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# Bounds for Anti-Distance*

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Given two matrices $A,\ B$, we give an upper and a lower bound for the anti–distance $\sup_U \|A - U^*BU\|_\infty$ where $U$ runs over the set of unitary matrices.

Let $\mathbb{M}_n$ be the space of $n \times n$ complex matrices, and $\mathbb{U}_n$ the group of unitary matrices. For each $B \in \mathbb{M}_n$, the set $\{U^*BU\ :\ U \in \mathbb{U}_n\}$ is the unitary similarity orbit through $B$. Thus $\sup_{U \in \mathbb{U}_n} \|A - U^*BU\|_\infty$ is considered as the *anti–distance* from $A$ to the orbit with respect to the *spectral norm* $\|\cdot\|_\infty$. In this note we give an upper and a lower bound for the anti–distance. Recall that

$$\|A\|_\infty \equiv sup_{\|x\|=1}\|Ax\| = sup_{\|x\|=\|y\|=1}|y^*Ax|,$$

where $y^*$ is the conjugate transposed (row) vector of $y$.

**Theorem 1.**

$$\sup_{U \in \mathbb{U}_n} \|A - U^*BU\|_\infty \le \sqrt{2}\|A \otimes I - I \otimes B^T\|_\infty. \tag{1}$$

**Proof.** Since for any $U \in \mathbb{U}_n$

$$\|A \otimes I - I \otimes B^T\|_\infty = \|A \otimes I - I \otimes (U^*BU)^T\|_\infty$$

it suffices to prove

$$\|A - B\|_\infty \le \sqrt{2}\|A \otimes I - I \otimes B^T\|_\infty, \tag{2}$$

or equivalently for any (column) vectors $x,\ y$

$$|y^*(A - B)x| \le \sqrt{2}\|A \otimes I - I \otimes B^T\|_\infty \cdot \|x\| \cdot \|y\|. \tag{3}$$

Remark that

$$y^*(A - B)x = \mathrm{tr}((Ax) \cdot y^* - x \cdot (B^*y)^*).$$

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Since the matrix $(Ax)\cdot y^* - x\cdot(B^*y)^*$ is of rank $\leq 2$, the number of its non-zero singular values $s_1 \geq s_2 \geq \cdots \geq s_n (\geq 0)$ is 2 at most. Therefore

$$\begin{aligned} |y^*(A-B)x| &\leq \|(Ax)\cdot y^* - x\cdot(B^*y)^*\|_1 \\ &= s_1 + s_2 \\ &\leq \sqrt{2}(s_1^2+s_2^2)^{1/2} \\ &= \sqrt{2}\|(Ax)\cdot y^* - x\cdot(B^*y)^*\|_2, \end{aligned} \tag{3}$$

where $\|\cdot\|_1$ and $\|\cdot\|_2$ denote the trace norm and the Hilbert–Schmidt norm respectively;

$$\|X\|_1 \equiv \mathrm{tr}(|X|) \quad \text{and} \quad \|X\|_2 \equiv (\mathrm{tr}(|X|^2))^{1/2},$$

with $|X|$, the positive semidefinite square root of $X^*X$.
Consider the linear map $\Delta_{A,B}$ on the Hilbert space $(\mathbb{M}_n, \|\cdot\|_2)$, defined by

$$\Delta_{A,B}(X) = AX - XB. \tag{4}$$

Then clearly we have

$$(Ax)\cdot y^* - x\cdot(B^*y)^* = \Delta_{A,B}(x\cdot y^*). \tag{5}$$

On the other hand, since $\|A\otimes I - I\otimes B^T\|_\infty$ coincides with the operator (spectral) norm of $\Delta_{A,B}$ on $(\mathbb{M}_n, \|\cdot\|_2)$, it follows from (3) and (5) that

$$\begin{aligned} |y^*(A-B)x| &\leq \sqrt{2}\|A\otimes I - I\otimes B^T\|_\infty \cdot \|x\cdot y^*\|_2 \\ &= \sqrt{2}\|A\otimes I - I\otimes B^T\|_\infty \cdot \|x\|\cdot\|y\|. \end{aligned}$$

□

The following special case was proved by Omladic and Semrl [2] by a different method.

**Corollary 2.** *If $A$, $B$ are normal matrices with eigenvalues $\{\alpha_1,\cdots,\alpha_n\}$ and $\{\beta_1,\cdots,\beta_n\}$ respectively, then*

$$\sup_{U\in\mathbb{U}_n} \|A - U^*BU\|_\infty \leq \sqrt{2}\max{}_{i,j}|\alpha_i - \beta_j|. \tag{6}$$

**Proof.** Remark that for general $A$, $B$, the set of eigenvalues of $A\otimes I - I\otimes B^T$ coincides with $\{\alpha_i - \beta_j : i,j = 1,2,\cdots,n\}$. If $A$ and $B$ are normal, so is $A\otimes I - I\otimes B^T$, hence

$$\|A\otimes I - I\otimes B^T\|_\infty = \max{}_{i,j}|\alpha_i - \beta_j|. \tag{7}$$

Now the assertion follows from (1) and (7). □

Again in view of (7), for normal $A$, $B$ the inequality

$$\|A\otimes I - I\otimes B^T\|_\infty \leq \sup_{U\in\mathbb{U}_n}\|A - U^*BU\|_\infty$$

is quite trivial. Let us show that this holds in general.

**Theorem 3.**

$$\|A \otimes I - I \otimes B^T\|_\infty \leq \sup_{U \in \mathbb{U}_n} \|A - U^* B U\|_\infty. \tag{8}$$

**Proof.** Let us consider the operator $\Delta_{A,B}$, defined by (4), on each of the Schatten $p$-space $(\mathbb{M}_n, \|\cdot\|_p)$ $(1 \leq p \leq \infty)$, where

$$\|X\|_p \equiv (\mathrm{tr}(|X|^p))^{1/p}.$$

The operator norm of $\Delta_{A,B}$ on $(\mathbb{M}_n, \|\cdot\|_p)$ will be denoted by $\|\Delta_{A,B}\|_{p \to p}$. We used already the fact

$$\|\Delta_{A,B}\|_{2 \to 2} = \|A \otimes I - I \otimes B^T\|_\infty. \tag{9}$$

Since $(\mathbb{M}_n, \|\cdot\|_p)_{1 \leq p \leq \infty}$ is an interpolation scale (see [1] Chap. III), it follows from the general theory that

$$\|\Delta_{A,B}\|_{2 \to 2} \leq \sqrt{\|\Delta_{A,B}\|_{1 \to 1} \cdot \|\Delta_{A,B}\|_{\infty \to \infty}}. \tag{10}$$

By (9) and (10), for (8) it suffices to prove that

$$\|\Delta_{A,B}\|_{1 \to 1} = \|\Delta_{A,B}\|_{\infty \to \infty} = \sup_{U \in \mathbb{U}_n} \|A - U^* B U\|_\infty. \tag{11}$$

First, since the unit ball of $(\mathbb{M}_n, \|\cdot\|_\infty)$ is the convex hull of $\mathbb{U}_n$,

$$\begin{aligned} \|\Delta_{A,B}\|_{\infty \to \infty} &= \sup_{U \in \mathbb{U}_n} \|UA - BU\|_\infty \\ &= \sup_{U \in \mathbb{U}_n} \|A - U^* B U\|_\infty. \end{aligned}$$

Next

$$\begin{aligned} \sup_{U \in \mathbb{U}_n} \|UA - BU\|_\infty &= \sup_{\|T\|_1 \leq 1} \sup_{U \in \mathbb{U}_n} |\mathrm{tr}(TUA - TBU)| \\ &= \sup_{\|T\|_1 \leq 1} \sup_{U \in \mathbb{U}_n} |\mathrm{tr}(U \cdot \Delta_{A,B}(T))| \\ &= \sup_{\|T\|_1 \leq 1} \|\Delta_{A,B}(T)\|_1 = \|\Delta_{A,B}\|_{1 \to 1}. \end{aligned}$$

These establish (11) and complete the proof. □

## References


[1] I. Gohberg, M.G. Krein: Introduction to the Theory of Nonselfajoint Operators, Translations of Mathematical Monographs, Vol. 18, American Mathematical Society, Providence, R.I., 1969.

[2] M. Omladic, P. Semrl: On the distance between normal matrices, Proc. Amer. Math. Soc. 110 (1990) 591–596.

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