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C-Minimal Pairs of Compact Convex Sets

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Pairs of compact convex sets naturally arise in quasidifferential calculus as a sub- and superdifferentials of a quasidifferentiable function (see [1]). Since the sub- and superdifferential are not uniquely determined, minimal representations are of special importance. For the 2-dimensional case, equivalent minimal pairs of compact convex sets are uniquely determined up to translations (see [3], [14]). For the 3-dimensional case, this is not longer true. J. Grzybowski [3] gave an example of finitely many equivalent minimal pairs of compact convex sets which are not connected by translations. A continuous family of equivalent minimal pairs of compact convex sets which are not connected by translation for different indices is given in [9]. In a recent paper R. Urbański [16] investiged the mimimality of pairs of compact convex sets which satisfy additional conditions, namely the minimal convex pairs. This paper is a continuation of this research direction. Here we study the minimality under a different type of conditions. Moreover we give a definition of a "convex hull" of a pair of sets.

Keywords: Pairs of convex sets, sublinear function, quasidifferential calculus

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1. Introduction

In this paper we consider the Rådström-Hörmander lattice [4] of equivalence classes of pairs of nonempty compact convex sets and investigate minimal representants which satisfy additional conditions. As in [6] we denote for a real topological vector space X the set of all nonempty compact convex subsets by $\mathcal{K}(X)$ and the set of all pairs of nonempty compact convex subsets by $\mathcal{K}^2(X)$, i.e. $\mathcal{K}^2(X) = \mathcal{K}(X) \times \mathcal{K}(X)$. The equivalence relation between pairs of compact convex sets is given by: "$(A,B) \sim (C,D)$ *if and only if* $A + D = B + C$" using the Minkowski-sum, and a partial order is given by the relation: "$(A,B) \leq (C,D)$ *if and only if* $A \subseteq C$ *and* $B \subseteq D$." The space $\mathcal{K}^2(X)$ has been

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investigated in series of papers, (see for instance [3], [6], [7], [8], [9], [14], [16]).
Pairs of compact convex sets arise in quasidifferential calculus as the sub- and superdifferentials of the directional derivative of a quasidifferentiable function (see [1], [13]).
Let us first set some notations: Let X be a real topological vector space, and X^* be the space of all continuous real valued linear functional. For two compact convex sets $A, B \in \mathcal{K}(X)$ we will use the notation

$$A \vee B := \operatorname{conv}(A \cup B),$$

where the operation "conv" denotes the convex hull. With $\overline{A}$ we denote the closure of a set A.
During the proofs, an easy identity for compact convex sets, which was first observed by A. Pinsker [10] will be used frequently, namely: For $A, B, C \in \mathcal{K}(X)$ we have:

$$(A + C) \vee (B + C) = C + (A \vee B). \tag{$*$}$$

This identity can be proved as follows: Every $x \in (A \vee B) + C$ can be represented as $x = \alpha \cdot a + (1-\alpha) \cdot b + c$ with $a \in A,\ b \in B,\ c \in C$ and $0 \leq \alpha \leq 1$. Now $x = \alpha \cdot a + (1-\alpha) \cdot b + c = \alpha \cdot (a+c) + (1-\alpha) \cdot (b+c)$ and hence $C + (A \vee B) \subseteq (A + C) \vee (B + C)$.
The converse inclusion can be seen as follows: Let $x \in (A+C) \vee (B+C)$. Then we have: $x = \alpha \cdot (a+c_1) + (1-\alpha) \cdot (b+c_2)$ with $a \in A,\ b \in B,\ c_1, c_2 \in C$ and $0 \leq \alpha \leq 1$. Now $x = \alpha \cdot (a+c_1) + (1-\alpha) \cdot (b+c_2) = \alpha \cdot a + (1-\alpha) \cdot b + \alpha \cdot c_1 + (1-\alpha) \cdot c_2$. Hence we see that the converse inclusion $(A+C) \vee (B+C) \subseteq C + (A \vee B)$ holds also.
We will use the abbreviation $A + B \vee C$ for $A + (B \vee C)$ and $C + d$ for $C + \{d\}$ for compact convex sets A, B, C and a point d. Moreover we will write $[a, b]$ instead of $\{a\} \vee \{b\}$.
Finally let us state explicitely the order cancellation law (see [4], [15]).
Let X be real topological vector space and $A, B, C \subseteq X$ compact convex subsets. Then the inclusion

$$A + B \subseteq A + C$$

implies

$$B \subseteq C.$$

Thus from the algebraic point of view the set $\mathcal{K}(X)$ of all nonempty compact convex subsets of a real topological vector space X is a *commutative semi-ring with cancellation property* endowed with the "*addition* $\#$ " given by:

$$A \# B := A \vee B$$

and the "*multiplication* $\star$" given by:

$$A \star B := A + B.$$

Within this context, the elements of $\mathcal{K}^2(X)$ with respect to the relation $\sim$ can be considered as *fractions*.

The starting point of this paper is a characterization, which is proved in [16], of those convex subsets of a compact convex set, which define a decomposition in the following sense:

In [8] the following notation was introduced: Let $A, B, S \in \mathcal{K}(X)$, then we say that S *"separates"* the sets A and B if for every $a \in A$ and $b \in B$ we have $[a, b] \cap S \neq \emptyset$. In this terminology we have:

Theorem 1.1. *Let X be a real topological vector space and $A, B \in \mathcal{K}(X)$. Then the following statements are equivalent:*

(i) *The set $A \cup B$ is convex*
(ii) *The set $A \cap B$ separates the sets A and B*
(iii) *The set $A \vee B$ is a summand of the set $A + B$*
(iv) $A + B = A \vee B + A \cap B$ *and* $A \cap B \neq \emptyset$.

Let us remark, that if $A \cap B$ separates the sets A and B, the basic relationship between the Minkowski-sum, the convex hull and the intersection is given by property (iv) of Theorem 1.1. The algebraic analogue of this formula is, that the product of two integers $a, b \in \mathbb{N}$ is equal to the product of its *greatest common divisor* $d(a, b)$ with its *smallest common multiplier* $m(a, b)$. i.e. $a \cdot b = d(a, b) \cdot m(a, b)$.
In [16] a pair $(A, B) \in \mathcal{K}^2(X)$ is called *"convex"* if $A \cup B$ is a convex set. Moreover it is shown, that every pair $(A, B) \in \mathcal{K}^2(X)$ is equivalent to a convex pair, namely:

Proposition 1.2. *Let X be a real topological vector space and $(A, B) \in \mathcal{K}^2(X)$. Then:*

(i) *The pair $(A + A \vee B, B + A \vee B) \in \mathcal{K}^2(X)$ is convex*
(ii) $(A + A \vee B) \cap (B + A \vee B) = A + B$
(iii) $(A, B) \sim (A + A \vee B, B + A \vee B)$.
(iv) *Let $(C, D) \in \mathcal{K}^2(X)$ and $(A, B) \sim (C, D)$ and assume that the pair (A, B) is convex. Then*
$$A + D = B + C = C \vee D + A \cap B.$$

Let us recall that for a real topological vector space X a pair $(A, B) \in \mathcal{K}^2(X)$ is *minimal* if and only if for every equivalent pair $(C, D) \in \mathcal{K}^2(X)$ the relation $(C, D) \leq (A, B)$ implies $C = A$ and $B = D$. Therefore we define:

Definition 1.3. Let X be a real topological vector space. Then a convex pair $(A, B) \in \mathcal{K}^2(X)$ is *minimal convex* if and only if for every equivalent convex pair $(C, D) \in \mathcal{K}^2(X)$ the relation $(C, D) \leq (A, B)$ implies $C = A$ and $B = D$.

It is shown in [16] that for every convex pair $(A, B) \in \mathcal{K}^2(X)$ there exists an equivalent minimal convex pair and the following characterization of minimal convex pairs holds.

Theorem 1.4. *Let X be a real topological vector space. Then the convex pair $(A, B) \in \mathcal{K}^2(X)$ is minimal convex if and only if the pair $(A \cap B, A \cup B)$ is minimal.*

2. Algebraic Properties of Compact Convex Sets

In this section we prove several algebraic properties of compact convex sets, which are usefull for further discussions and also interesting in itself. Let us start with the following observations:

Lemma 2.1. *Let X be a real topological vector space and $A, B, C \subset X$ nonempty subsets.*
Then

$$A \cup B \; + \; C \; = \; (A \; + \; C) \cup (B \; + \; C).$$

Proof. For $x \in A \cup B + C$, exists $c \in C$ and $d \in A \cup B$ such that $x = c + d$. Hence $x \in (A + C) \cup (B + C)$, i.e. $A \cup B + C \subseteq (A + C) \cup (B + C)$.
Conversely, for $x \in (A + C) \cup (B + C)$ there exist elements $c \in C$ and $d \in A$ or $d \in B$ such that $x = c + d$. Hence $x \in A \cup B + C$, i.e. $(A + C) \cup (B + C) \subseteq A \cup B + C$. □

Lemma 2.2. *Let X be a real topological vector space with* $\dim X > 1$ *and* $(A, B) \in \mathcal{K}^2(X)$.
If for every singleton $C = \{c\} \in \mathcal{K}(X)$

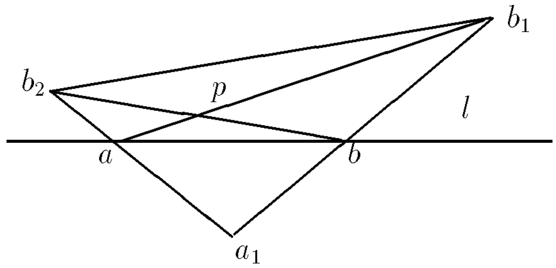
$$(A \vee C) \cup (B \vee C) \; = \; A \vee B \vee C \tag{2.2.1}$$

holds, then $A \cap B \neq \emptyset$.

Proof. Suppose that $A \cap B = \emptyset$. Then there exists $a \in A$ and $b \in B$ such that $(a, b) \cap (A \cup B) = \emptyset$, where (a, b) is the open line segment between the points a, b.
Let us denote by l the line passing through the points a and b.
Now we consider two cases:

(i) $A, B \subset l$, i.e. the sets A and B are intervalls lying in l. Then for every $c \in X \setminus l$ and $C := \{c\}$ we have $(A \vee C) \cap (B \vee C) = \{c\}$. But $\{c\} \cap [a, b] = \emptyset$. From this follows that $\{c\}$ does not separate the sets $A \vee C$ and $B \vee C$. Hence the set $(A \vee C) \cup (B \vee C)$ is not convex which is a contradiction to the assumption.

(ii) Now let us assume, that one of the sets A and B is not contained in the line l, for instance that $B \not\subset l$. Then there exists a point $b_1 \in B$ with $b_1 \notin l$ such that $(a, b_1) \cap B = \emptyset$. Now put $C_1 := \{b_1\}$. Then it follows from equation (2.2.1) and Theorem 1.1 (ii) that the set $(A \vee b_1) \cap B$ separates the sets $A \vee b_1$ and B. Hence there exist numbers $\alpha, \beta \geq 0, \; \alpha + \beta = 1$, such that $\alpha \cdot a \; + \; \beta \cdot b \in (A \vee b_1) \cap B$. But by assumption we have $(a, b) \cap (A \cup B) = \emptyset$, hence $\alpha = 0$ and $\beta = 1$, and we obtain that $b \in A \vee b_1$. From this follows, that $b = \alpha_1 \cdot b_1 \; + \; \beta_1 \cdot a_1$ for some $\alpha_1, \beta_1 \geq 0, \alpha_1 + \beta_1 = 1$ and $a_1 \in A$.



Now we proceed analogously for the set $C_2 := \{a_1\}$ and obtain that $a = \alpha_2 \cdot a_1 + \beta_2 \cdot b_2$ for some $\alpha_2, \beta_2 \geq 0, \alpha_2 + \beta_2 = 1$ and $b_2 \in B$. Hence we have $(a, b_1) \cap [b, b_2] \neq \emptyset$. Since $(a, b_1) \cap [b, b_2] = \{p\}$ for some $p \in (a, b_1)$, this is a contradiction to $(a, b_1) \cap B = \emptyset$. Hence we have $A \cap B \neq \emptyset$. □

Remark 2.3. If $\dim X = 1$, and $A, B \in \mathcal{K}(X)$ with $A \cap B = \emptyset$.
Then for every $C \in \mathcal{K}(X)$ we have $(A \vee C) \cup (B \vee C) = A \vee B \vee C$.

Lemma 2.4. *Let X be a real topological vector space and $A, B \in \mathcal{K}(X)$.*
Moreover assume, that $A \cap B \neq \emptyset$ and that for every segment $C = [a, b] \in \mathcal{K}(X)$

$$(A \vee C) \cap (B \vee C) = (A \cap B) \vee C$$

holds.
Then the set $A \cap B$ separates the sets A and B.

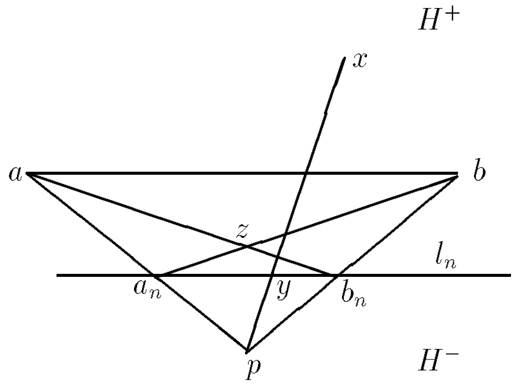
Proof. First observe, that if $\dim X = 1$ and $A \cap B \neq \emptyset$ then $A \cup B$ is convex and hence $A \cap B$ separates the sets A and B.
Now suppose that $\dim X > 1$ and that $A \cap B$ does not separate the sets A and B. Then there exist elements $a \in A$ and $b \in B$ such that $[a, b] \cap (A \cap B) = \emptyset$. For an arbitrary element $p \in A \cap B$ let us consider the sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ given by:

$$a_n := \frac{1}{n} \cdot p + (1 - \frac{1}{n}) \cdot a \quad \text{and} \quad b_n := \frac{1}{n} \cdot p + (1 - \frac{1}{n}) \cdot b \quad \text{for } n \in \mathbb{N}.$$

Denote $I_n := [a_n, b_n] = \frac{1}{n} \cdot p + (1 - \frac{1}{n}) \cdot [a, b]$. It is easy to observe that $I_{n_0} \cap [a, b] = \emptyset$ for some $n_0 \in \mathbb{N}$. Namely, assume that for every $n \in \mathbb{N}$ we have $I_n \cap [a, b] \neq \emptyset$. Then there exits a sequence $(c_n)_{n \in \mathbb{N}}$ with:

$$c_n := \frac{1}{n} \cdot p + (1 - \frac{1}{n}) \cdot [\alpha_n \cdot a + (1 - \alpha_n) \cdot b]$$

and $1 \geq \alpha_n \geq 0$. Since there exist a subsequence $(\alpha_{n_k})_{k \in \mathbb{N}}$ which converges to $\alpha_0 \in [0, 1]$, it follows, that the sequence $(c_{n_k})_{k \in \mathbb{N}}$ which converges to $c_0 = \alpha_0 \cdot a + (1 - \alpha_0) \cdot b$. Since $A \cap B$ is compact and for every $n \in \mathbb{N}$ we have $c_n \in A \cap B$ it follows additionally that $c_0 \in [a, b] \cap (A \cap B)$.



Now let H be the linear manifold generated by the points a, b, p. Moreover let l_n be the line which passes through the points a_n and b_n. This line devides the plane into the halfplanes H^+ and H^- where we assume that $p \in H^-$.
Suppose that $H^+ \cap (A \cap B) \neq \emptyset$. Then there exists $x \in H^+ \cap (A \cap B)$ and therefore $[x, p] \cap l \neq \emptyset$. Let us put $\{y\} := [x, p] \cap l$, then we have $I_n \cap A \cap B \neq \emptyset$. But this is a contradiction, hence $H^+ \cap A \cap B = \emptyset$.
Now put $\{z\} := [a_n, b] \cap [a, b_n]$, then $z \in H^+$. First observe that $z = (\{a\} \vee \{b_n\}) \cap (\{b\} \vee \{a_n\}) \subset (A \vee I_n) \cap (B \vee I_n)$, and that $z \notin I_n$.
Furthermore choose $u \in \{[(A \cap B) \vee I_n] \setminus I_n\} \cap H$. Then there exist $\alpha, \beta > 0, \alpha + \beta = 1$ such that $u = \alpha \cdot q \ + \ \beta \cdot v$ for some $q \in A \cap B$ and $v \in I_n \subset H^-$. This implies that $q = \frac{1}{\alpha} \cdot u - \frac{\beta}{\alpha} \cdot v$ and $\frac{1}{\alpha} - \frac{\beta}{\alpha} = 1$. Hence $q \in A \cap B \cap H \subset H^-$, and we obtain that $u \in \alpha \cdot H^- + \beta \cdot H^- \subset H^-$.
Now suppose that $z \in [(A \cap B) \vee I_n] \cap H$. Since $z \notin I_n$ this implies that $z \in \{[(A \cap B) \vee I_n] \setminus I_n\} \cap H \subset H^-$. But this contradicts the assumption that $z \in H^+$.
Now for $C := I_n$ we have that $(A \vee C) \cap (B \vee C) \neq (A \cap B) \vee C$, hence $A \cap B$ separates the sets A and B. □

Now we are able to prove the following equivalences:

Proposition 2.5. *Let X be a real topological vector space and $(A, B) \in \mathcal{K}^2(X)$. Then the following assertions are equivalent:*

(i) *$A \cup B$ is convex*
(ii) *For every $C \in \mathcal{K}(X)$ we have $(A + C) \cup (B + C) = A \vee B + C$*
(iii) *$A \cap B \neq \emptyset$ and for every $C \in \mathcal{K}(X)$ we have $(A + C) \cap (B + C) = A \cap B + C$*
(iv) *$A \cap B \neq \emptyset$ and for every segment $C \in \mathcal{K}(X)$ we have $(A \vee C) \cap (B \vee C) = (A \cap B) \vee C$*
(v) *For every singleton $C \in \mathcal{K}(X)$ we have $(A \vee C) \cup (B \vee C) = A \vee B \vee C$, if $A \cap B \neq \emptyset$ or $\dim X > 1$.*

Proof. (i) $\Longleftrightarrow$ (ii). If $A \cup B$ is convex then $A \cup B = A \vee B$. But from Lemma 2.1 follows that for every $C \in \mathcal{K}(X)$ we have $(A + C) \cup (B + C) = A \cup B + C$. Hence $(A + C) \cup (B + C) = A \vee B + C$. For the converse direction observe that (i) follows from (ii) for $C := \{0\}$, then $A \cup B = A \vee B$.
(ii) $\Longleftrightarrow$ (iii). By formula $(*)$ and Theorem 1.1 we have $A + C + B + C = A \vee B + C + (A + C) \cap (B + C)$. This implies $A + B + C = A \vee B + (A + C) \cap (B + C)$. But (ii) is equivalent to (i), hence we know that $A \cup B$ is convex and therefore we have $A + B = A \vee B + A \cap B$. From this we obtain that

$$A \vee B + A \cap B + C = A \vee B + (A + C) \cap (B + C).$$

Hence by the law of cancellation we get

$$A \cap B + C = (A + C) \cap (B + C).$$

(iii) $\Longrightarrow$ (i). Observe that for $C := A \vee B$, we get from (iii):

$$A + B \subset (A + A \vee B) \cap (B + A \vee B) = A \cap B + A \vee B.$$

Thus by [16] Theorem 2.3 we have that $A \cap B$ separates the sets A and B and hence $A \cup B$ is convex.

(i) $\Longrightarrow$ (iv) and (v). For $C \in \mathcal{K}(X)$, then we have:

$$\begin{aligned}
A \vee C + B \vee C \; &= \; (A \vee C + B) \vee (A \vee C + C) \\
&= \; (A + B) \vee (B + C) \vee (A + C) \vee (C + C) \\
&= \; (A \cup B + A \cap B) \vee (A \vee B + C) \vee (C + C) \\
&= \; (A \cup B + (A \cap B) \vee C) \vee (C + C) \\
&\subseteq \; (A \cup B + (A \cap B) \vee C) \vee ((A \cap B) \vee C + C) \\
&\subseteq \; (A \cup B) \vee C + (A \cap B) \vee C \\
&\subseteq \; (A \vee C) \vee (B \vee C) + (A \cap B) \vee C \\
&\subseteq \; (A \vee C) \vee (B \vee C) + (A \vee C) \cap (B \vee C).
\end{aligned}$$

Hence

$$A \vee C + B \vee C \; \subseteq (A \vee C) \vee (B \vee C) + (A \vee C) \cap (B \vee C),$$

and by [16] Theorem 2.3 and Theorem 1.1 we have:

$$A \vee C + B \vee C \; = \; (A \vee C) \vee (B \vee C) + (A \vee C) \cap (B \vee C).$$

Hence we have

$$(A \vee C) \cap (B \vee C) \; = \; (A \cap B) \vee C$$

and

$$(A \vee C) \cup (B \vee C) \; = \; (A \vee B) \vee C.$$

(iv) $\Longrightarrow$ (i). By Lemma 2.4 we have that $A \cap B$ separates the sets A and B, hence the set $A \cup B$ is convex.

(v) $\Longrightarrow$ (i). If $A \cap B \neq \emptyset$, then for $x \in A \cap B$ and $C := \{x\}$ we obtain $A \cup B \; = \; A \vee B$. If $\dim X \; > \; 1$, then from Lemma 2.2 we get $A \cap B \neq \emptyset$ and proceed as in the above mentioned case. $\square$

Proposition 2.6. *Let X be a real topological vector space and $(A, B), (C, D) \in \mathcal{K}^2(X)$ are two equivalent pairs and assume that the pair (C, D) is convex.*
Then the pairs $(A + C, B + C), (A + D, B + D)$ are convex.

Proof. From Proposition 1.2 we have that

$$A + D \; = \; B + C \; = \; A \vee B \; + \; C \cap D.$$

This implies that

$$A + C + B + C \; = \; A \vee B + C + A + C \cap D,$$

and

$$A + D + B + D \; = \; A \vee B + D + B + C \cap D.$$

Moreover,

$$(A + C) \vee (B + C) \; = \; A \vee B + C \quad \text{and} \quad (A + D) \vee (B + D) \; = \; A \vee B + D.$$

Now from Theorem 1.1 follows, that $(A + C, B + C)$ and $(A + D, B + D)$ are convex pairs. $\square$

3. C-minimal pairs

Definition 3.1. Let X be a real topological vector space and $C \in \mathcal{K}(X)$. Then the pair $(A, B) \in \mathcal{K}^2(X)$ is called *"C-minimal"* if the pair $(A + C, B + C)$ is convex, and if for every $C_1 \in \mathcal{K}(X)$ with $C_1 \subseteq C$ and such that $(A + C_1, B + C_1)$ is a convex pair it follows that $C_1 = C$.

Remark 3.2. The pair (A, B) is convex if and only if the pair (A, B) is C-minimal and the set C is a singleton.

Theorem 3.3. *Let X be a real topological vector space and $C \in \mathcal{K}(X)$. Then the pair $(A, B) \in \mathcal{K}^2(X)$ is C-minimal if and only if there exists a $D \in \mathcal{K}(X)$ such that the pair (C, D) is minimal and equivalent to $(A \vee B, A + B)$.*

Proof. "$\Longrightarrow$" Since the pair $(A, B) \in \mathcal{K}^2(X)$ is C-minimal, we know by Theorem 1.1 (iv) that

$$A + C + B + C \;=\; (A + C) \cup (B + C) + (A + C) \cap (B + C).$$

Since

$$(A + C) \cup (B + C) \;=\; (A + C) \vee (B + C) \;=\; (A \vee B) + C$$

we obtain that

$$A + B + C \;=\; A \vee B \;+\; (A + C) \cap (B + C).$$

If we put $D := (A + C) \cap (B + C)$, then it follows, that

$$(A \vee B, A + B) \sim (C, D). \tag{3.3.1}$$

Now suppose that there exists a $C_1 \subset C$ and $D_1 \subset D$ such that $(C, D) \sim (C_1, D_1)$. Then $(A \vee B, A + B) \sim (C_1, D_1)$. Hence we have $A \vee B + D_1 = A + B + C_1$, which implies $(A + C_1) \vee (B + C_1) + D_1 = (A + C_1) + (B + C_1)$. Thus $(A + C_1) \vee (B + C_1)$ is a summand of $(A + C_1) + (B + C_1)$, which means that the set $(A + C_1) \cup (B + C_1)$ is convex. Hence by the C-minimality of (A, B) it follows that $C = C_1$ and $D = D_1$.

"$\Longleftarrow$" Now if $(A \vee B, A + B) \sim (C, D)$ and (C, D) minimal, then $(A + C, B + C)$ is a convex pair. Then for every $(C_1, D_1) \leq (C, D)$ which is equivalent to (C, D) we have $C = C_1$ and $D = D_1$. □

Corollary 3.4. *For every pair $(A, B) \in \mathcal{K}^2(X)$ there exists a set $C \in \mathcal{K}(X)$ such that the pair (A, B) is C-minimal.*

Theorem 3.5. *Let X be a real topological vector space and $A, B, C \in \mathcal{K}(X)$. Then the pair $(A + C, B + C)$ is minimal convex if and only if there exists a $D \in \mathcal{K}(X)$ such that $(A \vee B, A + B)$ is equivalent to (C, D) and $(A \vee B + C, D)$ is a minimal pair.*

Proof. "$\Longleftarrow$" First observe, that the minimality of the pair $(A \vee B + C, D)$ implies, that (C, D) is minimal since

$$(A \vee B + C, D) = (A \vee B, 0) \;+\; (C, D)$$

and $(A \vee B, 0)$ is minimal (see [6]). Now by Theorem 3.3 the pair (A, B) is C-minimal. But then we have:

$$(A + C) \cup (B + C) \;=\; A \vee B + C \quad \text{and} \quad (A + C) \cap (B + C) \;=\; D.$$

Hence it follows from Theorem 1.4 that $(A+C, B+C)$ is a minimal convex pair.

"$\Longrightarrow$" Now let us assume that $(A+C, B+C)$ is a minimal convex pair. Then the set $(A+C)\cup(B+C)$ is convex and by Theorem 1.1 the pair $(A\vee B, A+B)$ is equivalent to (C, D), with $D=(A+C)\cap(B+C)$. Now from Theorem 1.4 follows that $(A\vee B+C, D)$ is a minimal pair. □

Corollary 3.6. *The pair $(A + A\vee B, B + A\vee B) \in \mathcal{K}^2(X)$ is minimal convex if and only if the pair $(A\vee B, A+B)$ is minimal.*

Proof. The minimality of the pair $(A\vee B, A+B)$ is equivalent to the minimality of the pair $(2\cdot(A\vee B), A+B)$. Hence we obtain from the above Theorem, that $(A+A\vee B, B+A\vee B)\in\mathcal{K}^2(X)$ is a minimal convex pair if and only if $(A\vee B, A+B)$ is a minimal pair. □

Remark 3.7.

(i) The set $D\in\mathcal{K}(X)$ of Theorem 3.5 is uniquely determined and coincides with $(A+C)\cap(B+C)$.

(ii) If the pair $(A+C, B+C)$ is minimal convex, then (A, B) is a C-minimal pair.

Proposition 3.8. *Let X be a real topological vector space and $(A,B),(C,D)\in\mathcal{K}^2(X)$ are two equivalent minimal pairs and assume that the sets $A\cup C$ and $B\cup D$ are convex. Then $A = C$ and $B = D$.*

Proof. Observe that by Proposition 2.5 (iii)

$$A+(B\cap D) = (A+B)\cap(A+D) = (A+B)\cap(B+C) = B + (A\cap C),$$

since the sets $A\cup C$ and $B\cup D$ are convex. By assumption the pair $(A,B)\in\mathcal{K}^2(X)$ is minimal. Hence

$$A\cap C = A \quad \text{and} \quad B\cap D = B.$$

Since the pair $(C,D)\in\mathcal{K}^2(X)$ is is also minimal, it follows that $A=C$ and $B=D$. □

In analogy to the above Proposition 3.8 and the results of J. Grzybowski [3] and S. Scholtes [14] we have:

Theorem 3.9. *If (A,B), $(C_1,C_2)\in\mathcal{K}^2(\mathbb{R}^n)$, $1\le n\le 2$ and the pair (A,B) is C_i-minimal for $i=1,2$, then there exists an $x\in\mathbb{R}^n$ such that $C_2=C_1+x$.*

Proof. Since for $i=1,2$ the pair (A,B) is C_i-minimal, there exists by Theorem 3.3 elements $D_i\in\mathcal{K}(\mathbb{R}^n)$, such that the pairs (C_i,D_i), $i=1,2$ are both minimal and equivalent to $(A\vee B, A+B)$. Hence it follows from the resultes of J. Grzybowski and S. Scholtes (see for instance: [3], Lemma 3.7) that $C_2=C_1+x$ for an $x\in\mathbb{R}^n$. □

Now we study some properties of C-minimal equivalence classes.

Lemma 3.10. *Let X be a topological vector space and $A, B_\alpha\in\mathcal{K}(X)$, for $\alpha\in\Lambda$. If the family $\{B_\alpha\}_{\alpha\in\Lambda}$ is a chain, then $\bigcap_{\alpha\in\Lambda} B_\alpha \neq \emptyset$ and*

$$A+\bigcap_{\alpha\in\Lambda} B_\alpha = \bigcap_{\alpha\in\Lambda}(A+B_\alpha).$$

Proof. Since the family $\mathcal{F} := \{B_\alpha\}_{\alpha\in\Lambda}$ has the finite intersection property, we have $\bigcap_{\alpha\in\Lambda} B_\alpha \neq \emptyset$ (see [2]).
If we define for $\alpha,\beta\in\Lambda$ a partial order by:

$$\alpha \le \beta \quad \text{if and only if} \quad B_\beta \subseteq B_\alpha$$

then the set Λ is directed by $\le$.
Now observe that

$$A + \bigcap_{\alpha\in\Lambda} B_\alpha \ \subseteq\ \bigcap_{\alpha\in\Lambda} (A + B_\alpha).$$

To prove the converse inclusion, let us choose an arbitrary element $x \in \bigcap_{\alpha\in\Lambda}(A + B_\alpha)$. Then for every $\alpha\in\Lambda$ there exist $a_\alpha\in A$ and $b_\alpha\in B_\alpha$ such that $x = a_\alpha + b_\alpha$. Since the set $\mathcal{S} := \{a_\alpha \mid \alpha\in\Lambda\}$ is a net in the compact set A, there exists a cluster point $a\in A$ of $\mathcal{S}$. Now $b_\alpha = x - a_\alpha$ and since X is a topological vector space, there exists an element $b\in X$ which is a cluster point of the net $\mathcal{S}' := \{b_\alpha \mid \alpha\in\Lambda\}$ with $x = a + b$.
To complete the proof, it suffices to show, that $b\in X$ belongs to all members of the family $\mathcal{F}$. Therefore take a $B_\alpha\in\mathcal{F}$. Then for every neighborhood V of b there exists an index $\beta\ge\alpha$ such that $b_\beta\in V$. Since $b_\beta\in B_\beta\subseteq B_\alpha$, we conclude that $B_\alpha\cap V\neq\emptyset$. Furthermore, the set B_α is closed which implies, that $b\in B_\alpha$. □

Definition 3.11. Let X be a real topological vector space and $A,B,C\in\mathcal{K}(X)$. The class $[A,B]$ is called *"C-convex"* if for every representant $(A_1,B_1)\in[A,B]$ the pair (A_1+C, B_1+C) is convex.

Definition 3.12. Let X be a real topological vector space and $A,B,C\in\mathcal{K}(X)$. The class $[A,B]$ is called *"C-minimal"* if for every $C_1\in\mathcal{K}(X)$ with $C_1\subseteq C$ for which the class $[A,B]$ is C_1-convex, follows that $C_1 = C$.

Theorem 3.13. *Let X be a real topological vector space.
Then for every class $[A_0,B_0]$, with $A_0,B_0\in\mathcal{K}(X)$, there exists a set $C_0\in\mathcal{K}(X)$ such that the class $[A_0,B_0]$ is C_0-minimal.*

Proof. For the pair $(A_0,B_0)\in\mathcal{K}^2(X)$ let us consider the family

$$\mathcal{C}_{[A_0,B_0]} := \{C \mid C\in\mathcal{K}(X) \quad \text{such that for every } (A,B)\in[A_0,B_0] \\ \text{the pair } (A+C, B+C) \text{ is convex } \}.$$

This family is non empty, namely $A_0 + A_0\vee B_0 \in \mathcal{C}_{[A_0,B_0]}$, which can be seen as follows: Observe first that the pair (A_0,B_0) is equivalent to the convex pair $(A_0 + A_0\vee B_0, B_0 + A_0\vee B_0)$. If we put now $C^* := A_0 + A_0\vee B_0$ then the rest follows from Proposition 2.6. Now we can order the class $\mathcal{C}_{[A_0,B_0]}$ by inclusion, namely $C_\alpha\subseteq C_\beta \iff \alpha\le\beta$. Now consider an ordered chain $\{C_\alpha\}_{\alpha\in\Lambda}$. Then by Lemma 3.10 we have:

$$A + \bigcap_{\alpha\in\Lambda} C_\alpha \ =\ \bigcap_{\alpha\in\Lambda}(A + C_\alpha)$$

and

$$B + \bigcap_{\alpha\in\Lambda} C_\alpha \ =\ \bigcap_{\alpha\in\Lambda}(B + C_\alpha).$$

for arbitrary elements $A, B \in \mathcal{K}(X)$.
Now consider $(A,B) \in [A_0,B_0]$. From the definition of the class $\mathcal{C}_{[A_0,B_0]}$ follows, that for every $\alpha \in \Lambda$ the pair $(A+C_\alpha, B+C_\alpha)$ is convex. Therefore we have: $(A \vee B, A+B) \sim (C_\alpha, (A+C_\alpha) \cap (B+C_\alpha))$, which means that

$$A+B+C_\alpha \;=\; A \vee B + (A+C_\alpha) \cap (B+C_\alpha).$$

Now by Lemma 3.10 we have:

$$\bigcap_{\alpha\in\Lambda} (A+B+C_\alpha) \;=\; A+B+\bigcap_{\alpha\in\Lambda} C_\alpha,$$

and also

$$\bigcap_{\alpha\in\Lambda} (A \vee B + (A+C_\alpha) \cap (B+C_\alpha)) \;=\; A \vee B + \bigcap_{\alpha\in\Lambda} (A+C_\alpha) \cap (B+C_\alpha).$$

Denote

$$C_0 \;=\; \bigcap_{\alpha\in\Lambda} C_\alpha.$$

Hence

$$A \;+\; B \;+\; C_0 \;=\; A \vee B \;+\; (A+C_0) \cap (B+C_0).$$

Now from $C_0 \in \mathcal{C}_{[A_0,B_0]}$ and the Lemma of Kuratowski-Zorn follows, that $\mathcal{C}_{[A_0,B_0]}$ has a minimal element. $\square$

Proposition 3.14. *Let X be a real topological vector space $C \in \mathcal{K}(X)$ and $(A_0,B_0) \in \mathcal{K}^2(X)$ be a C-minimal pair. Moreover let us assume that A_0 or B_0 is a summand of $(A_0+C) \cap (B_0+C)$.*
Then the class $[A_0,B_0]$ is C-minimal.

Proof. For every $(A,B) \in [A_0,B_0]$ we have $(A,B) \sim (A_0+C, B_0+C)$. Hence

$$A \;+\; B_0 \;+\; C \;=\; A_0 \;+\; C + \; B \;=\; A \vee B \;+\; (A_0+C) \cap (B_0+C). \tag{3.14.1}$$

Now assume that B_0 is a summand of $(A_0+C) \cap (B_0+C)$. Hence we have $(A_0+C) \cap (B_0+C) \;=\; B_0 \;+\; S$, for some $S \in \mathcal{K}(X)$. Now we obtain from (3.14.1) that

$$B \;+\; C \;=\; A \vee B \;+\; S,$$

and by adding $A+C$ we get $A \;+\; B \;+\; C \;+\; C \;=\; A \vee B \;+\; C \;+\; A \;+\; S$, and therefore we have:

$$(A \;+C) \;+\; (B \;+\; C) \;=\; (A \vee B \;+C) \;+\; (A \;+\; S)$$

i.e.

$$(A \;+C) \;+\; (B \;+\; C) \;=\; (A \;+\; C) \vee (B \;+\; C) \;+\; (A \;+\; S).$$

From Theorem 1.1 and the formula $(A \;+C) \vee (B \;+\; C) \;=\; (A \vee B \;+C)$, follows that $(A \;+C) \cup (B \;+\; C)$ is convex. Since $(A_0,B_0) \in \mathcal{K}^2(X)$ a C-minimal pair it follows, that the class $[A_0,B_0]$ is C-minimal. $\square$

Proposition 3.15. *Let X be a real topological vector space.*
Then a convex pair $(A_0, B_0) \in \mathcal{K}^2(X)$ is minimal convex if and only if for every convex pair $(A, B) \in \mathcal{K}^2(X)$ equivalent to (A_0, B_0) with $A \cup B \subseteq A_0 \cup B_0$ follows that $A = A_0$ and $B = B_0$.

Proof. "$\Longrightarrow$" Let (A, B) be a convex pair which is equivalent to (A_0, B_0) and such that $A \cup B \subseteq A_0 \cup B_0$. Then

$$A \dotplus B_0 = A_0 \dotplus B = A \cup B \dotplus A_0 \cap B_0 \subseteq A_0 \cup B_0 \dotplus A_0 \cap B_0 = A_0 \dotplus B_0.$$

Now from the order cancellation law we get that $A \subset A_0$ and $B \subset B_0$. From the convex minimality we deduce that $A = A_0$ and $B = B_0$.
"$\Longleftarrow$" Now let $(A, B) \sim (A_0, B_0)$ be a convex pair with $A \subset A_0$ and $B \subset B_0$. Then $A \cup B \subset A_0 \cup B_0$, and therefore by assumption we have $A \cup B = A_0 \cup B_0$. This means by Theorem 1.1 that

$$A \dotplus B_0 = B \dotplus A_0 = A_0 \cup B_0 \dotplus A_0 \cap B_0 = A_0 \dotplus B_0,$$

and hence $A = A_0$ and $B = B_0$. □

4. Convex hull of pairs of compact convex sets

In this section we give a definition of a convex hull of a pair of compact convex sets. The intention is, that the convex hull can be considered as a *smallest convexification* of the set, and this is the motivation for our definition:

Definition 4.1. Let X be a real topological vector space and and $(A, B) \in \mathcal{K}^2(X)$. A convex pair $(C, D) \in \mathcal{K}^2(X)$ is called a *"convex hull"* of the pair (A, B) if $(A, B) \leq (C, D) \sim (A, B)$ and if for every convex pair $(C_1, D_1) \in \mathcal{K}^2(X)$ with $(C_1, D_1) \sim (C, D)$ and $(A, B) \leq (C_1, D_1) \leq (C, D)$, it follows that $(C, D) = (C_1, D_1)$.

Proposition 4.2. *Let X be a real topological vector space and $(C, D) \in \mathcal{K}^2(X)$ a convex pair, which is equivalent to $(A, B) \in \mathcal{K}^2(X)$. Then the following statements are equivalent:*
(i) *The set $C \cap D$ separates the sets A and B,*
(ii) $(A, B) \leq (C, D)$,
(iii) $A \cup B \subseteq C \cup D$.

Proof. "(i) $\Longrightarrow$ (ii)" Since the pair (C, D) is convex and equivalent to (A, B) by Theorem 1.2 we have that $A \dotplus D = B \dotplus C = A \vee B \dotplus C \cap D$. Since $C \cap D$ separates the sets A and B we have by (see [8]) that $A \dotplus B \subseteq A \vee B \dotplus C \cap D$. Therefore $A \dotplus B \subseteq A \dotplus D = B \dotplus C$ and from the order law of cancellation we get that $A \subset C$ and $B \subset D$.
"(ii) $\Longrightarrow$ (i)" is obvious by (see [8]).
"(iii) $\Longrightarrow$ (ii)" Since the pair $(A \dotplus A \vee B, B \dotplus A \vee B)$ is convex and equivalent to the convex pair (C, D) it follows immediately from Lemma 1.2 that

$$A \dotplus A \vee B \dotplus D = B \dotplus A \vee B \dotplus C = A \dotplus B \dotplus C \cup D \supseteq A \dotplus B \dotplus A \vee B.$$

The inclusions $A \subseteq C$ and $B \subseteq D$ follow now from the order law of cancellation.
"(ii) $\Longrightarrow$ (iii)" This implication is obvious. □

Theorem 4.3. *Let X be a real topological vector space. Then for every pair $(A,B) \in \mathcal{K}^2(X)$ there exists a convex hull of (A,B).*

Proof. For the pair $(A,B) \in \mathcal{K}^2(X)$ let us consider the family

$$\mathcal{C}_{(A,B)} := \{(C,D) \mid (C,D) \in \mathcal{K}^2(X) \text{ such that } (A,B) \leq (C,D) \sim (A,B) \text{ and } (C,D) \text{ is convex}\}.$$

This family is non empty, namely for every arbitrary $x \in A \vee B$ we have by Proposition 1.2 that the pair $(A + A \vee B - x, B + A \vee B - x)$ is convex and equivalent to (A,B). Moreover

$$(A,B) \leq (A + A \vee B - x, B + A \vee B - x).$$

Analogously to the proof of Theorem 3.13 we can show that for an arbitrary totally ordered set $\mathcal{B} \subseteq \mathcal{C}_{(A,B)}$, with $\mathcal{B} := \{(A_\alpha, B_\alpha) \mid \alpha \in \Lambda\}$, the pair (A_0, B_0) is equivalent to (A,B), where

$$A_0 = \bigcap_{\alpha\in\Lambda} A_\alpha \quad \text{and} \quad B_0 = \bigcap_{\alpha\in\Lambda} B_\alpha.$$

It is obvious that $(A,B) \leq (A_0, B_0)$. Hence by the Lemma of Kuratowski-Zorn there exists a minimal element in $\mathcal{C}_{(A,B)}$ and this is a convex hull of the pair (A,B). □

Remark 4.4. If the pair (A,B) is equivalent to (C,D) and is such that $(A,B) \leq (C,D)$ and (C,D) is minimal convex, then (C,D) is a convex hull of (A,B).

5. Reduction of pairs of compact convex sets

In this part of the paper we discuss algorithmic concepts of reducing a pair of compact convex sets within its equivalence class and such that given properties of a pair remain preserved under the reduction.
In [16] the following result is proved:

Theorem 5.1. *Let X be a real topological vector space and $(A,B) \in \mathcal{K}^2(X)$ be a convex pair. Then for every pair $(F,G) \in \mathcal{K}^2(X)$ which is equivalent to $(A \cap B, A \cup B)$ such that $(F,G) \leq (A \cap B, A \cup B)$ there exists a convex pair $(A_0,B_0) \sim (A,B)$ with $A_0 \cap B_0 = F$ and $A_0 \cup B_0 = G$.*

Now we show:

Proposition 5.2. *If $(A,B) \in \mathcal{K}^2(X)$ is a convex pair and $F,G \in \mathcal{K}(X)$ such that $A \cup B + F = A \cap B + G$, $F \subseteq A \cap B$. Moreover let us assume that $A \cup G = A \cup B$ or $B \cup G = A \cup B$.*
Then $A \cup B$ is a summand of $A + G$ or $A \cup B$ is a summand of $B + G$.

Proof. From Theorem 5.1 follows, that there exist $A_0, B_0 \in \mathcal{K}(X)$ such that

$$A + B_0 = B + A_0 = A \cup B + F = A \cap B + G,$$

with

$$B_0 \cap A_0 = F \quad \text{and} \quad B_0 \cup A_0 = G.$$

Hence we have:

$$B + A_0 \; + A \cup B \; = \; A \cap B + A \cup B + G \; = \; A + B + G,$$

and

$$A_0 + A \cup B \; = \; A + G. \tag{5.2.1}$$

Analogously

$$B_0 + A \cup B \; = \; B + G. \tag{5.2.2}$$

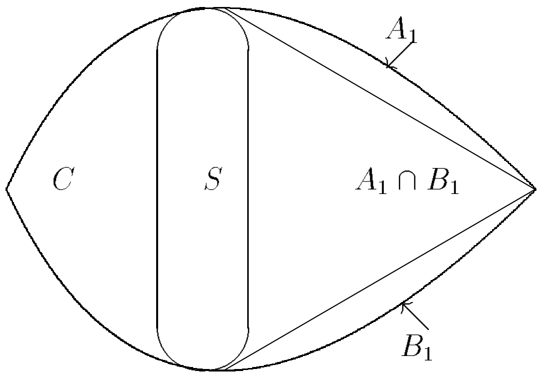
If $A \cup G \; = \; A \cup B$ then $A + G \; = \; A \cap G + A \cup B$, and we obtain that

$$A_0 + A \cup B \; = \; A \cap G + A \cup B,$$

which gives $A_0 \; = \; A \cap G$. From the assumption $B \cup G \; = \; A \cup B$ follows analogously, that $B_0 \; = \; B \cap G$. □

Proposition 5.3. *Let X be a real topological vector space and $(A, B) \in \mathcal{K}^2(X)$ a convex pair. Moreover let us assume that there exist sets $A_1, B_1, C \in \mathcal{K}(X)$, such that $A = A_1 \cup C$, $B = B_1 \cup C$ and $C \cap A_1 = C \cap B_1$.*
Then (A_1, B_1) is a convex pair which is equivalent to (A, B).

Proof.



Let us put $S := C \cap A_1$. Then we have:

$$A + S \; = \; C + A_1 \qquad B + S \; = \; C + B_1. \tag{5.3.1}$$

Hence

$$A \cup B + S \; = \; C + A_1 \vee B_1 \qquad A + B + 2S \; = \; 2C + A_1 + B_1. \tag{5.3.2}$$

From the last equation we get:

$$A \cup B + S + A \cap B + S \; = \; 2C + A_1 + B_1. \tag{5.3.3}$$

But

$$C + A_1 \cap B_1 \; = \; A \cap B + S, \tag{5.3.4}$$

hence from (5.3.3) and (5.3.4) follows, that

$$A_1 + B_1 \;=\; A_1 \vee B_1 + A_1 \cap B_1.$$

Now by Theorem 5.1 we know that the pair (A_1, B_1) is convex and from (5.3.1) follows that $A + B_1 \;=\; B + A_1$. □

Corollary 5.4. *Let $(A, B) \in \mathcal{K}^2(X)$ be a convex pair and let $G \in \mathcal{K}(X)$ be such that $A \cup G = B \cup G = A \cup B$.*
Then the pair $(A \cap G, B \cap G)$ is convex and equivalent to (A, B).

Proof. Put $A_1 := A \cap G$, $B_1 := B \cap G$, and $C := A \cap B$. Then $C \cup A_1 = (A \cap B) \cup (A \cap G) = A \cap (B \cup G) = A$, and $C \cup B_1 = B$. Moreover $C \cup (A_1 \cap B_1) = (A \cap B) \cup (A \cap B \cap G) = A \cap B$ and $C \cap A_1 = C \cap B_1 = A \cap B \cap G$. Hence by the above Proposition 5.3 we obtain that $(A \cap G, B \cap G)$ is a convex pair, which is equivalent to the pair (A, B). □

Corollary 5.5. *Let $(A, B) \in \mathcal{K}^2(X)$ be a convex pair and assume that $\overline{\mathrm{conv}}(A \cup B \setminus A \cap B) \neq A \cup B$.*
Then (A, B) is not convex minimal.

Proof. Denote

$$F := (A \cap B) \cap \overline{\mathrm{conv}}(A \cup B \setminus A \cap B) \quad \text{and} \quad G := \overline{\mathrm{conv}}(A \cup B \setminus A \cap B).$$

Then $A \cup B + F = A \cap B + G$ and since $A \cup G = B \cup G = A \cup B$, we obtain from Proposition 5.2 that $(A \cap G, B \cap G) \sim (A, B)$. Since $A \cap G$ is an essential subset of A, we see that the pair (A, B) is not convex minimal. □

Proposition 5.6. *Let X be a real topological vector space and $(A_1, B_1), (A_2, B_2) \in \mathcal{K}^2(X)$ convex pairs such that:*

$$\begin{aligned} &(A_1 \cap B_1 + x) \cup (A_2 \cup B_2) = A_1 \cup B_1, \\ &(A_1 \cap B_1 + x) \cap (A_2 \cup B_2) = A_2 \cap B_2 \;+\; x, \quad \text{for some} \quad x \in X. \end{aligned}$$

Moreover assume that

$$A_1 \cup B_2 \;=\; A_2 \cup B_2 \quad \text{and} \quad A_1 \cap B_2 \;=\; A_1 \cap B_1 \tag{5.6.1}$$

or

$$A_2 \cup B_1 \;=\; A_1 \cup B_1 \quad \text{and} \quad A_2 \cap B_1 \;=\; A_2 \cap B_2. \tag{5.6.2}$$

Then

$$(A_1, B_1) \sim (A_2, B_2) \quad \text{and} \quad (A_2, B_2) \;\leq\; (A_1, B_1).$$

Proof. Let us first observe that

$$(A_1 \cap B_1 + x) \;+\; (A_2 \cup B_2) \;=\; A_1 \cup B_1 \;+\; (A_1 \cap B_1 + x) \cap (A_2 \cup B_2),$$

and hence

$$A_1 \cap B_1 \;+\; A_2 \cup B_2 \;=\; A_1 \cup B_1 \;+\; A_2 \cap B_2. \tag{5.6.3}$$

Without loss of generality we may assume that assumption (5.6.1) is satisfied. Then by Proposition 1.2 we have:

$$A_1 \; + \; B_2 \; = \; A_1 \cup B_2 \; + \; A_1 \cap B_2 \; = \; A_2 \cup B_2 \; + \; A_1 \cap B_1. \tag{5.6.4}$$

Now let us put $F := \; A_2 \cap B_2$ and $G := \; A_2 \cup B_2$. Then we have $G \subseteq \; A_1 \cup B_1$ and $F \subseteq \; A_1 \cap B_1$, and by Theorem 5.1 there exists a pair $(A_0, B_0) \in \mathcal{K}^2(X)$ which is equivalent to (A_1, B_1) and such that

$$A_1 \; + \; B_0 \; = \; A_1 \cup B_1 \; + \; F \; = \; A_1 \cap B_1 \; + \; G \; = \; B_1 \; + \; A_0 \tag{5.6.5}$$

and moreover $A_0 \cap B_0 \; = \; F$ and $A_0 \cup B_0 \; = \; G$.
From equation (5.6.4) we obtain $A_1 \; + \; B_0 \; = \; A_1 \; + \; B_2$, and hence $B_0 \; = \; B_2$.
Now we observe that $A_0 + B_0 = F + G = A_2 + B_2$, and hence we have $A_0 = A_2$. □

6. Application to the quasidifferential calculus

For an arbitrary real topological vector space X a characterization of a convex pair $(A, B) \in \mathcal{K}^2(X)$ has been given in [16] and is stated in Theorem 1.1. In the following we restrict our attention to the case of a locally convex topological vector space X and will describe the charactization of a convex pair in terms of its support functions.
Therefore let X be a locally convex topological vector space and X^* the topological dual, i.e. the linear space of all continuous linear functional defined on X, endowed with the weak-*-topology. Moreover let us denote by

$$\langle \cdot, \cdot \rangle \; : \; X^* \times X \to \mathbb{R}$$

be the dual pairing given by

$$\langle \; v, x \; \rangle \; := \; v(x).$$

Then the *"support function"* of $A \in \mathcal{K}(X)$ is given by

$$p_A : X^* \longrightarrow \mathbb{R}$$

with

$$p_A(x) = \max_{a \in A} \langle a, x \rangle.$$

It was shown by L. Hörmander [4] that the support function is sublinear and continuous with respect to the weak-*-topology on X^*.
There exists a partial order on the set of support functions, namely

$$p_A \le p_B \quad \text{if and only if for every} \quad x \in X^* \quad p_A(x) \le p_B(x) \quad \Longleftrightarrow \quad A \subseteq B.$$

For a locally convex topological vector space X we will now give a characterization of a convex pair $(A, B) \in \mathcal{K}^2(X)$ in terms of the support functions p_A and p_B.

Let us furthermore denote by

$$\mathcal{D}(X^*) := \{\varphi = p_A - p_B \mid (A, B) \in \mathcal{K}^2(X)\}$$

the real vector space of differences of support functions.

Theorem 6.1. *Let X be a locally convex topological vector space and $(A, B) \in \mathcal{K}^2(X)$. Then the pair (A, B) is convex if and only if*

$$\min\{p_A, p_B\} : X^* \longrightarrow \mathbb{R}$$

is a convex function.

Proof. "$\Longrightarrow$" By Theorem 1.1 the pair $(A, B) \in \mathcal{K}^2(X)$ is convex if and only if $A \vee B$ is a summand of $A+B$. This means that there exists an $S \in \mathcal{K}(X)$ such that $A+B = A \vee B + S$. From property (iv) of Theorem 1.1 follows easily that $S = A \cap B$. Hence we have:

$$A + B = A \vee B + A \cap B.$$

But since for the support function of $A \vee B$ the formula (see [4]) $p_{A \vee B} = \max\{p_A, p_B\}$ holds this equation can be formulated in terms of support functions as:

$$p_A + p_B = \max\{p_A, p_B\} + p_{A \cap B}.$$

Now we have $\min\{p_A, p_B\} = p_{A \cap B}$, since for arbitrary real numbers $a, b \in \mathbb{R}$ the formula $a + b = \max\{a, b\} + \min\{a, b\}$ holds.

"$\Longleftarrow$" Now assume that $\min\{p_A, p_B\}$ is convex. Since this function is also positively homogeneous there exists an $S \in \mathcal{K}(X)$ such that $\min\{p_A, p_B\} = p_S$. Hence we have

$$p_A + p_B = \max\{p_A, p_B\} + \min\{p_A, p_B\} = p_{A \vee B} + p_S,$$

which means that

$$p_{A + B} = p_{A \vee B} + p_S = p_{A \vee B + S}.$$

From this follows that

$$A + B = A \vee B + S,$$

(see [4]), which means that $A \vee B$ is a summand of $A + B$. Hence by Theorem 1.1 the set $A \cup B$ is convex. $\square$

Let us remark, that for $A, B \in \mathcal{K}(X)$ with $A \cap B \neq \emptyset$ the support function of $A \cap B$ is the *"infimal convolution"* (see [11]) $p_A \square p_B$ given by:

$$p_A \square p_B : X^* \longrightarrow \mathbb{R}$$

with

$$(p_A \square p_B)(x) := \inf\{p_A(x - v) + p_B(v) \mid v \in X^*\}.$$

Hence we have the following

Corollary 6.2. *Let X be a locally convex topological vector space and $(A, B) \in \mathcal{K}^2(X)$. Then the pair (A, B) is convex if and only if*

$$\min\{p_A, p_B\} = p_A \square p_B.$$

Observe that the representation of the elements of the space $\mathcal{D}(X^*)$ is not unique. From the obove mentioned characterization of a convex pair, we are able to show, that some special types of representations exist.

Proposition 6.3. *Let X be a locally convex topological vector space and*

$$\mathcal{D}(X^*) := \{\varphi = p_A - p_B \mid (A,B) \in \mathcal{K}^2(X)\}$$

the real vector space of differences of support functions, defined on X^*.
Then for every element $\varphi \in \mathcal{D}(X^*)$ there exist the following types of representations:

(i) $\varphi \in \mathcal{D}(X^*)$ *can be represented as* $\varphi = p_A - p_B$ *with* $\min\{p_A, p_B\} \geq 0$

(ii) $\varphi \in \mathcal{D}(X^*)$ *can be represented as* $\varphi = p_{A_0} - p_{B_0}$ *such that* $\min\{p_{A_0}, p_{B_0}\}$ *is a sublinear function and that* $p_{A_0} + p_{B_0}$ *is minimal, i.e.* $p_{A_0} + p_{B_0} = \min\{p_A + p_B \mid \varphi = p_A - p_B\}$

(iii) *There exists a* $C \in \mathcal{K}(X)$ *such that for every representation of* $\varphi \in \mathcal{D}(X^*)$ *as* $\varphi = p_A - p_B$ *the function* $\min\{p_A, p_B\} + p_C$ *is sublinear.*

Proof. (i) The first assertion follows immediately from the Theorem of Hahn-Banach. Namely let

$$\varphi(h) := \sup_{v \in A} \langle\, h, v \,\rangle - \sup_{v \in B} \langle\, h, v \,\rangle.$$

Then we can add to the first summand a continuous linear functional $f_1 \in X^{\star\star}$, given by $f_1(h) := \langle u_1, h \rangle,\ u_1 \in X$, with

$$\sup_{v \in A} \langle\, h, v \,\rangle \geq f_1(h)$$

and, analogously, to the second summand a $f_2 \in X^{\star\star}$, $f_2(h) := \langle u_2, h \rangle,\ u_2 \in X$ with

$$\sup_{v \in B} \langle\, h, v \,\rangle \geq f_2(h).$$

Now we take a representation:

$$\varphi(h) = (\sup_{v \in A} \langle\, h, v \,\rangle - f_1(h) + \max\{f_1(h) - f_2(h), 0\}) - (\sup_{v \in B} \langle\, h, v \,\rangle - \min\{f_2(h) - f_1(h), 0\}).$$

Now we prove part (ii)

Since every pair $(A,B) \in \mathcal{K}^2(X)$ is equivalent to a convex pair, namely $(A,B) \sim (A + A \vee B, B + A \vee B)$ there exists a minimal convex pair (A_0, B_0) equivalent to (A,B). From Theorem 1.1 and Proposition 3.15 follows, that $\min\{p_{A_0}, p_{B_0}\}$ is convex and $p_{A_0} + p_{B_0}$ is minimal.

Part (iii) follows immediately from Theorem 3.5. Namely for the class $[A,B]$ there exists a minimal element $C \in \mathcal{K}(X)$ such that for all $(A', B') \in [A,B]$ the pair $(A' + C, B' + C)$ is convex. Hence by Theorem 1.1 we have that $\min\{p_{A'+C}, p_{B'+C}\} = \min\{p_{A'}, p_{B'}\} + p_C$, which means that the function $\min\{p_{A'}, p_{B'}\} + p_C$ is sublinear. □

Now let us turn our attention to the class of quasidifferentiable functions which where introduced by V.F. Demyanov and A.M. Rubinov [1]. We will start with the definition of a quasidifferentiable function defined on an open subset of a normed vector spaces. (see: [1]).

Let $(X, \|\cdot\|)$ be a real normed vector space, let X^* be its topological dual, and let $U \subseteq X$ be an open subset of X.

Definition 6.4. A continuous real valued function $f : U \to \mathbb{R}$ is said to be quasidifferentiable at $x_0 \in U$ if the following two conditions are satisfied:

(a) For every $g \in X \setminus \{0\}$ the directional derivative

$$\left.\frac{df}{dg}\right|_{x_0} = \lim_{t \to 0+} \frac{f(x_0 + tg) - f(x_0)}{t}$$

exists.

(b) There exist two sets $\underline{\partial} f|_{x_0},\ \overline{\partial} f|_{x_0} \in \mathcal{K}(X^*)$ such that

$$\left.\frac{df}{dg}\right|_{x_0} = \max_{v \in \underline{\partial} f|_{x_0}} \langle v, g\rangle + \min_{w \in \overline{\partial} f|_{x_0}} \langle w, g\rangle.$$

Here $\mathcal{K}(X^*)$ denotes the collection of all nonempty weak-*-compact convex subsets of X^*. We remark that, by the Theorem of Alaoglu (cf. [12], p. 228) the elements of $\mathcal{K}(X^*)$ are bounded in the dual norm.

Observe that the condition (b) is equivalent to the requirement that the directional derivative as a function of the direction g can be expressed as the difference of two sublinear functions.

Although the quasidifferential $Df|_{x_0} = (\ \underline{\partial} f|_{x_0}, \overline{\partial} f|_{x_0}\ )$ is not uniquely determined it preserves certain properties within an equivalence class.

In [1], chapter 17, it is shown that for a quasidifferentiable function all steepest ascent- and descent directions can be explicitely determined.

Namely, if for a given quasidifferentiable function $f : U \longrightarrow \mathbb{R}$ a quasidifferential $Df|_{x_0} = (\ \underline{\partial} f|_{x_0}, \overline{\partial} f|_{x_0}\ )$, in $x_0 \in U \subset X$ is known, then every steepest descent direction of the function f in x_0 is given by

$$g^* := -\frac{w_0 \ + \ v_0}{\|w_0 \ + \ v_0\|^*} \tag{6.4.1}$$

with

$$\|w_0 \ + \ v_0\|^* \ = \ \sup_{w \in \overline{\partial} f|_{x_0}} \ \inf_{v \in \underline{\partial} f|_{x_0}} \|w \ + \ v\|^*, \tag{6.4.2}$$

where $\|.\|^*$ denotes the dual norm.

An analogue formula holds for the steepest ascent direction.

If the quasidifferentiable function $f : U \longrightarrow \mathbb{R}$ has a local maximum in $x_0 \in U$ then the formulas (6.4.1) and (6.4.2) imply that

$$-\overline{\partial} f|_{x_0} \ \subseteq \ \underline{\partial} f|_{x_0} \tag{6.4.3}$$

and analogously for a local minimum in $x_0 \in U$ of the function $f : U \longrightarrow \mathbb{R}$ in $x_0 \in U$ one has

$$-\underline{\partial} f|_{x_0} \ \subseteq \ \overline{\partial} f|_{x_0}. \tag{6.4.4}$$

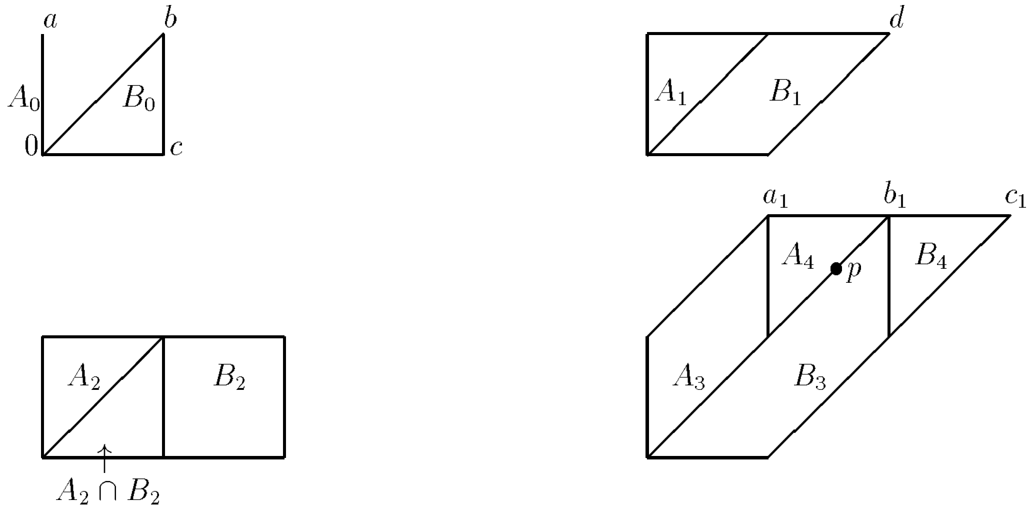
Hence we see, that in the case of a local extremum of a quasidifferentiable function $f : U \longrightarrow \mathbb{R}$ in $x_0 \in U$ the class $[\ \underline{\partial} f|_{x_0}, -\overline{\partial} f|_{x_0}\ ]$ is convex and that it is C-minimal for every $C := \{p\},\ p \in X$.

7. Examples of different types of minimality

Example 7.1. Let $X := \mathbb{R}^2$ and let us put $0 := (0,0)$, $a := (0,1)$, $b := (1,1)$, $c := (1,0)$ and put $d := b + c$. Now we define the following compact convex sets:

$$\begin{array}{lcl} A_0 := 0 \vee a & , & B_0 := 0 \vee b \vee c \\ A_1 := A_0 \vee b & , & B_1 := B_0 \vee d \\ A_2 := A_0 + I_0 & , & B_2 := B_0 + I_0 \\ A_3 := A_0 + I_1 & , & B_3 := B_0 + I_1 \\ A_4 := A_1 + I_1 & , & B_4 := B_1 + I_1 \end{array}$$

where $I_0 := 0 \vee c$ and $I_1 := 0 \vee b$. Moreover put $b_1 := 2b$ and $c_1 := c + b_1$.



It is obvious that (A_0, B_0) is equivalent to (A_2, B_2) and (A_3, B_3). Moreover (A_4, B_4) is equivalent to (A_1, B_1). Now we show that (A_0, B_0) is equivalent to (A_1, B_1). This can be seen as follows:
First we have:

$$(A_1 + c) \cup B_0 \;=\; B_1 \quad \text{and} \quad (A_1 + c) \cap B_0 \;=\; A_0 + c.$$

Now from [16] Corollary 2.4 follows that

$$B_0 + A_1 + C \;=\; A_0 + c + B_1 \quad \text{and hence} \quad B_0 + A_1 \;=\; A_0 + B_1,$$

which means that $(A_0, B_0) \sim (A_i, B_i)$ for $i \in \{1, 2, 3, 4\}$. Moreover it follows immediately from [7] Theorem 2.1 that the pair (A_0, B_0) is minimal.
The pair (A_1, B_1) is convex and $(A_1 \cup B_1, A_1 \cap B_1) \;=\; (A_1 \cup B_1, I_1)$. Since the pair $(A_1 \cup B_1, I_1)$ is minimal, it follows from Theorem 1.4 that the pair (A_1, B_1) is minimal convex. Now observe that

$$(A_0 \vee B_0, A_0 + B_0) \sim (A_0 + I_0, A_0 + B_0) \sim (I_0, B_0),$$

and since (I_0, B_0) is a minimal pair it follows by Theorem 3.3 that the pair (A_0, B_0) is C-minimal with $C := I_0$. Moreover we have $A_2 = A_0 + C$ and $B_2 = B_0 + C$. Since $B_0 = (A_0 + C) \cap (B_0 + C)$, it follows from Proposition 3.14 that the class $[A_0, B_0]$ is C-minimal.

Now we will prove that the pair (A_4, B_4) is a convex hull of the pair (A_3, B_3). Therefore suppose that there exists a convex pair $(C, D) \in \mathcal{K}^2(X)$ such that

$$(A_3, B_3) \leq (C, D) \leq (A_4, B_4).$$

Then

$$A_3 \cap B_3 = I_1 \subset C \cap D \subset A_4 \cap B_4 = 2 \cdot I_1.$$

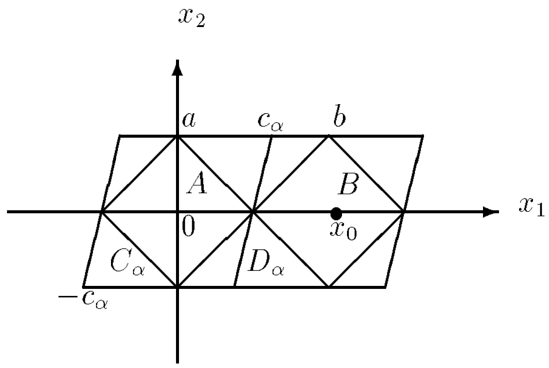
If we assume that $C \cap D \neq A_4 \cap B_4$, then there exists a point $p \in 2 \cdot I_1,\ p \neq b_1$ such that $C \cap D = [(0,0), p]$. But since $C \cup D$ is convex, the intersection $C \cap D$ separates the sets C and D. Now for the two points $a_1 \in C$ and $b_1 \in D$ we have $[a_1, b_1] \cap C \cap D = \emptyset$. This is a contradiction and therefore we have $p = b_1$ and $C \cap D = A_3 \cap B_3$. From [16] Corollary 3.7 it follows, that $C = A_4$ and $D = B_4$.

Moreover we have that $A_3 + x \subseteq A_4$ implies that $x = 0$, and hence the pair $(A_4 + x, B_4 + x)$ is not a convex hull of the pair (A_3, B_3) for $x \neq 0$.

Example 7.2. Let $X := \mathbb{R}^2$ and let us put

$$A := \{(-1,0)\} \vee \{(0,-1)\} \vee \{(1,0)\} \vee \{(1,1)\} \quad \text{and} \quad B := A + x_0 \quad \text{with} \quad x_0 := (2,0).$$

Then the pair $(A, B) \in \mathcal{K}^2(\mathbb{R}^2)$ is not convex.



Given any $c_\alpha := \alpha \cdot a + (1-\alpha) \cdot b,\ 0 \leq \alpha \leq 1$, where $a := (0,1)$ and $b := (2,1)$.
Now we consider

$$C_\alpha := \{-c_\alpha\} \vee \{c_\alpha\} \vee \{(c_\alpha - x_0)\} \vee \{(x_0 - c_\alpha)\}, \quad \text{and} \quad D_\alpha := C_\alpha + x_0.$$

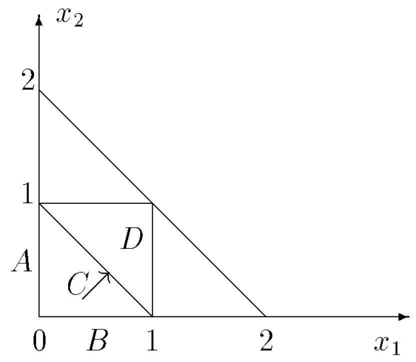
The pair (C_α, D_α) is a convex hull of the pair (A, B). We observe that for every $x \neq 0$ the pair $(C_\alpha + x, D_\alpha + x)$ is not a convex hull of (A, B). Moreover

$$\bigcap_{0 \leq \alpha \leq 1} C_\alpha = A \quad \text{and} \quad \bigcap_{0 \leq \alpha \leq 1} D_\alpha = B.$$

A minimal pair which is equivalent to the pair (A, B) is the pair $(\{0\}, \{x_0\})$ and a convex minimal pair of (A, B) is the pair $([0, x_0], [x_0, 2 \cdot x_0])$.

Example 7.3. Let $X := \mathbb{R}^2$ and let us put

$$A := \{(0,0)\} \vee \{(0,1)\}\ ,\ B := \{(0,0)\} \vee \{(1,0)\}$$
$$C := \{(0,1)\} \vee \{(1,0)\}\ ,\ D := C \vee \{(1,1)\}.$$

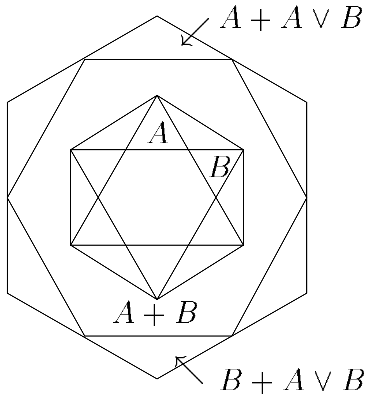


It is easy to see that the pair $(A \vee B, A + B)$ is equivalent to the pair (C, D). Moreover by Theorem 1.4 the pair $(A + C, B + C)$ is convex minimal.

Example 7.4. Let $X := \mathbb{R}^2$ and let us put

$$A := \{(0,1)\} \vee \{(\frac{1}{2}\sqrt{3}, -\frac{1}{2})\} \vee \{(-\frac{1}{2}\sqrt{3}, -\frac{1}{2}, 0)\}, \quad \text{and} \quad B := -A.$$

Then the pair $(A, B) \in \mathcal{K}(\mathbb{R}^2)$ is minimal.



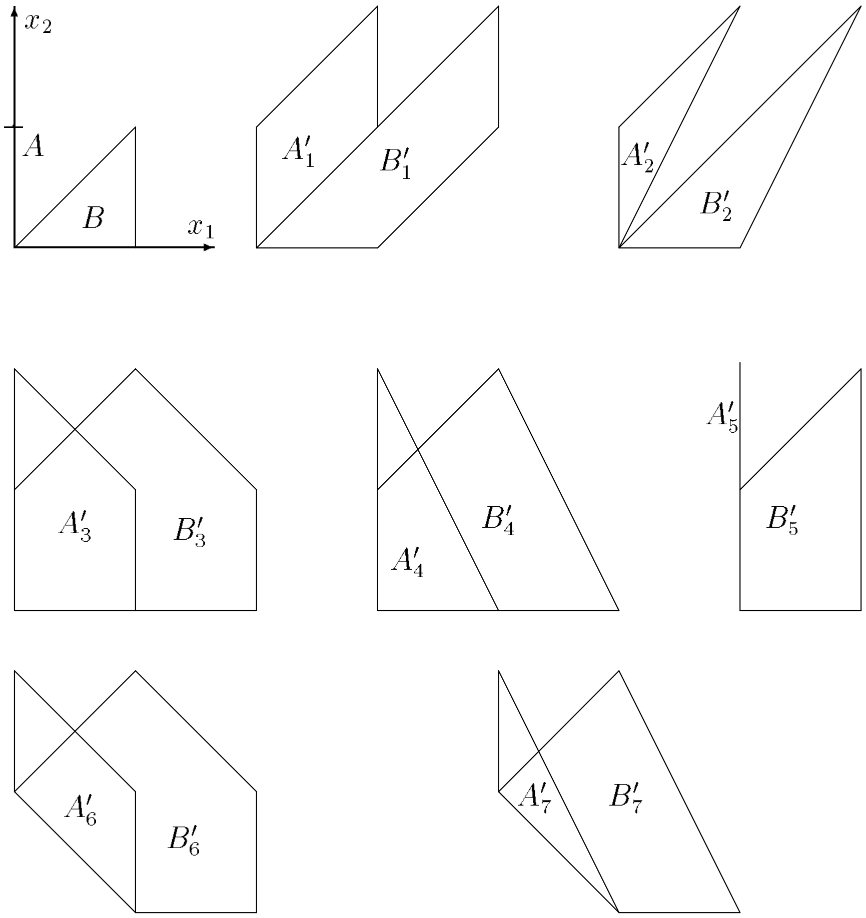
Also $(A \vee B, A + B)$ is minimal. Hence by Theorem 1.4 we see that the pair $(A + A \vee B, B + A \vee B)$ is minimal convex.
Since $(A + A \vee B) \cap (B + A \vee B) = A + B$, it follows from Proposition 3.14 that the class $[A, B]$ is C-minimal with $C = A \vee B$.

Example 7.5. Let $X := \mathbb{R}^2$ and let us put

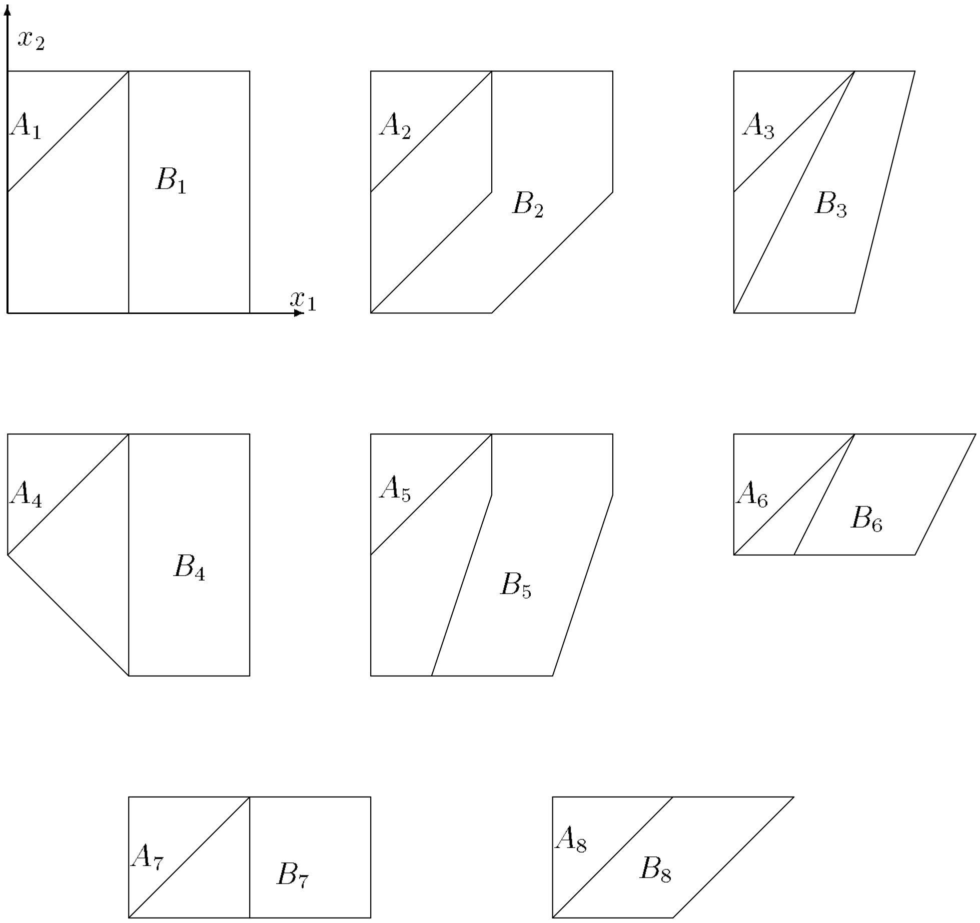
$$A := \{(0,0)\} \vee \{(0,1)\} \quad \text{and} \quad B := \{(0,0)\} \vee \{(1,0)\} \vee \{(1,1)\}.$$

Then the following pairs (A'_j, B'_j) for $j \in \{1, ..., 7\}$ and the convex pairs (A_k, B_k) for $k \in \{1, ..., 8\}$ are equivalent to the pair (A, B) and the pair (A_8, B_8) is minimal convex.

The first assertion can be seen by using subsequently the reduction technique for pairs of compact convex sets (see [8] Theorem 2.6).



Now we construct the convex pairs (A_k, B_k) for $k \in \{1, ..., 8\}$. Let us put $A_1 := A + A \vee B$ and $B_1 := B + A \vee B$. Then the pair (A_1, B_1) is convex and equivalent to the pair (A, B). Moreover the following convex pairs (A_k, B_k) for $k \in \{2, ..., 8\}$ are equivalent to (A_1, B_1) and from Theorem 1.4 follows, that the pair (A_8, B_8) is minimal convex.



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Leere Seite
26