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A Systematization of Convexity Concepts for Sets and Functions

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The paper surveys several convexity concepts, referring to sets and to functions respectively, with the purpose to put them in some kind of order according to the same principle. First it examines the connections between six convexity concepts regarding sets in topological linear spaces and points out the most general concept among these concepts. On the basis of this analysis it is then revealed that twelve convexity concepts concerning functions, that take values in topological linear spaces, can be naturally defined by reduction to the investigated convexities for sets. The most general convexity concept for functions is also found. It is applied to establish an alternative theorem as well as necessary optimality conditions for weak multiobjective optimization problems.

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1. Introduction

The investigations of scalar optimization problems in topological linear spaces use above all ordinary convex sets, but also two types of generalized convex sets, called nearly convex sets and closely convex sets, respectively.

Multiobjective optimization requires additional convexity concepts for sets. Tackling decision problems with multiobjectives, P. L. Yu [23] has given a generalization of the convex sets by introducing sets that are convex with respect to another set. The point of his idea was to consider the translate of the original set by a given set K. In the present paper we show that both the nearly convex sets and the closely convex sets can be generalized in the same way. In consequence, the number of convexity concepts induced by a given set will increase to three.

The main purpose of the present paper is to reveal the connections between the three basic convexity concepts mentioned at the beginning, on the one hand, and the three convexity concepts induced by a given set K, on the other hand. The major results highlighting these connections are proved in Section 2. They show that, if K is a convex set, then the concept of a closely convex set with respect to K is the most general concept among the six convexity concepts considered here for sets. Some interesting characterizations of

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the sets, that are closely convex with respect to a convex cone with nonempty interior, are also established in Section 2. As an application of these sets we derive in Section 3 a characterization of the elements that are weakly minimal with respect to a convex cone. After that we leave the convexity concepts regarding sets and turn in Section 4 to a discussion of convexity concepts regarding functions that take values in topological linear spaces. In order to define a convexity concept for such a function we start either from the graph or from the range of the function. Then we apply the six convexity concepts, examined in Section 2, to each of these two particular sets associated with the function and obtain in this way in each case six convexity concepts for functions. The names that we attribute to these convexities concerning functions are in such a manner chosen that they distinctly emphasize the relationship with one of the six convexity concepts concerning sets. It will be seen that this procedure of introducing convexity concepts for functions by reduction to sets allows to find easily the connections between the twelve classes of functions we have obtained. The connections that we will reveal will show that if K is a convex set, then the functions that are called closely convexlike with respect to K are the most general ones among all the functions considered in this paper. By using closely convexlike functions with respect to a convex cone with nonempty interior we state in the last section of the paper an alternative theorem as well as necessary conditions for the solutions of weak multiobjective optimization problems.
It should be mentioned that most of the convexities concerning functions, that occur in the present paper, are not new. They have already been introduced in earlier papers, mostly separately by other approaches and under other less logical names. But here, for the first time, there is given a simple unitary scheme, based on convexities for sets, in which they all can be naturally arranged and which clearly emphasizes the relations between them. Besides it is shown that the familiar technique of separating convex sets by closed hyperplanes can be directly used to derive some results, that relate to classes of generalized convex sets or generalized convex functions, considered in this paper, and that are applicable in nonconvex multiobjective optimization.

Throughout the paper X and Y denote topological linear spaces over the field $\mathbb{R}$ of real numbers. The zero-elements in X and Y are denoted by o_X and o_Y, respectively. The topological dual of Y is denoted by Y' and its zero-element by o'.
The addition and scalar multiplication on the family of subsets of Y are defined by

$$aM + bN = \{z \in Y \mid \exists (x, y) \in M \times N \ : \ z = ax + by\},$$

where M, N are subsets of Y, and a, b are real numbers.
If M is any set in a topological space, then $\operatorname{int} M$ denotes the interior of M, while $\operatorname{cl} M$ denotes the closure of M.

2. Convexity Concepts for Sets

We start by recalling the definitions of the basic convexity concepts concerning subsets of topological linear spaces.

A subset S of Y is said to be:
(a) *convex* if $(1-a)S + aS \subseteq S$ for all $a \in \,]0,1[$;
(b) *nearly convex* if there is an $a \in \,]0,1[$ such that $(1-a)S + aS \subseteq S$;
(c) *closely convex* if $\operatorname{cl} S$ is a convex set.

Introduced in ancient times for the needs of geometry, in the last century the convex sets have demonstrated their utility also in analysis, functional analysis and optimization theory (see, for instance, [9], [19] and [5]). Unlike convex sets, nearly convex sets have not been frequently used. Properties of the sets of this type may be found in [1], [14] and [20]. From the three convexity concepts defined above, that of a closely convex set is the most recent. It has been introduced in [2].

Within the class of convex sets the convex cones play an important role. We say that a subset S of Y is a *convex cone* if it is not empty, convex and satisfies $aS \subseteq S$ for all $a \in]0, \infty[$.

A simple example of a convex cone is the set $\mathbb{R}^n_+$ consisting of all vectors $(\alpha_1, \ldots, \alpha_n)$ in $\mathbb{R}^n$ with $\alpha_i \geq 0$ for each $i \in \{1, \ldots, n\}$. In particular, the set $\mathbb{R}_+$ of all nonnegative real numbers is a convex cone.
If S is a subset of Y, then

$$S^* = \{y' \in Y' \mid \forall y \in S \ : \ y'(y) \geq 0\}$$

is a convex cone in Y' called the *dual cone* of S.

Next let us fix a subset K of Y. By means of this set and the three basic convexity concepts we can define three other convexity concepts for sets as follows.

A subset S of Y is said to be:

(a′) *convex with respect to* K (or shortly *K-convex*) if $S + K$ is a convex set;

(b′) *nearly convex with respect to* K (or shortly *nearly K-convex*) if $S + K$ is a nearly convex set;

(c′) *closely convex with respect to* K (or shortly *closely K-convex*) if $S + K$ is a closely convex set.

Obviously, a set S in Y is convex (respectively nearly convex, closely convex) if and only if it is convex (respectively nearly convex, closely convex) with respect to $\{o_Y\}$.
Under the assumption that K is a convex cone, the K-convex sets have been introduced in [23]. So far as we know, the other two convexity concepts induced by the set K have not been explored before.

In what follows we discuss the connections between the six convexity concepts mentioned in this section. In order to simplify the formulations, we denote by

$$\mathbf{C}(Y), \ \mathbf{NC}(Y) \ \text{ and } \ \mathbf{CC}(Y)$$

the families of all convex sets, nearly convex sets, and closely convex sets in Y, respectively.
Similarly, we denote by

$$\mathbf{C}_K(Y), \ \mathbf{NC}_K(Y) \ \text{ and } \ \mathbf{CC}_K(Y)$$

the families of all K-convex sets, nearly K-convex sets, and closely K-convex sets in Y, respectively.

Theorem 2.1. *If K is a convex subset of Y and S is a subset of Y, then the implications indicated by the arrows in the diagram below are true:*

$$\begin{array}{ccccc} S \in \mathbf{C}(Y) & \Rightarrow & S \in \mathbf{NC}(Y) & \Rightarrow & S \in \mathbf{CC}(Y) \\ \Downarrow & & \Downarrow & & \Downarrow \\ S \in \mathbf{C}_K(Y) & \Rightarrow & S \in \mathbf{NC}_K(Y) & \Rightarrow & S \in \mathbf{CC}_K(Y). \end{array} \tag{2.1}$$

Proof. We establish this theorem in a sequence of five steps.

Step 1. The implication $S \in \mathbf{C}(Y) \Rightarrow S \in \mathbf{NC}(Y)$ follows immediately from the definitions of the convex and nearly convex sets, while the implication $S \in \mathbf{NC}(Y) \Rightarrow S \in \mathbf{CC}(Y)$ is a consequence of Corollary 2.3 stated in [1]. Consequently, the implications in the first row of diagram (2.1) are true.

Step 2. By applying the implications of the first row of diagram (2.1) to the set $S + K$ instead of S, we obtain the implications of the second row of the above diagram.

Step 3. Suppose that $S \in \mathbf{C}(Y)$. Because K is also convex, it follows that

$$(1-a)(S+K) + a(S+K) \subseteq (1-a)S + aS + (1-a)K + aK \subseteq S + K \tag{2.2}$$

for all $a \in]0,1[$. Hence $S + K$ is a convex set. Thus the implication $S \in \mathbf{C}(Y) \Rightarrow S \in \mathbf{C}_K(Y)$ is true.

Step 4. Suppose that $S \in \mathbf{NC}(Y)$. Then there exists a number $a \in]0,1[$ such that $(1-a)S + aS \subseteq S$. Consequently, this number satisfies (2.2). Hence $S + K$ is a nearly convex set. Thus the implication $S \in \mathbf{NC}(Y) \Rightarrow S \in \mathbf{NC}_K(Y)$ is true.

Step 5. Suppose that $S \in \mathbf{CC}(Y)$. This means that $\mathrm{cl}\, S$ is a convex set. On the other hand, taking into account that K is a convex set, it follows by Step 1 that $\mathrm{cl}\, K$ is a convex set. Since both the sets $\mathrm{cl}\, S$ and $\mathrm{cl}\, K$ are convex, it follows that the set $\mathrm{cl}\, S + \mathrm{cl}\, K$ is also convex. By applying again Step 1, we conclude that $\mathrm{cl}\,(\mathrm{cl}\, S + \mathrm{cl}\, K)$ is a convex set. But, in view of

$$S + K \subseteq \mathrm{cl}\, S + \mathrm{cl}\, K \subseteq \mathrm{cl}\,(S+K),$$

we have

$$\mathrm{cl}\,(S+K) = \mathrm{cl}\,(\mathrm{cl}\, S + \mathrm{cl}\, K).$$

Consequently, $\mathrm{cl}\,(S+K)$ must be a convex set. Thus the implication $S \in \mathbf{CC}(Y) \Rightarrow S \in \mathbf{CC}_K(Y)$ is true. □

Corollary 2.2. *If K and S are subsets of Y and L is a convex subset of Y, then the following implications are true:*

$$S \in \mathbf{C}_K(Y) \Rightarrow S \in \mathbf{C}_{K+L}(Y), \quad S \in \mathbf{NC}_K(Y) \Rightarrow S \in \mathbf{NC}_{K+L}(Y),$$

$$S \in \mathbf{CC}_K(Y) \Rightarrow S \in \mathbf{CC}_{K+L}(Y).$$

Proof. Apply the vertical implications of diagram (2.1). □

By invoking Corollary 2.2 we get the next corollary.

Corollary 2.3. *Let K be a subset of Y containing o_Y, let L be a convex cone in Y such that $K \subseteq L$, and let S be a subset of Y. Then the following implications are true:*

$$S \in \mathbf{C}_K(Y) \Rightarrow S \in \mathbf{C}_L(Y), \quad S \in \mathbf{NC}_K(Y) \Rightarrow S \in \mathbf{NC}_L(Y),$$

$$S \in \mathbf{CC}_K(Y) \Rightarrow S \in \mathbf{CC}_L(Y).$$

Proof. Note that $K + L = L$, and then apply Corollary 2.2. □

Remark 2.4. By reversing the arrows in diagram (2.1) we obtain implications that are not generally true. This claim results from the following simple examples involving only subsets of $\mathbb{R}^2$.

1. The set $M = \{(x,y) \in \mathbb{R}^2 \mid y > 0\} \cup \{(x,0) \in \mathbb{R}^2 \mid x \in \mathbb{Q}\}$ is nearly convex, but it is not convex. Hence, the implication $S \in \mathbf{NC}(Y) \Rightarrow S \in \mathbf{C}(Y)$ is not always true. Consequently, the implication $S \in \mathbf{NC}_K(Y) \Rightarrow S \in \mathbf{C}_K(Y)$ is not always true either.
2. The set $M = \{(x,y) \in \mathbb{R}^2 \mid y > 0\} \cup \{(x,0) \in \mathbb{R}^2 \mid x \in \mathbb{R}\setminus\mathbb{Q}\}$ is closely convex. Next note that, for every $a \in]0,1[$, the irrational numbers

   $$x_1 = \begin{cases} \sqrt{2} & \text{if } a \in \mathbb{Q} \\ 1/(a-1) & \text{if } a \in \mathbb{R}\setminus\mathbb{Q} \end{cases} \quad \text{and} \quad x_2 = (1 - 1/a)x_1$$

   satisfy $(1-a)x_1 + ax_2 = 0$. Thus there is no $a \in]0,1[$ such that

   $$(1-a)(\mathbb{R}\setminus\mathbb{Q}) + a(\mathbb{R}\setminus\mathbb{Q}) \subseteq \mathbb{R}\setminus\mathbb{Q}.$$

   This means that $\mathbb{R}\setminus\mathbb{Q}$ is not nearly convex. Hence the set M is not nearly convex either. Consequently, the implication $S \in \mathbf{CC}(Y) \Rightarrow S \in \mathbf{NC}(Y)$ is not always true. Of course, the implication $S \in \mathbf{CC}_K(Y) \Rightarrow S \in \mathbf{NC}_K(Y)$ is not always true either.
3. The set $M = \{(x,0) \in \mathbb{R}^2 \mid x \geq 0\} \cup \{(0,y) \in \mathbb{R}^2 \mid y \geq 0\}$ is $\mathbb{R}^2_+$-convex, because $M + \mathbb{R}^2_+ = \mathbb{R}^2_+$. Taking into account the implications in the second row of diagram (2.1), it follows that the set M is also nearly $\mathbb{R}^2_+$-convex and closely $\mathbb{R}^2_+$-convex. But, obviously, M is not closely convex. Thus, due to the implications in the first row of diagram (2.1), the set M cannot be neither convex nor nearly convex. In consequence, the implications

   $$S \in \mathbf{C}_K(Y) \Rightarrow S \in \mathbf{C}(Y), \quad S \in \mathbf{NC}_K(Y) \Rightarrow S \in \mathbf{NC}(Y),$$
   $$S \in \mathbf{CC}_K(Y) \Rightarrow S \in \mathbf{CC}(Y)$$

   are not always true.

Provided that K is a convex set, it follows from Theorem 2.1 and Remark 2.4 that the concept of a closely K-convex set is the most general concept among the six convexities concerning sets that we have considered in this section. In the case when K is a convex cone with nonempty interior we can give some nice characterizations of the closely K-convex sets. For establishing these characterizations we need the following lemma.

Lemma 2.5. *Let K be a convex cone in Y with* $\operatorname{int} K \neq \emptyset$, *and let S be a subset of Y. Then the following equalities hold:*

$$\operatorname{int}[\operatorname{cl}(S+K)] = S + \operatorname{int} K; \tag{2.3}$$

$$\operatorname{cl}(S+K) = \operatorname{cl}(S + \operatorname{int} K). \tag{2.4}$$

Proof. Let y be any point in $\operatorname{int}[\operatorname{cl}(S+K)]$. Then there exists a neighbourhood U of o_Y such that

$$y - U \subseteq \operatorname{cl}(S+K). \tag{2.5}$$

Now we choose a point $k_0 \in \operatorname{int} K$. Since U is absorbing, we can find an $a \in]0, \infty[$ such that $ak_0 \in U$. In view of (2.5) it follows that the point $z = y - ak_0$ lies in $\operatorname{cl}(S + K)$. On the other hand, $y - \operatorname{int} K$ is a neighbourhood of z, because this set is open and contains z. Consequently, we have

$$(S + K) \cap (y - \operatorname{int} K) \neq \emptyset.$$

Thus there exist $s \in S$ and $k \in K$ such that $s + k \in y - \operatorname{int} K$. From this we get

$$y \in s + k + \operatorname{int} K \subseteq S + K + \operatorname{int} K \subseteq S + \operatorname{int} K.$$

Since y was arbitrarily chosen in $\operatorname{int}[\operatorname{cl}(S + K)]$, we have proved that

$$\operatorname{int}[\operatorname{cl}(S + K)] \subseteq S + \operatorname{int} K. \tag{2.6}$$

Next note that

$$S + \operatorname{int} K \subseteq S + K \subseteq \operatorname{cl}(S + K).$$

Taking into account that the set $S + \operatorname{int} K$ is open, it follows that

$$S + \operatorname{int} K \subseteq \operatorname{int}[\operatorname{cl}(S + K)]. \tag{2.7}$$

From (2.6) and (2.7) we obtain (2.3).
Now let y be any point in $S + K$. Further let V be any neighbourhood of y. From $y \in S + K$ it follows that $y - s \in K$ for some $s \in S$. Next we choose a point $k_0 \in \operatorname{int} K$. Since V is a neighbourhood of y and

$$\lim_{a \to 1} [s + (1 - a)k_0 + a(y - s)] = y,$$

we can choose a number $a_0 \in [0, 1[$ such that the point $z = s + (1 - a_0)k_0 + a_0(y - s)$ lies in V. On the other hand, a well-known property of the convex sets in topological linear spaces (see, for instance, formula (11.1) in [9, p. 59]) implies that

$$(1 - a_0)k_0 + a_0(y - s) \in \operatorname{int} K.$$

Therefore z lies also in $S + \operatorname{int} K$. Consequently, we have $V \cap (S + \operatorname{int} K) \neq \emptyset$. Since y was arbitrarily chosen in $S + K$ and V was an arbitrary neighbourhood of y, we have shown that

$$S + K \subseteq \operatorname{cl}(S + \operatorname{int} K).$$

Taking into account that the set $\operatorname{cl}(S + \operatorname{int} K)$ is closed, it follows that

$$\operatorname{cl}(S + K) \subseteq \operatorname{cl}(S + \operatorname{int} K). \tag{2.8}$$

Finally note that $S + \operatorname{int} K \subseteq S + K$ implies

$$\operatorname{cl}(S + \operatorname{int} K) \subseteq \operatorname{cl}(S + K). \tag{2.9}$$

From (2.8) and (2.9) we obtain (2.4). □

By applying this lemma, we now can obtain the announced characterizations of the closely K-convex sets.

Theorem 2.6. *Let K be a convex cone in Y with $\operatorname{int} K \neq \emptyset$, let $L = \operatorname{int} K$, and let S be a subset of Y. Then the implications indicated by the arrows in the diagram below are true:*

$$\begin{array}{ccc} S \in \mathbf{CC}_K(Y) & \Rightarrow & S \in \mathbf{C}_L(Y) \\ \Uparrow & & \Downarrow \\ S \in \mathbf{CC}_L(Y) & \Leftarrow & S \in \mathbf{NC}_L(Y). \end{array}$$

Proof. The key observation for the proof is the well-known fact that the interior of a convex set in a topological linear space is a convex set (see, for instance, [19, Lemma 2, p. 13]).
Suppose that $S \in \mathbf{CC}_K(Y)$. Thus the set $\operatorname{cl}(S+K)$ is convex, and hence $\operatorname{int}[\operatorname{cl}(S+K)]$ is also convex. By applying (2.3), we conclude that $S + \operatorname{int} K$ is convex. In other words, we have $S \in \mathbf{C}_L(Y)$. Consequently, the implication $S \in \mathbf{CC}_K(Y) \Rightarrow S \in \mathbf{C}_L(Y)$ is true.
Next we note that the implications

$$S \in \mathbf{C}_L(Y) \Rightarrow S \in \mathbf{NC}_L(Y) \Rightarrow S \in \mathbf{CC}_L(Y)$$

are true in virtue of Theorem 2.1.
Finally, suppose that $S \in \mathbf{CC}_L(Y)$. Thus $\operatorname{cl}(S + \operatorname{int} K)$ is a convex set. Taking into account (2.4), it follows that $\operatorname{cl}(S+K)$ is also convex, i.e. $S \in \mathbf{CC}_K(Y)$. Therefore the implication $S \in \mathbf{CC}_L(Y) \Rightarrow S \in \mathbf{CC}_K(Y)$ is true. □

3. A Characterization of Weakly *K*-Minimal Elements

Since their introduction, the K-convex sets have been successfully applied in multiobjective optimization (see [23], [8], [11]). Here we will illustrate that the more general closely K-convex sets, we dealt with in the preceding section, are also useful in optimization. When K is a convex cone in Y with $\operatorname{int} K \neq \emptyset$, then we intend to characterize those points s_0 of a given closely K-convex subset S of Y that satisfy

$$(s_0 - \operatorname{int} K) \cap S = \emptyset. \tag{3.1}$$

Any element $s_0 \in S$ satisfying (3.1) is said to be *weakly minimal with respect to K* (or shortly *weakly K-minimal*).

Lemma 3.1. *Let K be a convex cone in Y with $\operatorname{int} K \neq \emptyset$, let S be a nonempty subset of Y, and let s_0 be an element of Y. Then the following assertions are true:*

1. *If s_0 satisfies (3.1) and S is closely K-convex, then there exists a $y_0' \in K^* \setminus \{o'\}$ such that*
$$y_0'(s_0) \leq y_0'(s) \quad \textit{for all } s \in S. \tag{3.2}$$
2. *If there exists a $y_0' \in K^* \setminus \{o'\}$ satisfying (3.2), then (3.1) holds.*

Proof. 1. First we observe that

$$(s_0 - \operatorname{int} K) \cap (S+K) = \emptyset. \tag{3.3}$$

Indeed, if we assume the contrary, then there exist elements $s \in S$ and $k \in K$ such that $s + k \in s_0 - \operatorname{int} K$. From this we obtain $s \in s_0 - (k + \operatorname{int} K) \subseteq s_0 - \operatorname{int} K$. Consequently, we have $s \in (s_0 - \operatorname{int} K) \cap S$, which contradicts (3.1). Hence (3.3) must hold.

From (3.3) it follows that

$$S+K \subseteq Y \setminus (s_0 - \operatorname{int} K). \tag{3.4}$$

Since $s_0 - \operatorname{int} K$ is an open set, it follows that $Y \setminus (s_0 - \operatorname{int} K)$ is a closed set. Therefore (3.4) implies

$$\operatorname{cl}(S+K) \subseteq Y \setminus (s_0 - \operatorname{int} K),$$

i.e.

$$(s_0 - \operatorname{int} K) \cap [\operatorname{cl}(S+K)] = \emptyset.$$

Since the sets $s_0 - \operatorname{int} K$ and $\operatorname{cl}(S+K)$ are convex, we can apply a well-known separation theorem (see [9, p. 63]) and conclude that there is a $y_0' \in Y' \setminus \{o'\}$ such that

$$\sup \{y_0'(s_0 - k) \mid k \in K\} \leq \inf \{y_0'(y) \mid y \in \operatorname{cl}(S+K)\}. \tag{3.5}$$

Now let s be any element in S, and let k be any element in K. Since K is a cone, we have $ak \in K$ for all $a \in \,]0,\infty[$. Consequently, (3.5) implies

$$y_0'(s_0 - k) \leq y_0'(s + \frac{1}{a}k) \quad \text{for all } a \in \,]0,\infty[,$$

on the one hand, and

$$y_0'(s_0 - ak) \leq y_0'(s + ak) \quad \text{for all } a \in \,]0,\infty[,$$

on the other hand. These inequalities can be rewritten as follows:

$$y_0'(k) \geq a[y_0'(s_0 - k) - y_0'(s)] \quad \text{for all } a \in \,]0,\infty[,$$

$$y_0'(s) \geq y_0'(s_0) - 2a y_0'(k) \quad \text{for all } a \in \,]0,\infty[.$$

Letting $a \to 0$, we then obtain $y_0'(k) \geq 0$ and $y_0'(s) \geq y_0'(s_0)$. Hence y_0' lies in K^* and satisfies (3.2).

2. Let $y_0' \in K^* \setminus \{o'\}$ satisfy (3.2). Suppose that there is an element $s \in S$ such that $s - s_0 \in -\operatorname{int} K$. Then we can choose a neighbourhood U of o_Y which satisfies

$$s - s_0 + U \subseteq -K. \tag{3.6}$$

Next we select an element $y \in Y$ for which $y_0'(y) > 0$. Since $y_0' \neq o'$, we can find such an element. Then take into consideration that U is absorbing and choose a number $a \in \,]0,\infty[$ such that $ay \in U$. From (3.6) it follows that $s - s_0 + ay$ lies in $-K$, and therefore we have $y_0'(s - s_0 + ay) \leq 0$. This inequality implies

$$y_0'(s) \leq y_0'(s_0) - a y_0'(y) < y_0'(s_0),$$

which contradicts (3.2). Thus (3.1) must hold. □

By applying Lemma 3.1 we obtain the following characterization theorem for the weakly K-minimal elements.

Theorem 3.2. *Let K be a convex cone in Y with $\operatorname{int} K \neq \emptyset$, let S be a nonempty subset of Y which is closely K-convex, and let s_0 be an element of S. Then s_0 is weakly K-minimal if and only if there exists a $y_0' \in K^* \setminus \{o'\}$ such that*

$$y_0'(s_0) = \min \{y_0'(s) \mid s \in S\}.$$

This theorem generalizes Theorem 1 given in [21, p. 104].

4. Convexity Concepts for Functions

Let D be a nonempty subset of X. The convexity concepts regarding sets, studied in Section 2, allow us to introduce convexity concepts for functions defined on D and taking values in Y. The procedure is quite simple. It is based on assigning a determined subset $S(f)$ of a topological linear space to each function $f : D \to Y$. If the set $S(f)$ has one of the six convexity properties defined in Section 2, then we say that the function f has that one same convexity property. Consequently, the set-valued mapping $f \longmapsto S(f)$ will yield six convexity concepts regarding functions. We illustrate this technique, which can be called defining by reduction to sets, in two cases, namely when the set $S(f)$ assigned to f is the graph

$$\operatorname{gr} f = \{(x, f(x)) \mid x \in D\}$$

and the range

$$\operatorname{rng} f = \{f(x) \mid x \in D\}$$

of f, respectively.

Let K be a subset of Y. Set $K_0 = \{o_X\} \times K$.
Depending on the convexity properties of $\operatorname{gr} f$ in the topological linear space $X \times Y$, we say that $f : D \to Y$ is:

(a_1) *convex* if $\operatorname{gr} f \in \mathbf{C}(X \times Y)$;

(b_1) *nearly convex* if $\operatorname{gr} f \in \mathbf{NC}(X \times Y)$;

(c_1) *closely convex* if $\operatorname{gr} f \in \mathbf{CC}(X \times Y)$;

(a_1') *convex with respect to K* (or shortly *K-convex*) if $\operatorname{gr} f \in \mathbf{C}_{K_0}(X \times Y)$;

(b_1') *nearly convex with respect to K* (or shortly *nearly K-convex*) if $\operatorname{gr} f \in \mathbf{NC}_{K_0}(X \times Y)$;

(c_1') *closely convex with respect to K* (or shortly *closely K-convex*) if $\operatorname{gr} f \in \mathbf{CC}_{K_0}(X \times Y)$.

Similarly, depending on the convexity properties of $\operatorname{rng} f$ in the topological linear space Y, we say that $f : D \to Y$ is:

(a_2) *convexlike* if $\operatorname{rng} f \in \mathbf{C}(Y)$;

(b_2) *nearly convexlike* if $\operatorname{rng} f \in \mathbf{NC}(Y)$;

(c_2) *closely convexlike* if $\operatorname{rng} f \in \mathbf{CC}(Y)$;

(a_2') *convexlike with respect to K* (or shortly *K-convexlike*) if $\operatorname{rng} f \in \mathbf{C}_K(Y)$;

(b_2') *nearly convexlike with respect to K* (or shortly *nearly K-convexlike*) if $\operatorname{rng} f \in \mathbf{NC}_K(Y)$;

(c_2') *closely convexlike with respect to K* (or shortly *closely K-convexlike*) if $\operatorname{rng} f \in \mathbf{CC}_K(Y)$.

In virtue of the definitions (a$'_1$), (b$'_1$) and (c$'_1$) we have to check whether the set $\operatorname{gr} f + K_0$ is convex, nearly convex or closely convex, so that we can decide whether f is K-convex, nearly K-convex or closely K-convex. This set $\operatorname{gr} f + K_0$ is called the *epigraph of f with respect to K* and denoted by $\operatorname{epi}_K f$. Obviously, we have

$$\operatorname{epi}_K f = \{(x,y) \in X \times Y \mid x \in D, y \in f(x) + K\}.$$

Similarly, in virtue of the definitions (a$'_2$), (b$'_2$) and (c$'_2$) we have to examine whether the set $\operatorname{rng} f + K$ is convex, nearly convex or closely convex, so that we can decide whether f is K-convexlike, nearly K-convexlike or closely K-convexlike. This set $\operatorname{rng} f + K$ is called the *epirange of f with respect to K.*

By analogy with the notations introduced in Section 2 for the families of all subsets of Y having the same convexity property, we denote by

$$\mathbf{C}(D,Y),\ \mathbf{NC}(D,Y),\ \mathbf{CC}(D,Y),\ \mathbf{C}_K(D,Y),\ \mathbf{NC}_K(D,Y),\ \mathbf{CC}_K(D,Y)$$

the families of all functions from D into Y that are convex, nearly convex, closely convex, K-convex, nearly K-convex and closely K-convex, respectively. Similarly, we denote by

$$\mathbf{CL}(D,Y),\ \mathbf{NCL}(D,Y),\ \mathbf{CCL}(D,Y),\ \mathbf{CL}_K(D,Y),\ \mathbf{NCL}_K(D,Y),\ \mathbf{CCL}_K(D,Y)$$

the families of all functions from from D into Y that are convexlike, nearly convexlike, closely convexlike, K-convexlike, nearly K-convexlike and closely K-convexlike, respectively.

According to the definitions of the above introduced twelve convexity concepts regarding functions, Theorem 2.1 implies the following theorem proclaiming some connections between these concepts.

Theorem 4.1. *If K is a convex subset of Y and f is a function from D into Y, then the implications indicated by the arrows in the diagrams below are true:*

$$\begin{array}{ccccc} f \in \mathbf{C}(D,Y) & \Rightarrow & f \in \mathbf{NC}(D,Y) & \Rightarrow & f \in \mathbf{CC}(D,Y) \\ \Downarrow & & \Downarrow & & \Downarrow \\ f \in \mathbf{C}_K(D,Y) & \Rightarrow & f \in \mathbf{NC}_K(D,Y) & \Rightarrow & f \in \mathbf{CC}_K(D,Y); \end{array} \tag{4.1}$$

$$\begin{array}{ccccc} f \in \mathbf{CL}_K(D,Y) & \Rightarrow & f \in \mathbf{NCL}_K(D,Y) & \Rightarrow & f \in \mathbf{CCL}_K(D,Y) \\ \Uparrow & & \Uparrow & & \Uparrow \\ f \in \mathbf{CL}(D,Y) & \Rightarrow & f \in \mathbf{NCL}(D,Y) & \Rightarrow & f \in \mathbf{CCL}(D,Y). \end{array} \tag{4.2}$$

The relationship between the diagrams (4.1) and (4.2) is given by the next theorem.

Theorem 4.2. *Let K be a subset of Y, and let f be a function from D into Y. Then the following implications are true:*

$$f \in \mathbf{C}_K(D,Y) \Rightarrow f \in \mathbf{CL}_K(D,Y), \quad f \in \mathbf{NC}_K(D,Y) \Rightarrow f \in \mathbf{NCL}_K(D,Y),$$
$$f \in \mathbf{CC}_K(D,Y) \Rightarrow f \in \mathbf{CCL}_K(D,Y).$$

Proof. Suppose that $f \in \mathbf{C}_K(D,Y)$. We claim that under this assumption the set $\operatorname{rng} f + K$ is convex. To prove this, let y_1 and y_2 be elements of $\operatorname{rng} f + K$, and let a be in

]0,1[. We can associate with y_1 and y_2 ordered pairs $(x_1, k_1), (x_2, k_2) \in D \times K$ such that $y_i = f(x_i) + k_i$ for $i \in \{1, 2\}$. Then we have $(x_i, y_i) \in \operatorname{epi}_K f$ for $i \in \{1, 2\}$. Taking into account that the set $\operatorname{epi}_K f$ is convex, it follows that

$$((1-a)x_1 + ax_2, (1-a)y_1 + ay_2) = (1-a)(x_1, y_1) + a(x_2, y_2) \in \operatorname{epi}_K f.$$

Thus

$$(1-a)x_1 + ax_2 \in D \quad \text{and} \quad (1-a)y_1 + ay_2 \in f((1-a)x_1 + ax_2) + K.$$

These relations show that $(1-a)y_1 + ay_2 \in \operatorname{rng} f + K$. Therefore $\operatorname{rng} f + K$ is convex, as claimed. In other words, the function f is K-convexlike. Consequently, the implication $f \in \mathbf{C}_K(D, Y) \Rightarrow f \in \mathbf{CL}_K(D, Y)$ is true.
Now suppose that $f \in \mathbf{NC}_K(D, Y)$. This means that the set $\operatorname{epi}_K f$ is nearly convex. Hence there exists a number $a \in]0, 1[$ such that

$$(1-a) \operatorname{epi}_K f + a \operatorname{epi}_K f \subseteq \operatorname{epi}_K f.$$

By reasoning as in the proof of the implication $f \in \mathbf{C}_K(D, Y) \Rightarrow f \in \mathbf{CL}_K(D, Y)$, we immediately see that this number a satisfies

$$(1-a)(\operatorname{rng} f + K) + a(\operatorname{rng} f + K) \subseteq \operatorname{rng} f + K.$$

Therefore $\operatorname{rng} f + K$ is nearly convex, and whence f is nearly K-convexlike. Consequently, the implication $f \in \mathbf{NC}_K(D, Y) \Rightarrow f \in \mathbf{NCL}_K(D, Y)$ is also true.
Finally, suppose that $f \in \mathbf{CC}_K(D, Y)$. We claim that in this case the set $\operatorname{cl}(\operatorname{rng} f + K)$ is convex. To prove this, let y_1 and y_2 be elements of $\operatorname{cl}(\operatorname{rng} f + K)$, and let a be in]0,1[. Further, fix any neighbourhood U of the element $y = (1-a)y_1 + ay_2$. Since the mapping

$$(t, u) \in Y \times Y \longmapsto (1-a)t + au \in Y$$

is continuous at (y_1, y_2), we can choose an open neighbourhood V_1 of y_1 and an open neighbourhood V_2 of y_2 such that

$$(1-a)V_1 + aV_2 \subseteq U. \tag{4.3}$$

Taking into account that $y_1, y_2 \in \operatorname{cl}(\operatorname{rng} f + K)$, it results that

$$(\operatorname{rng} f + K) \cap V_i \neq \emptyset \quad \text{for } i \in \{1, 2\}.$$

Thus there exist $x_1, x_2 \in D$ and $k_1, k_2 \in K$ such that

$$z_1 = f(x_1) + k_1 \in V_1 \quad \text{and} \quad z_2 = f(x_2) + k_2 \in V_2.$$

Now, note that (x_1, z_1) and (x_2, z_2) lie in $\operatorname{cl}(\operatorname{epi}_K f)$. Then it follows that

$$((1-a)x_1 + ax_2, (1-a)z_1 + az_2) \in \operatorname{cl}(\operatorname{epi}_K f),$$

because the set $\operatorname{cl}(\operatorname{epi}_K f)$ is convex. Since $W = X \times [(1-a)V_1 + aV_2]$ is a neighbourhood of the element $((1-a)x_1 + ax_2, (1-a)z_1 + az_2)$, we must have $(\operatorname{epi}_K f) \cap W \neq \emptyset$. Hence there exist an $x_0 \in D$ and a $k_0 \in K$ such that

$$f(x_0) + k_0 \in (1-a)V_1 + aV_2. \tag{4.4}$$

From (4.3) and (4.4) it results that $(\operatorname{rng} f + K) \cap U \neq \emptyset$. Since U was an arbitrary neighbourhood of y, we have $y \in \operatorname{cl}(\operatorname{rng} f + K)$. This shows that $\operatorname{cl}(\operatorname{rng} f + K)$ is a convex set. Hence the function f is closely K-convexlike. Consequently, the last implication of our theorem, namely $f \in \mathbf{CC}_K(D, Y) \Rightarrow f \in \mathbf{CCL}_K(D, Y)$, is also true. □

Remark 4.3. By reversing the arrows in the diagrams (4.1) and (4.2) we obtain implications that are not necessarily true. This assertion results from the following examples.

1. Let D be a nonempty subset of X. Define $\varphi : D \to X$ by $\varphi(x) = x$. Then it is immediately seen that the following equivalences are true:

$$D \in \mathbf{C}(X) \;\Leftrightarrow\; \varphi \in \mathbf{C}(D, X);$$
$$D \in \mathbf{NC}(X) \;\Leftrightarrow\; \varphi \in \mathbf{NC}(D, X);$$
$$D \in \mathbf{CC}(X) \;\Leftrightarrow\; \varphi \in \mathbf{CC}(D, X).$$

   These equivalences together with Remark 2.4 show that the implications that result from reversing the arrows in the first row of diagram (4.1) are not always true. Consequently, the implications that result from reversing the arrows in the second row of diagram (4.1) are not always true either.

2. Let $\phi : \mathbb{R} \to \mathbb{R}$ be the function defined by $\phi(x) = x^2$. This function is $\mathbb{R}_+$-convex. Taking into account the implications in the second row of diagram (4.1), it follows that ϕ is also nearly $\mathbb{R}_+$-convex and closely $\mathbb{R}_+$-convex. But ϕ is not closely convex. Thus, due to the implications in the first row of diagram (4.1), the function ϕ cannot be neither convex nor nearly convex. Consequently, the implications

$$f \in \mathbf{C}_K(D, Y) \Rightarrow f \in \mathbf{C}(D, Y), \quad f \in \mathbf{NC}_K(D, Y) \Rightarrow f \in \mathbf{NC}(D, Y),$$
$$f \in \mathbf{CC}_K(D, Y) \Rightarrow f \in \mathbf{CC}(D, Y)$$

   are not generally true.

3. Consider again the function φ, defined in example 1. For this function we have $\operatorname{rng} \varphi = D$. Therefore the following equivalences are obvious:

$$D \in \mathbf{C}(X) \;\Leftrightarrow\; \varphi \in \mathbf{CL}(D, X);$$
$$D \in \mathbf{NC}(X) \;\Leftrightarrow\; \varphi \in \mathbf{NCL}(D, X);$$
$$D \in \mathbf{CC}(X) \;\Leftrightarrow\; \varphi \in \mathbf{CCL}(D, X).$$

   Moreover, if K is a subset of X, then we also have

$$D \in \mathbf{C}_K(X) \;\Leftrightarrow\; \varphi \in \mathbf{CL}_K(D, X);$$
$$D \in \mathbf{NC}_K(X) \;\Leftrightarrow\; \varphi \in \mathbf{NCL}_K(D, X);$$
$$D \in \mathbf{CC}_K(X) \;\Leftrightarrow\; \varphi \in \mathbf{CCL}_K(D, X).$$

   Taking into account all these equivalences and Remark 2.4, it follows that in diagram (4.2) the reversed implications are not always true.

Remark 4.4. The implications that are obtained by reversing the arrows in the implications stated in Theorem 4.2 do not always hold either. To see this, consider the function

$\phi : \mathbb{R} \to \mathbb{R}$, defined by $\phi(x) = x^3$. This function is $\mathbb{R}_+$-convexlike, because $\operatorname{rng}\phi + \mathbb{R}_+ = \mathbb{R}$. Taking into account the implications in the first row of diagram (4.2), it follows that the function ϕ is also nearly $\mathbb{R}_+$-convexlike and closely $\mathbb{R}_+$-convexlike. But ϕ is not closely $\mathbb{R}_+$-convex, because the closed set

$$\operatorname{gr}\phi + \{0\} \times \mathbb{R}_+ = \{(x, y) \in \mathbb{R}^2 \mid x^3 \le y\}$$

is not convex. Thus, due to the implications in the second row of diagram (4.1), the function ϕ cannot be neither $\mathbb{R}_+$-convex nor nearly $\mathbb{R}_+$-convex. Consequently, the implications

$$f \in \mathbf{CL}_K(D, Y) \Rightarrow f \in \mathbf{C}_K(D, Y), \quad f \in \mathbf{NCL}_K(D, Y) \Rightarrow f \in \mathbf{NC}_K(D, Y),$$

$$f \in \mathbf{CCL}_K(D, Y) \Rightarrow f \in \mathbf{CC}_K(D, Y)$$

are not generally true.

When K is a convex subset of Y, then the Theorems 4.1 and 4.2 together with the Remarks 4.3 and 4.4 show that the concept of a closely K-convexlike function is the most general concept among the twelve convexities concerning functions studied in this section. From Theorem 2.6 we immediately obtain the following characterizations of the closely K-convexlike functions.

Theorem 4.5. *Let K be a convex cone in Y with $\operatorname{int} K \neq \emptyset$, let $L = \operatorname{int} K$, and let f be a function from D into Y. Then the implications indicated by the arrows in the diagram below are true:*

$$\begin{array}{ccc} f \in \mathbf{CCL}_K(D, Y) & \Rightarrow & f \in \mathbf{CL}_L(D, Y) \\ \Uparrow & & \Downarrow \\ f \in \mathbf{CCL}_L(D, Y) & \Leftarrow & f \in \mathbf{NCL}_L(D, Y). \end{array}$$

Next we deal with a weakened convexity for functions that has been discussed in [20]. It is defined as follows.

If K is a subset of Y, then a function $f : D \to Y$ is called *K-subconvexlike* if there exists a $k_0 \in \operatorname{int} K$ such that for any $a \in]0, 1[$ and any $b \in]0, \infty[$ one has

$$(1 - a) \operatorname{rng} f + a \operatorname{rng} f + bk_0 \subseteq \operatorname{rng} f + K. \tag{4.5}$$

In the special case when $Y = \mathbb{R}^n$, this concept has been introduced in [13]. It has been applied not only in [13], but also in [22]. Its relationship with the convexity concepts investigated in the present paper is given by our next result.

Corollary 4.6. *Let K be a convex cone in Y with $\operatorname{int} K \neq \emptyset$, and let f be a function from D into Y. Then the following assertions are equivalent:*

1. *f is K-subconvexlike.*
2. *There exist a $k_0 \in \operatorname{int} K$ and an $a \in]0, 1[$ such that (4.5) holds for all $b \in]0, \infty[$.*
3. *f is closely K-convexlike.*

Proof. Obviously, the implication 1. $\Rightarrow$ 2. is true. Next we assume that 2. holds. For short we set $L = \operatorname{int} K$. Let y_1 and y_2 be in $\operatorname{rng} f + L$. Then there exist elements $k_1, k_2 \in L$ such that $y_i \in \operatorname{rng} f + k_i$ for $i \in \{1, 2\}$. The convexity of L implies that $(1-a)k_1 + ak_2 \in L$. But, in addition, the set L is open. Hence there exists a neighbourhood U of o_Y such that $(1-a)k_1+ak_2-U \subseteq L$. Since U is absorbing, we can find a number $b \in]0,\infty[$ such that $bk_0 \in U$. Therefore we have

$$(1-a)k_1+ak_2-bk_0 \in L. \tag{4.6}$$

From (4.5) and (4.6) we obtain

$$\begin{aligned}(1-a)y_1+ay_2 &\in (1-a)\ \mathrm{rng}\, f + a\ \mathrm{rng}\, f + (1-a)k_1+ak_2\\ &= [(1-a)\ \mathrm{rng}\, f + a\ \mathrm{rng}\, f + bk_0] + [(1-a)k_1+ak_2-bk_0]\\ &\subseteq \mathrm{rng}\, f + K + L\\ &\subseteq \mathrm{rng}\, f + L.\end{aligned}$$

Since y_1 and y_2 were arbitrarily chosen in $\mathrm{rng}\, f + L$, we have shown that $\mathrm{rng}\, f + L$ is a nearly convex set. In other words, f is nearly L-convexlike. By Theorem 4.5 it follows that f is closely K-convexlike. In conclusion, the implication 2. $\Rightarrow$ 3. is true.
Finally, if the function f is closely K-convexlike, then by Theorem 4.5 it is also L-convexlike, where as before $L = \mathrm{int}\, K$. Hence $\mathrm{rng}\, f + L$ is a convex set. We choose any $k_0 \in L$. Then we have

$$\begin{aligned}(1-a)\ \mathrm{rng}\, f + a\ \mathrm{rng}\, f + bk_0 &= (1-a)(\mathrm{rng}\, f + bk_0) + a(\mathrm{rng}\, f + bk_0)\\ &\subseteq (1-a)(\mathrm{rng}\, f + L) + a(\mathrm{rng}\, f + L)\\ &\subseteq \mathrm{rng}\, f + L\\ &\subseteq \mathrm{rng}\, f + K\end{aligned}$$

for all $a \in]0,1[$ and all $b \in]0,\infty[$. Thus f is K-subconvexlike. In conclusion, the implication 3. $\Rightarrow$ 1. is also true. □

From Corollary 4.6, which completes the results of Section 3 of the paper [20], we conclude that the class $\mathbf{CCL}_K(D,Y)$ of all closely K-convexlike functions $f : D \to Y$ coincides with the class of all K-subconvexlike functions $f : D \to Y$ when K is a convex cone in Y with $\mathrm{int}\, K \neq \emptyset$.
Finally, it should be remarked that some of the convexities regarding functions, defined here by reduction to sets, can also be defined otherwise. This is shown by the following elementary theorem.

Theorem 4.7. *Let K be a convex cone in Y containing o_Y, and let f be a function from D into Y. Then the following assertions are true:*

1. $f \in \mathbf{C}_K(D,Y)$ *if and only if for all $a \in]0,1[$ and all $x_1, x_2 \in D$ the conditions*

$$(1-a)x_1+ax_2 \in D, \tag{4.7}$$

$$(1-a)f(x_1)+af(x_2) \in f((1-a)x_1+ax_2)+K \tag{4.8}$$

are satisfied.

2. $f \in \mathbf{NC}_K(D,Y)$ *if and only if there exists a number $a \in]0,1[$ such that for all $x_1, x_2 \in D$ the conditions* (4.7) *and* (4.8) *are satisfied.*
3. $f \in \mathbf{CL}_K(D,Y)$ *if and only if for all $a \in]0,1[$ and all $x_1, x_2 \in D$ the condition*

$$(1-a)f(x_1)+af(x_2) \in \mathrm{rng}\, f + K \tag{4.9}$$

is satisfied.

4. $f \in \mathbf{NCL}_K(D, Y)$ *if and only if there exists a number* $a \in \,]0,1[$ *such that for all* $x_1, x_2 \in D$ *the condition* (4.9) *is satisfied.*

Proof. 1. *Necessity.* Let x_1 and x_2 be points in D. Since $o_Y \in K$, we have

$$(x_i, f(x_i)) \in \operatorname{gr} f \subseteq \operatorname{epi}_K f \quad \text{for } i \in \{1,2\}.$$

Taking into account that the set $\operatorname{epi}_K f$ is convex, it follows that

$$((1-a)x_1 + ax_2, (1-a)f(x_1) + af(x_2)) = (1-a)(x_1, f(x_1)) + a(x_2, f(x_2)) \in \operatorname{epi}_K f$$

for all $a \in \,]0,1[$. Thus (4.7) and (4.8) hold for all $a \in \,]0,1[$.
Sufficiency. Let (x_1, y_1) and (x_2, y_2) be elements of $\operatorname{epi}_K f$. Then (4.7) holds for all $a \in \,]0,1[$. On the other hand, taking into account that $y_i \in f(x_i) + K$ for $i \in \{1,2\}$ and that (4.8) holds for all $a \in \,]0,1[$, we obtain

$$\begin{aligned}(1-a)y_1 + ay_2 &\in (1-a)f(x_1) + af(x_2) + (1-a)K + aK \\ &\subseteq f((1-a)x_1 + ax_2) + K\end{aligned}$$

for all $a \in \,]0,1[$. Thus

$$(1-a)(x_1, y_1) + a(x_2, y_2) = ((1-a)x_1 + ax_2, (1-a)y_1 + ay_2) \in \operatorname{epi}_K f$$

for all $a \in \,]0,1[$. Consequently, the set $\operatorname{epi}_K f$ is convex. This means that f is K-convex.
2. By reasoning as in the proof of assertion 1., but for a fixed number $a \in \,]0,1[$, we immediately see that assertion 2. is true.
3. *Necessity.* Let x_1 and x_2 be points in D. Since $o_Y \in K$, we have

$$f(x_i) \in \operatorname{rng} f \subseteq \operatorname{rng} f + K \quad \text{for } i \in \{1,2\}.$$

Due to the convexity of the set $\operatorname{rng} f + K$ it follows that (4.9) holds for all $a \in \,]0,1[$.
Sufficiency. Let y_1 and y_2 be points in $\operatorname{rng} f + K$. Then there exist elements $x_1, x_2 \in D$ such that $y_i \in f(x_i) + K$ for $i \in \{1,2\}$. In view of (4.9) it follows that

$$\begin{aligned}(1-a)y_1 + ay_2 &\in (1-a)f(x_1) + af(x_2) + (1-a)K + aK \\ &\subseteq \operatorname{rng} f + K\end{aligned}$$

for all $a \in \,]0,1[$. Consequently, the set $\operatorname{rng} f + K$ is convex. This means that the function f is K-convexlike.
4. By reasoning as in the proof of assertion 3., but again for a fixed number $a \in \,]0,1[$, we see that assertion 4. is true. □

Corollary 4.8. *Let f be a function from D into Y. Then the following assertions are true:*

1. $f \in \mathbf{C}(D, Y)$ *if and only if for all* $a \in \,]0,1[$ *and all* $x_1, x_2 \in D$ *the conditions* (4.7) *and*

$$(1-a)f(x_1) + af(x_2) = f((1-a)x_1 + ax_2) \tag{4.10}$$

are satisfied.

2. $f \in \mathbf{NC}(D,Y)$ *if and only if there exists a number* $a \in]0,1[$ *such that for all* $x_1, x_2 \in D$ *the conditions* (4.7) *and* (4.10) *are satisfied.*

Remark 4.9. The assertions 3. and 4. of Theorem 4.7 are mentioned without proofs in [20, Lemma 3.2].

Remark 4.10. The definitions (a$_2$), (b$_2$), (c$_2$), (a$'_2$), (b$'_2$) and (c$'_2$) as well as the definition of the K-subconvexlike functions do not require that D is a subset of the topological linear space X. All these definitions and the results referring to the functions corresponding to these definitions remain valid even if D is any nonempty set.

5. Characterizations of Weakly K-Minimizers

The K-convex functions as well as their generalizations that we have called here K-convexlike and nearly K-convexlike functions, respectively, have been employed for deriving minimax theorems, theorems of the alternative, optimality conditions and duality results (e.g. see [4], [16], [3], [7], [6], [24], [12], [17], [14], [18], [15], [10]). Some of the results involving such functions are also valid for the closely K-convexlike functions. To see this it suffices to transpose the results, stated for sets in Section 3, to functions defined on a nonempty set D and taking values in Y.
Indeed, Lemma 3.1 yields the following theorem of the alternative of Gordan type.

Theorem 5.1. *Let K be a convex cone in Y with* $\operatorname{int} K \neq \emptyset$*, let D be a nonempty set, and let $f : D \to Y$ be a closely K-convexlike function. Then exactly one of the following assertions is true:*

1. *There exists an $x \in D$ such that $-f(x) \in \operatorname{int} K$.*
2. *There exists a $y' \in K^* \setminus \{o'\}$ such that $y'(f(x)) \geq 0$ for all $x \in D$.*

Proof. Take $S = \operatorname{rng} f$ and $s_0 = o_Y$. Then apply Lemma 3.1. □

This theorem has been stated in [15]. According to Corollary 4.6 it generalizes Theorem 2.1 proved in [13]. In virtue of Theorem 4.1 it is also a generalization of several other results of the same type (Theorem 2.5.1 in [3], Lemma 2.1 in [7], Lemma 3 in [25], Theorem 4.1 in [10]).

When K is a convex cone in Y with $\operatorname{int} K \neq \emptyset$ and f is a function from D into Y, then we say that an element $x_0 \in D$ is a *weak K-minimizer of f over a subset E* of D if $x_0 \in E$ and if $f(x_0)$ is a weakly K-minimal element of the set $f(E)$. By taking into account this definition, Theorem 3.2 yields the next theorem which generalizes several results given in the literature (see, for instance, [21, Theorem 2, p. 105], [17, Theorem 2.10, p. 91] and [22, Theorem 3.1]).

Theorem 5.2. *Let K be a convex cone in Y with* $\operatorname{int} K \neq \emptyset$*, let D be a nonempty set, let $f : D \to Y$ be a closely K-convexlike function, and let x_0 be an element of D. Then x_0 is a weak K-minimizer of f over D if and only if there exists a $y'_0 \in K^* \setminus \{o'\}$ such that*

$$y'_0(f(x_0)) = \min \{y'_0(f(x)) \mid x \in D\}.$$

Lemma 3.1 can also be used to derive necessary conditions for weak solutions of constrained multiobjective optimization problems. To see this, let Y_1 and Y_2 be real topological linear spaces. Let Y_1' and Y_2' be their topological duals, respectively. The zero-elements of the spaces Y_1, Y_2, Y_1' and Y_2' are denoted by o_1, o_2, o_1' and o_2', respectively.

Theorem 5.3. *Let $K_1 \subseteq Y_1$ and $K_2 \subseteq Y_2$ be convex cones with*

$$\operatorname{int} K_1 \neq \emptyset \quad \textit{and} \quad \operatorname{int} K_2 \neq \emptyset,$$

let D be a nonempty set, let $f = (f_1, f_2) : D \to Y_1 \times Y_2$ be a closely $K_1 \times K_2$-convexlike function, and let $x_0 \in D$ be a weak K_1-minimizer of f_1 over the set

$$E = \{x \in D \mid -f_2(x) \in K_2\}. \tag{5.1}$$

Then the following assertions are true:

1. *There exists a $(y_1', y_2') \in K_1^* \times K_2^* \setminus \{(o_1', o_2')\}$ such that*

$$y_1'(f_1(x_0)) = \min\ \{y_1'(f_1(x)) + y_2'(f_2(x)) \mid x \in D\}. \tag{5.2}$$

2. *If $(\operatorname{rng} f_2) \cap (-\operatorname{int} K_2) \neq \emptyset$, then there exists a $(y_1', y_2') \in K_1^* \times K_2^*$ with $y_1' \neq o_1'$ such that (5.2) holds.*

Proof. 1. If we set $y_0 = (f_1(x_0), o_2)$ and $K = K_1 \times K_2$, then we have

$$(y_0 - \operatorname{int} K) \cap \operatorname{rng} f = \emptyset.$$

By applying assertion 1. of Lemma 3.1 in the space $Y_1 \times Y_2$, we conclude that there is a $y_0' \in K^* \setminus \{o'\}$ such that

$$y_0'(y_0) \leq y_0'(y) \ \text{ for all } y \in \operatorname{rng} f. \tag{5.3}$$

Now we define the functions $y_1' : Y_1 \to \mathbb{R}$ and $y_2' : Y_2 \to \mathbb{R}$ by

$$y_1'(y_1) = y_0'(y_1, o_2) \ \text{ and } \ y_2'(y_2) = y_0'(o_1, y_2),$$

respectively. Then we have

$$y_0'(y) = y_1'(y_1) + y_2'(y_2) \ \text{ for all } y = (y_1, y_2) \in Y_1 \times Y_2. \tag{5.4}$$

In view of this equality, (5.3) can be rewritten as follows:

$$y_1'(f_1(x_0)) \leq y_1'(f_1(x)) + y_2'(f_2(x)) \ \text{ for all } x \in D. \tag{5.5}$$

On the other hand, due to (5.4) it is also seen that

$$(y_1', y_2') \in K_1^* \times K_2^* \setminus \{(o_1', o_2')\}.$$

Since $-f_2(x_0) \in K_2$ and $y_2' \in K_2^*$, we have $y_2'(f_2(x_0)) \leq 0$. Taking into account this inequality, it follows from (5.5) that

$$y_1'(f_1(x_0)) + y_2'(f_2(x_0)) \leq y_1'(f_1(x_0)) \leq y_1'(f_1(x)) + y_2'(f_2(x))$$

for all $x \in D$. Consequently, (5.2) must hold.

2. According to assertion 1. there is a $(y_1', y_2') \in K_1^* \times K_2^* \setminus \{(o_1', o_2')\}$ satisfying (5.2). We must have $y_1' \neq o_1'$. Indeed, if we suppose that $y_1' = o_1'$, then we have $y_2' \neq o_2'$ as well as

$$0 = \min \{y_2'(f_2(x)) \mid x \in D\}.$$

By applying assertion 2. of Lemma 3.1 it follows that

$$(\operatorname{rng} f_2) \cap (-\operatorname{int} K_2) = \emptyset,$$

which is absurd. Hence we have $y_1' \neq o_1'$, as claimed. □

Remark 5.4. If x_0 is an element of the set E, defined by (5.1), and if there is a $(y_1', y_2') \in K_1^* \times K_2^*$ with $y_1' \neq o_1'$ such that (5.2) holds, then x_0 is a weak K_1-minimizer of the function f_1 over E even when the function $f = (f_1, f_2) : D \to Y_1 \times Y_2$ is not closely $K_1 \times K_2$-convexlike. Indeed, for any $x \in E$ we have $-f_2(x) \in K_2$, and thus $y_2'(f_2(x)) \leq 0$. Therefore (5.2) implies

$$y_1'(f_1(x_0)) \leq y_1'(y) \text{ for all } y \in f_1(E).$$

By applying assertion 2. of Lemma 3.1, it follows that

$$[f_1(x_0) - \operatorname{int} K_1] \cap f_1(E) = \emptyset.$$

This means that x_0 is a weak K_1-minimizer of f_1 over E.

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