_1 = 0$, and if $||x_n - y_v|| \leq \delta$ then $\tau_2 = n$.*

**Proof.** By Lemma 3.3 there exists $\delta \in (0, 4^{-1}\epsilon)$ such that for each integer $n \geq 1$ and each sequence $\{x_i\}_{i=0}^n \subset K$ which satisfies

$$||x_i - y_v|| \leq 4\delta,\ i = 0, n, \quad \sum_{i=0}^{n-1} v_\gamma(x_i, x_{i+1}) \leq \sigma(v_\gamma, 0, n, x_0, x_n) + 4\delta \tag{3.19}$$

the following relation holds

$$||x_i - y_v|| \leq \epsilon, \quad i = 0, \dots n. \tag{3.20}$$

By Lemma 3.2 there exist an integer $N \geq 1$ and a neighborhood $\mathfrak{U}_1$ of $v_\gamma$ in $\mathfrak{A}$ such that for each $u \in \mathfrak{U}_1$ and each sequence $\{x_i\}_{i=0}^N \subset K$ satisfying

$$\sum_{i=0}^{N-1} u(x_i, x_{i+1}) \leq \sigma(u, 0, N, x_0, x_N) + 3 \tag{3.21}$$

there is $j \in \{0, \dots N-1\}$ for which

$$||x_j - y_v||, \quad ||x_{j+1} - y_v|| \leq \delta. \tag{3.22}$$

Define

$$\mathfrak{U} = \{u \in \mathfrak{U}_1 : \rho(u, v_\gamma) \leq (16N)^{-1}\delta\}. \tag{3.23}$$

Assume that $u \in \mathfrak{U}$, an integer $n \geq 2N$ and a sequence $\{x_i\}_{i=0}^n \subset K$ satisfies (3.17). By (3.17) and the definition of $\mathfrak{U}_1$, $N$ (see (3.21), (3.22)) there exist integers $\tau_1$, $\tau_2$ such that:

$$\tau_1 \in \{0, \dots N\}, \quad \tau_2 \in \{n-N, \dots n\}, \quad ||x_{\tau_i} - y_v|| \leq \delta,\ i = 1, 2; \tag{3.24}$$

if $||x_0 - y_v|| \le \delta$ then $\tau_1 = 0$; if $||x_n - y_v|| \le \delta$ then $\tau_2 = n$.

We will show that (3.18) is valid. Let us assume the converse. Then there is an integer $s \in (\tau_1, \tau_2)$ for which

$$||x_s - y_v|| > \epsilon. \tag{3.25}$$

By (3.17), (3.24) and the definition of $\mathfrak{U}_1$, $N$ (see (3.21), (3.22)) there exist integers $t_1, t_2$ such that

$$\sup\{\tau_1, s - N\} \le t_1 < s, \quad s < t_2 \le \inf\{\tau_2, s + N\}, \quad ||x_{t_i} - y_v|| \le \delta,\ i = 1, 2. \tag{3.26}$$

(3.23), (3.26) and (3.17) imply that

$$\sum_{i=t_1}^{t_2-1} v_\gamma(x_i, x_{i+1}) \le \sigma(v_\gamma, t_1, t_2, x_{t_1}, x_{t_2}) + 2\delta. \tag{3.27}$$

It follows from (3.26), (3.27) and the definition of $\delta$ (see (3.19), (3.20)) that

$$||x_t - y_t|| \le \epsilon, \quad t = t_1, \dots t_2.$$

This is contradictory to (3.25). The obtained contradiction proves that (3.18) is valid. This completes the proof of the lemma. □

Clearly the set $\{v_\gamma : v \in \mathfrak{F}_0,\ \gamma \in (0,1)\}$ is everywhere dense in $\mathfrak{A}$.

Let $v \in \mathfrak{F}_0$, $\gamma \in (0,1)$ and let $j \ge 1$ be an integer. There exist an integer $N(v,\gamma,j) \ge 1$, an open neighborhood $\mathfrak{U}_0(v,\gamma,j)$ of $v_\gamma$ in $\mathfrak{A}$ and a number $\delta(v,\gamma,j) \in (0, 2^{-j})$ such that Lemma 3.4 holds with $v, \gamma, \epsilon = 2^{-j}$, $\delta = \delta(v,\gamma,j)$, $\mathfrak{U} = \mathfrak{U}_0(v,\gamma,j)$, $N = N(v,\gamma,g)$.

There are an open neighborhood $\mathfrak{U}(v,\gamma,j)$ of $v_\gamma$ in $\mathfrak{A}$ and an integer $N_1(v,\gamma,j) \ge 1$ such that $\mathfrak{U}(v,\gamma,j) \subset \mathfrak{U}_0(v,\gamma,j)$ and Lemma 3.2 holds with $v, \gamma$, $\mathfrak{U} = \mathfrak{U}(v,\gamma,j)$, $n = N_1(v,\gamma,j)$, $\epsilon = 4^{-j}\delta(v,\gamma,j)$.

Define

$$\mathfrak{F} = [\cap_{q=1}^{\infty} \cup \{\mathfrak{U}(v,\gamma,j) : v \in \mathfrak{F}_0, \gamma \in (0,1), j = q, q+1, \dots\}] \cap \mathfrak{F}_0.$$

Clearly $\mathfrak{F}$ is a countable intersection of open everywhere dense subsets of $\mathfrak{A}$.

It is easy to see that Theorem 1.2 follows from Proposition 2.3 and the definition of $\mathfrak{F}$.

**Proof of Theorem 1.3.** Let $w \in \mathfrak{F}$, $\epsilon > 0$. We may assume that $\epsilon < 1$. Choose an integer $q \ge 1$ such that

$$64 \cdot 2^{-q} < \epsilon. \tag{3.28}$$

There exist $v \in \mathfrak{F}_0$, $\gamma \in (0,1)$ and an integer $j \ge q$ such that

$$w \in \mathfrak{U}(v,\gamma,j). \tag{3.29}$$

By (3.29), Lemma 3.2 which holds with $\mathfrak{U} = \mathfrak{U}(v,\gamma,j)$, $n = N_1(v,\gamma,j)$, $\epsilon = 4^{-j}\delta(v,\gamma,j)$, $v, \gamma$, and relation $\sigma(w, 0, N_1(v,\gamma,j), y_w, y_w) = N_1(v,\gamma,j)\mu(w)$

$$||y_w - y_v|| \le 4^{-j}\delta(v,\gamma,j). \tag{3.30}$$

Set

$$\mathfrak{U} = \mathfrak{U}(v, \gamma, j), \quad N = N(v, \gamma, j), \quad \delta = 4^{-j}\delta(v, \gamma, j). \tag{3.31}$$

Assume that $u \in \mathfrak{U}$, an integer $n \geq 2N$ and a sequence $\{x_i\}_{i=0}^{n} \subset K$ satisfies

$$\sum_{i=0}^{n-1} u(x_i, x_{i+1}) \leq \sigma(u, 0, n, x_0, x_n) + \delta. \tag{3.32}$$

It follows from (3.32), (3.31), the definition of $\mathfrak{U}_0(v, \gamma, j)$, $N(v, \gamma, j)$, $\delta(v, \gamma, j)$ and Lemma 3.4 that there exist $\tau_1 \in \{0, \dots N\}$, $\tau_2 \in \{n - N, \dots n\}$ such that

$$||x_i - y_v|| \leq 2^{-j}, \quad t = \tau_1, \dots \tau_2.$$

Moreover if $||x_0 - y_v|| \leq \delta(v, \gamma, j)$ then $\tau_1 = 0$, and if $||x_n - y_v|| \leq \delta(v, \gamma, j)$ then $\tau_2 = n$. Together with (3.30), (3.28), (3.31) this implies that:

$$||x_i - y_w|| \leq 2^{1-j} < \epsilon, \quad i = \tau_1 \dots \tau_2;$$

if $||x_0 - y_w|| \leq \delta$ then $||x_0 - y_v|| \leq \delta(v, \gamma, j)$ and $\tau_1 = 0$; if $||x_n - y_w|| \leq \delta$ then $||x_n - y_v|| \leq \delta(v, \gamma, j)$ and $\tau_2 = n$. This completes the proof of the theorem. □

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