Journal of Convex Analysis
Volume 3 (1996), No. 2, 269–281

Monotone Trajectories of Differential Inclusions in Banach Spaces[1]

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Received January 23, 1995
Revised manuscript received October 12, 1995

Two existence results for monotone trajectories of differential inclusions $x'(t) \in F\big(t, x(t)\big)$ in a separable Banach space are obtained; they extend in two directions previous ones due to *Aubin-Cellina, Deimling* and *Haddad.*

Keywords : Nonautonomous differential inclusions, tangent cones, monotone trajectories.

1991 Mathematics Subject Classification: 34A60, 49K24

1. Introduction

Let X be a given nonempty compact subset of a separable Banach space E. A preorder $\succeq$ on X, that is a reflexive and transitive relation, is defined by a set-valued map P which to any $x \in X$ associates

$$P(x) = \{y \in X : y \succeq x\};$$

we recall that $\succeq$ is said to be a *continuous preorder* (see e.g. [9]) whenever P is a lower semicontinuous correspondence with a closed graph.

Let $F : [0, T] \times X \to E$ be a nonempty convex weakly compact set-valued map. Given x_0 in X we look for Lipschitz solutions of the differential inclusion:

$$w'(t) \in F(t, w(t)) \qquad w(0) = x_0 \tag{1.1}$$

which are viable, i.e. $w(t) \in X$ for all $t \in [0, T]$ and *monotone* with respect to the preorder P, that is

$$\text{for any } s, t \in [0, T],\ s < t \text{ implies } w(t) \in P(w(s)).$$

The same problem has been investigated by Haddad [9] (see also [1]) when E is a finite dimensional space. In [9] F is assumed to be globally upper semicontinuous and satisfies a tangential condition involving Bouligand's contingent cone $T_{P(x)}(x)$, where

$$T_{P(x)}(x) = \Big\{v \in E : \liminf_{\gamma \to 0^+} \frac{d_{P(x)}(x + \gamma v)}{\gamma} = 0\Big\};$$

[1] This research was supported by M.U.R.S.T. within the project "Analisi reale".

ISSN 0944-6532 / $ 2.50

It is well known (see e.g. [1]), that $T_{P(x)}(x)$ is a nonempty closed cone which is larger than the tangent cone introduced by Clarke [7] which is given by

$$C_{P(x)}(x) = \Big\{ v \in E : \limsup_{\gamma \to 0^+,\, y \to x} \frac{d_{P(x)}(y + \gamma v) - d_{P(x)}(y)}{\gamma} = 0 \Big\}.$$

Tallos [10] studied the existence of viable trajectories for (1.1) in a finite dimensional space, when F is integrably bounded, measurable in t upper semicontinuous in x and satisfies a tangential condition involving Clarke's cone.

Existence theorems of viable solutions in Banach spaces where proven by Benabdellah-Castaing-Gamal Ibrahim [2] and by Castaing-Moussaoui-Syam [5] when F is measurable in t and upper semicontinuous in x. We also refer to [1] and [8] for a wide bibliography on the subject.

The present paper extends in several directions the results obtained by Aubin-Cellina [1], Deimling [8] and Haddad [9]. Even in the particular case when F is scalarly globally upper semicontinuous, we need a careful proof of the convergence of the approximated solutions via new compactness results in L^1_E (see e.g. [2], Thm.5.4).

The second difficulty is due to the various weak measurability assumptions on F. The above mentioned difficulties are solved by approximations techniques involving a careful proof of the convergence of approximated solutions via a result of convergence due to Castaing-Moussaoui-Syam [5], Lemma 6.5 (see also Lemma 2.1 below), a multivalued version of Scorza-Dragoni Theorem [4], Thm.2.2 (see also Theorem 2.3) and a multivalued version of Dugundji's "single-valued" extension Theorem [4], Thm.2.3 (see also Theorem 2.4). So Theorem 3.3 and Theorem 3.4 are new achievements on this subject.

In the sequel we shall denote by $cwk(E)$ the set of all nonempty convex weakly compact subsets of E and by B the closed unit ball of E; further we shall put $|A| = \sup\{||x|| : x \in A\}$ for any subset $A \subset E$ and refer to λ as the Lebesgue measure on the real line R.

Finally we remind that a multifunction F from a measurable space (S, Σ) to bounded subsets of a Banach space E is said to be *scalarly Σ-measurable*, see e.g. [5], if for any e' in the dual E' of E the scalar function

$$\delta^*(e', F(x)) = \sup_{v \in F(x)} \langle e', v \rangle$$

is $\Sigma-$measurable; analogously, when S is also a topological space, F is said to be *scalarly upper semicontinuous*, see e.g. [5], whenever the scalar function $\delta^*(e', F(x))$ is upper semicontinuous for each $e' \in E'$.

The author takes the opportunity to thank Professor Charles Castaing for his helpful suggestions and comments.

2. Preliminary Results

In this section we summarize some fundamental results useful later on.

We begin with a lemma due to Castaing-Moussaoui-Syam [5]; we shall apply it to show the convergence of a sequence of approximated solutions.

Lemma 2.1. ([5], Lemma 6.5.) *Let (S, d) be a Souslin metrizable space. Let $F : [0, T] \times S \to cwk(E)$ satisfying:*

(i) *F is scalarly $\tau_\lambda([0,T]) \otimes \mathcal{B}(S)$–measurable;*
(ii) *for any t in $[0,T]$, $F(t,\cdot)$ is scalarly upper semicontinuous;*
(iii) $\sup_{(t,x)\in[0,T]\times S} |F(t,x)| < \infty$.

Let $(r_n)_n$ be a sequence of strictly positive numbers with $\lim_{n\to\infty} r_n = 0$. Let $(X_n)_{n\geq 1}$ be a sequence of λ-measurable mappings from $[0,T]$ to S which converges pointwisely to a λ-measurable mapping X, $(Y_n)_{n\geq 1}$ be a sequence in $L^1_E([0,T],\lambda)$ which $\sigma(L^1, L^\infty)$ converges to Y in $L^1_E([0,T],\lambda)$ and such that
$Y_n(t) \in \frac{1}{r_n}\int_{I_{t,r_n}} F(s, X_n(t))\, ds$ a.e. with $I_{t,r_n} = [0,T] \cap [t, t+r_n]$. Then

$$Y(t) \in F(t, X(t)) \quad a.e.$$

Next lemma is due to Haddad [9]; more precisely he gave his result in a finite dimensional space, but the same proof holds in an arbitrary Banach space; we shall apply it to construct a sequence of approximated solutions.

Lemma 2.2. *Let X be a locally compact subset of a Banach space E, $P : X \to X$ a given continuous preorder and $F : X \to cwk(E)$ a scalarly upper semicontinuous multifunction satisfying the following tangential condition:*

$$F(x) \cap T_{P(x)}(x) \neq \emptyset \qquad \text{for all} x \in X.$$

Given $x_0 \in X$, let $R > 0$ be such that $X_0 = X \cap (x_0 + RB)$ is compact and let $\alpha > 0$ be such that $\alpha \geq \mathrm{Sup}_{x\in X_0}|F(x)|$.
Then for every $\beta > 0$ it is possible to find two finite sequences
$0 = t_0 < t_1 < \ldots < t_{m-1} < \frac{R}{\alpha+\beta} \leq t_m$ and $\{x_0, x_1, \ldots, x_m\}$ such that for each $k = 0, 1, \ldots, m-1$ we have $t_{k+1} - t_k < \beta$, $x_k \in X_0$, $x_{k+1} \in P(x_k)$ and the existence of $y_k \in X_0$ depending on x_k and $v_k \in F(y_k)$ satisfying

$$||x_k - y_k|| < \beta, \qquad ||\frac{x_{k+1} - x_k}{t_{k+1} - t_k} - v_k|| < \beta. \tag{2.1}$$

Proof. We recall that, for convex weakly compact nonempty set valued maps, scalar upper semicontinuity is equivalent to weak upper semicontinuity, that is upper semicontinuity when E is endowed with weak topology (see [6], Thm.II.20); therefore $F(X_0)$ is weakly compact. The result then follows by the same reasoning given in [9], Lemma I-1. □

The following result, due to Castaing-Monteiro Marques [4], is a multivalued version of Scorza-Dragoni Theorem.

Theorem 2.3. *Let X be a Polish space and Y be a convex compact metrizable subset of a Hausdorff locally convex space.*
Let $F : [0,T] \times X \to ck(Y)$ (nonempty convex compact subsets of Y) be a multifunction satisfying:
(i) *for all $t \in [0,T]$, graph $F_t = \{(x,y) \in X \times Y : y \in F(t,x)\}$ is closed in $X \times Y$;*

(ii) *for any $x \in X$, $F(\cdot, x)$ admits a $(\tau_\lambda([0,T]), \mathcal{B}(Y))$-measurable selection.*

Then, there exists a measurable multifunction $F_0 : [0,T] \times X \to ck(Y) \cup \{\emptyset\}$ which has the following properties:

(1) *there is a λ-null set M, independent of (t,x), such that*

$$F_0(t,x) \subset F(t,x), \text{ for all} t \notin M \text{ and} x \in X;$$

(2) *if $u : [0,T] \to X$ and $v : [0,T] \to Y$ are $\tau_\lambda([0,T])$-measurable functions with $v(t) \in F(t,u(t))$ a.e., then $v(t) \in F_0(t,u(t))$ a.e.;*

(3) *for every $\epsilon > 0$, there is a compact subset $I_\epsilon \subset [0,T]$ such that $\lambda([0,T] \setminus I_\epsilon) < \epsilon$, the graph of the restriction $F_{0 \setminus I_\epsilon \times X}$ is closed and $\emptyset \neq F_0(t,x) \subset F(t,x)$, for all $(t,x) \in I_\epsilon \times X$.*

Reference. Castaing-Monteiro Marques ([4], Thm.2.2).

We also need the following multivalued version of Dugundji's extension Theorem.

Theorem 2.4. *Let X and E be Banach spaces and $I \subset X$, $D \subset E$ be nonempty closed sets. Let E_σ be the vector space E endowed with the $\sigma(E,E')$-topology. Let $F : I \times D \to E_\sigma$ be an upper semicontinuous multifunction with nonempty convex compact values in E_σ such that*

$$F(t,x) \subset c(t)(1 + \|x\|)B \quad \text{for all } (t,x) \in I \times D,$$

and some positive function c defined on I.

Let $(U_\lambda)_{\lambda \in \Lambda}$ be a locally finite open covering of $X \setminus I$ such that, for all $\lambda \in \Lambda$, $0 < diam\, U_\lambda \leq \mathrm{d}\ (U_\lambda, I)$, where

$$d(U_\lambda, I) = \inf\{\|t_\lambda - s\| : t_\lambda \in U_\lambda, s \in I\}.$$

Let $(\psi_\lambda)_{\lambda \in \Lambda}$ be a continuous partition of unity of $X \setminus I$ associated to the covering $(U_\lambda)_{\lambda \in \Lambda}$. For every $\lambda \in \Lambda$, choose $t_\lambda \in I$ such that $d_{U_\lambda}(t_\lambda) < 2d(U_\lambda, I)$ where

$$d_{U_\lambda}(t_\lambda) = \inf\{\|t_\lambda - s\| : s \in U_\lambda\}.$$

Then the multifunction $\tilde{F}$ defined on $X \times D$ by

$$\tilde{F}(t,x) = \begin{cases} F(t,x), & \text{if } t \in I, x \in D \\ \sum\limits_{\lambda \in \Lambda} \psi_\lambda(t) F(t_\lambda, x), & \text{if } t \in X \setminus I, x \in D \end{cases}$$

is an upper semicontinuous extension of F from $X \times D$ to E_σ with convex compact values in E_σ. Moreover, we have $\tilde{F}(X \times D) \subset coF(I \times D)$ (convex hull of the set $F(I \times D)$) and, if c is constant, $\tilde{F}(t,x) \subset c(1 + \|x\|)B$. In particular, if $F(t,x) \subset K$ for all (t,x), where K is a convex set, then $\tilde{F}(t,x) \subset K$.

Reference. Proof is a trivial adaptation of ([4], Thm.2.3) and it is omitted.

3. Existence results

We first give a local existence result (Theorem 3.1) for an autonomous r.h.s. $F = F(x)$, followed by a global existence one (Theorem 3.2) for $F = F(t,x)$, both of them under tangential conditions for the preorder P involving Bouligand's cone; in such results F is assumed to be scalarly upper semicontinuous.
In the sequel, under stronger tangential conditions involving Clarke's cone, we deal with the cases when $F = F(t,x)$ is scalarly measurable in (t,x) and scalarly upper semicontinuous in x (Theorem 3.3) and when $F = F(t,x)$ is separately measurable in t and scalarly upper semicontinuous in x (Theorem 3.4).

Theorem 3.1. *Let X be a locally compact subset of a separable Banach space E, $P : X \to X$ a given continuous preorder and $F : X \to cwk(E)$ a scalarly upper semicontinuous multifunction. Assume the following tangential condition:*

$$F(x) \cap T_{P(x)}(x) \neq \emptyset \qquad \text{for all } x \in X. \tag{3.1}$$

Then, for every $x_0 \in X$, there exist $T_0 > 0$ and a Lipschitz function $w : [0,T_0] \to X$ satisfying

$$\begin{aligned} &w(0) = x_0, \qquad w'(t) \in F(w(t)) \ \text{ for almost all } t \in [0,T_0] \\ &w \text{ is monotone with respect to the preorder } P. \end{aligned} \tag{3.2}$$

Proof. Given $x_0 \in X$, let $R > 0$ be such that $X_0 = X \cap (x_0 + RB)$ is compact and let $\alpha > 0$ be such that $\alpha \geq \mathrm{Sup}_{x \in X_0} |F(x)|$.

Fix a strictly positive integer n; applying Lemma 2.2 with $\beta = \frac{1}{n}$, we can associate three finite sequences $\{t_k^{(n)}\}_{k=0,1,...,m}$, $\{x_k^{(n)}\}_{k=0,1,...,m}$ and $\{y_k^{(n)}\}_{k=0,...m-1}$ with the properties given by the lemma; notice that it is not restrictive to put $t_m^{(n)} = \frac{R}{\alpha + \frac{1}{n}}$. Let $\frac{R}{\alpha} = T_0$ and define the function $w_n : [0,T_0] \to E$ as follows

$$w_n(t) = \begin{cases} x_k^{(n)} + (t - t_k^{(n)}) \frac{x_{k+1}^{(n)} - x_k^{(n)}}{t_{k+1}^{(n)} - t_k^{(n)}} & \text{for } t \in [t_k^{(n)}, t_{k+1}^{(n)}) \text{ and } k = 0,1,...,m-1 \\ x_m^{(n)} & \text{for } t \in [t_m^{(n)}, T_0]. \end{cases}$$

More precisely, on every interval $[t_k^{(n)}, t_{k+1}^{(n)}]$, with $k = 0,1,...,m-1$, w_n is the linear function interpolating $x_k^{(n)}$ and $x_{k+1}^{(n)}$, while w_n is extended with continuity on $[t_m^{(n)}, T_0]$.

We shall show that $(w_n)_n$ admits a subsequence which pointwisely converges in norm topology on $[0,T_0]$, to a Lipschitz function w satisfying both (3.2) and the requested monotonicity.

First of all it is convenient to introduce, for every $n \in N$, the following piecewise constant function $y_n : [0,T_0] \to X_0$ given by

$$y_n(t) = \begin{cases} y_k^{(n)} & \text{for } t \in [t_k^{(n)}, t_{k+1}^{(n)}) \text{ and } k = 0,1,...,m-1 \\ y_{m-1}^{(n)} & \text{for } t \in [t_m^{(n)}, T_0]. \end{cases}$$

Since $w_n'(t)$ is piecewise constant with

$$w_n'(t) = \begin{cases} \frac{x_{k+1}^{(n)} - x_k^{(n)}}{t_{k+1}^{(n)} - t_k^{(n)}} & \text{for } t \in (t_k^{(n)}, t_{k+1}^{(n)}) \text{ and } k = 0, 1, ..., m-1 \\ 0 & \text{for } t \in (t_m^{(n)}, T_0), \end{cases}$$

from (2.1) it yields

$$w_n'(t) \in F(y_n(t)) \cup \{0\} + \frac{1}{n} B \quad \text{for almost all } t \in [0, T_0], \tag{3.3}$$

and this implies

$$||w_n'(t)|| \le \alpha + \frac{1}{n} \le \alpha + 1 \quad \text{a.e.} \tag{3.4}$$

Moreover, for $t \in [t_k^{(n)}, t_{k+1}^{(n)}]$ and $k = 0, 1, ..., m-1$, since $t_{k+1}^{(n)} - t_k^{(n)} < \frac{1}{n}$, we have

$$\begin{aligned} &||w_n(t) - x_k^{(n)}|| = ||w_n(t) - w_n(t_k^{(n)})|| \le (t - t_k^{(n)})(\alpha + 1) \le \\ &\le (t_{k+1}^{(n)} - t_k^{(n)})(\alpha + 1) \le \frac{\alpha + 1}{n} \end{aligned}$$

and $w_n(t) \equiv x_m^{(n)}$ on $[t_m^{(n)}, T_0]$. Therefore we obtain

$$w_n(t) \in X_0 + \left(\frac{\alpha + 1}{n}\right) B \quad \text{for all } t \in [0, T_0] \text{ and } n \in N. \tag{3.5}$$

Let us consider now the sequence $(w_n')_n$; by (3.4) it is bounded and uniformly integrable in $L_E^1([0, T_0], \lambda)$; by (3.3) it holds

$$w_n'(t) \in F(X_0) \cup \{0\} + \frac{1}{n} B \quad \text{a.e. in } [0, T_0];$$

denote with Φ the balanced convex hull of $F(X_0)$, then Φ is convex weakly compact and we have

$$w_n'(t) \subset \Phi + \frac{1}{n} B \quad \text{a.e. in } [0, T_0].$$

Using measurable selections, $w_n'(t)$ can be expressed as $w_n'(t) = v_n(t) + h_n(t)$, for all $t \in [0, T_0]$, where v_n belongs to the set S_Φ of all measurable selections of Φ and h_n is measurable with $h_n(t) \in \frac{1}{n} B, \forall t$. Since S_Φ is weakly compact, $(v_n)_n$ is relatively weakly compact. Since $h_n \to 0$ in L^1, $(w_n')_n$ is relatively $\sigma(L^1, L^\infty)$ compact. So we can extract a subsequence, again denoted $(w_n')_n$ converging to $w' \in L_E^1([0, T_0], \lambda)$ in $\sigma(L^1, L^\infty)$-topology.

Let $w : [0, T_0] \to E$ be given by $w(t) = x_0 + \int_0^t w'(s)\, ds$; by (3.4) w is an $\alpha-$Lipschitz function satisfying $w_n(t) \rightharpoonup w(t)$ in the $\sigma(E, E')$-topology, for all $t \in [0, T_0]$.

Notice that, by (3.5) the following set $W = \{w_n(t) : t \in [0, T_0] \text{ and } n \in N\}$ is relatively compact for the norm topology, hence we have $w_n(t) \to w(t)$ in norm for all $t \in [0, T_0]$.

Therefore, by the inequalities after (3.4) and by Lemma 2.2, also the sequence $(y_n)_n$ converges pointwisely to the function w in $[0, T_0]$, since, for every $t \in [t_k^{(n)}, t_{k+1}^{(n)})$ with $k = 0, 1, ..., m-1$, it holds

$$||y_n(t) - w(t)|| \leq ||y_k^{(n)} - x_k^{(n)}|| + ||w_n(t_k^{(n)}) - w_n(t)|| + ||w_n(t) - w(t)||.$$

We shall prove now that w satisfies the differential inclusion (3.2) for almost all $t \in [0, T_0]$. Let $(e'_k)_k$ be a dense sequence in E' for the Mackey topology. Let $k, n \in N$ and $A \in \tau_\lambda([0, T_0])$; by (2.1) we get, in particular

$$w'_n(t) \in F(y_n(t)) + \frac{1}{n}B \quad \text{for almost every } t \in [0, t_m^{(n)}],$$

we also recall that $w'_n(t) \equiv 0$ on $(t_m^{(n)}, T_0)$ with $T_0 - t_m^{(n)} \to 0$ when $n \to +\infty$; therefore we have

$$\int_A \langle e'_k, w'_n(t)\rangle \, dt \leq$$
$$\leq \int_A \delta^*\Big(e'_k, F(y_n(t))\Big)\, dt + \frac{\lambda(A)}{n}||e'_k|| - \lambda(A \cap [t_m^{(n)}, T_0])\delta^*(e'_k, F(y_{m-1}^{(n)}));$$

since w'_n converges to w' in $\sigma(L^1, L^\infty)$, y_n converges pointwisely to w and F is scalarly upper semicontinuous, applying Fatou's Lemma we obtain

$$\int_A \langle e'_k, w'(t)\rangle \, dt \leq \int_A \delta^*\Big(e'_k, F(w(t))\Big)\, dt.$$

Hence we can find a negligible set M in $[0, T_0]$ such that

$$\langle e'_k, w'(t)\rangle \leq \delta^*\Big(e'_k, F(w(t))\Big) \quad \text{for all } t \in [0, T_0]\backslash M \text{ and } k \in N;$$

this implies (see [6], Lemma III.34)

$$w'(t) \in F(w(t)) \quad \text{for almost all } t \in [0, T_0]$$

and (3.2) holds.

To complete the proof it remains to show that w is a monotone function with respect to the preorder P. To this end take $s, t \in [0, T_0]$ with $s < t$.

First suppose $t < T_0$; in this case, for n large enough, we have $s \in [t_h^{(n)}, t_{h+1}^{(n)})$ and $t \in [t_k^{(n)}, t_{k+1}^{(n)})$ with $h + 1 \leq k$ and $h, k \in \{1, ..., m\}$. Then by the transitivity of P we deduce from Lemma 2.2 $x_k^{(n)} \in P(x_{h+1}^{(n)})$. By pointwise convergence of $(w_n)_n$ to w and Lemma 2.2 it is easy to prove that $(t_h^{(n)}, x_h^{(n)}) \to (s, w(s))$ and $(t_k^{(n)}, x_k^{(n)}) \to (t, w(t))$ as $n \to +\infty$; as the graph of P is closed we then get $w(t) \in P(w(s))$.

When $t = T_0$, we obtain $w(T_0) \in P(w(s))$ by the continuity of w.

The proof is then complete. □

Theorem 3.2. *Let X be a compact subset of a separable Banach space E; $P : X \to X$ a given continuous preorder and $F : [0,T] \times X \to cwk(E)$ a scalarly upper semicontinuous correspondence satisfying the following tangential condition*

$$F(t,x) \cap T_{P(x)}(x) \neq \emptyset \quad \textit{for all } (t,x) \in [0,T] \times X. \tag{3.6}$$

Then, for every $x_0 \in X$, there exists a Lipschitz function $w : [0,T] \to X$ such that

$$\begin{aligned} &w(0) = x_0, \qquad w'(t) \in F(t,w(t)) \, \textit{for almost all } t \in [0,T] \\ &w \quad \textit{is monotone with respect to the preorder } P. \end{aligned}$$

Proof. Consider the following multifunctions

$$H : [0,+\infty) \times X \to R \times E \quad \text{defined by}$$

$$H(t,x) = \begin{cases} \{1\} \times F(t,x) & \text{when } t \leq T \\ \{1\} \times F(T,x) & \text{when } t > T \end{cases}$$

and

$$\hat{P} : [0,+\infty) \times X \to [0,+\infty) \times X \quad \text{given by}$$

$$\hat{P}(t,x) = [t,+\infty) \times P(x).$$

It is easy to show that H is a nonempty convex weakly compact set-valued map; moreover, for any η' in $(R \times E)'$ one has $\delta^*(\eta', H(t,x)) = \eta'((1,0_E)) + \delta^*(e', F(t,x))$ for all $t \in [0,T]$ and $x \in X$, where e' denotes the restriction of η' to $\{0_R\} \times E$; hence H is also scalarly upper semicontinuous. Finally $\hat{P}$ is a continuous preorder on $[0,+\infty) \times X$.
We recall (see e.g. [1]) that an element v of E belongs to the cone $T_{P(x)}(x)$ if and only if there exists a sequence $(\gamma_n)_n$ of positive numbers converging to zero and a sequence $(x_n)_n$ in $P(x)$ such that $\frac{x_n - x}{\gamma_n} \to v$ as $n \to +\infty$, hence

$$\{1\} \times T_{P(x)}(x) \subset T_{\hat{P}(t,x)}(t,x), \quad \text{for every } (t,x) \in [0,+\infty) \times X;$$

thus condition (3.6) implies the following tangential condition on H

$$H(t,x) \cap T_{\hat{P}(t,x)}(t,x) \neq \emptyset \quad \text{for all } (t,x) \in [0,+\infty) \times X$$

and we have verified that all the assumptions of Theorem 3.1 hold.
Therefore, given $x_0 \in X$, it is possible to find a positive constant μ and a Lipschitz function $u : [0,\mu] \to [0,+\infty) \times X$ satisfying

$$\begin{aligned} &u(0) = (0,x_0), \qquad u'(\xi) \in H(u(\xi)) \, \text{for almost every } \xi \in [0,\mu] \\ &u \text{ is monotone with respect to the preorder } \hat{P}. \end{aligned}$$

Let $T_0 = \min\{\mu, T\}$; the previous condition then implies the existence of a Lipschitz function $w : [0,T_0] \to X$ such that

$$\begin{aligned} &w(0) = x_0, \qquad w'(t) \in F(t,w(t)) \, \text{for almost all } t \in [0,T_0] \\ &w \text{ is monotone with respect to the preorder } P; \end{aligned}$$

in fact it is enough to put $u(\xi) = (t(\xi), w(\xi))$ and notice that $t'(\xi) \equiv 1$.

Observe that H is bounded on $[0, +\infty) \times X$, and let $\alpha = \mathrm{Sup}_{(t,x)\in[0,+\infty)\times X}|H(t,x)|$; if $T_0 < T$, Theorem 3.1 can be applied again as from the initial condition $(T_0, w(T_0))$, then w can be extended to $[0,T]$ in such a way that it remains α-Lipschitz and monotone with respect to P and this completes the proof. □

Theorem 3.3. *Let X be a compact subset of a separable Banach space E, $P : X \to X$ a given continuous preorder and $F : [0,T] \times X \to cwk(E)$ a correspondence satisfying:*

(i) *F is scalarly $\tau_\lambda([0,T]) \otimes \mathcal{B}(X)$–measurable;*

(ii) *$F(t,\cdot)$ is scalarly upper semicontinuous on X, for any $t \in [0,T]$;*

(iii) *there exists a balanced convex weakly compact set K in E such that $F(t,x) \subset K$ for all $(t,x) \in [0,T] \times X$;*

(iv) *$F(t,x) \cap C_{P(x)}(x) \neq \emptyset$ for all $(t,x) \in [0,T] \times X$.*

Then, for every $x_0 \in X$, there exists a Lipschitz function $w : [0,T] \to X$ satisfying

$$w(0) = x_0, \quad w'(t) \in F(t, w(t)) \textit{ for almost all } t \in [0,T]$$

w is monotone with respect to the preorder P.

Proof. The following method will be used: for $h > 0$ we shall define an approximation F_h of the set-valued map F which enjoys more regularity than F, in fact F_h is globally scalarly upper semicontinuous and apply to F_h Theorem 3.2; thanks to new results due to Castaing-Moussaoui-Syam [5] (see also Lemma 2.1) we then pass to the limit when $h \to 0$ and obtain a monotone trajectory satisfying the original inclusion.

First notice that conditions (i) and (iii) imply τ_λ–measurability of $F(\cdot,x)$ on $[0,T]$ for every $x \in X$; indeed, since $\tau_\lambda([0,T])$ is a complete tribe, the class of universally measurable sets originated from $\tau_\lambda([0,T])$ coincides with $\tau_\lambda([0,T])$ (see [6], page 73), hence the assertion follows from [6], Thm.III-37.

Let $(r_n)_n$ be a sequence of strictly positive numbers converging to zero.

For every $n \in N$ and $t \in [0,T]$, put $I_{t,r_n} = [0,T] \cap [t, t+r_n]$ and consider the correspondence $F_n : [0,T] \times X \longmapsto E$ given by

$$F_n(t,x) = \frac{1}{r_n} \int_{I_{t,r_n}} F(s,x)\, ds,$$

where $\int_{I_{t,r_n}} F(s,x)\, ds$ is the Aumann integral of $F(\cdot,x)$ on I_{t,r_n}.

Since F is convex and weakly compact, by [5], Thm. 3.2 we derive that the set S_x of all integrable selections of $F(\cdot,x)$ is convex weakly compact in $L^1_E([0,T],\lambda)$; hence $F_n(t,x)$ is a convex weakly compact subset of E, for all $n \in N$ and $(t,x) \in [0,T] \times X$; moreover, reasoning as in [5], Prop.5.3, we obtain that F_n is scalarly upper semicontinuous.

Given $(t,x) \in [0,T] \times X$, consider the measurable multifunction $F(\cdot,x) \cap C_{P(x)}(x)$ defined on $[0,T]$; in consequence of (iv) it is nonempty valued; let $v : [0,T] \to E$ be an integrable selection of $F(\cdot,x) \cap C_{P(x)}(x)$; we have $\frac{1}{r_n} \int_{I_{t,r_n}} v(s)\, ds \in F_n(t,x)$ and since $C_{P(x)}(x)$ is a closed convex cone we also get $\frac{1}{r_n}\int_{I_{t,r_n}} v(s)\,ds \in C_{P(x)}(x)$, hence tangential condition (3.6) holds for each F_n.

Therefore F_n satisfies all the assumptions of Theorem 3.2 and for every $x_0 \in X$ it is possible to find a Lipschitz function $w_n : [0,T] \to X$ such that

$$\begin{array}{l} w_n(0) = x_0, \quad w_n'(t) \in F_n(t, w_n(t)) \text{ a.e.} \\ w_n \text{ is monotone with respect to } P. \end{array} \tag{3.7}$$

For each $n \in N$, by (iii) and the definition of F_n, we obtain $F_n(t,x) \subset K$ for all $(t,x) \in [0,T] \times X$, so given $\alpha > 0$ with $|K| \le \alpha$, the sequence $(w_n)_n$ turns out to be equi–α–Lipschitz.

Now recall that the set S_K of all measurable selections of K is convex and $\sigma(L^1, L^\infty)$ compact (see [6], Corollary V.4); by (3.7) $w_n'(t) \in K$ a.e., we can then apply Eberlein-Šmulian's Theorem to extract a subsequence, again denoted $(w_n')_n$, converging, for the $\sigma(L^1, L^\infty)$–topology, to a function $w' \in S_K$, hence $(w_n(t))_n$ weakly converges to $w(t) = x_0 + \int_0^t w'(s)\,ds$, for every $t \in [0,T]$.

Notice now that, for all $n \in N$ and $t \in [0,T]$, $w_n(t)$ belongs to the compact set X, therefore the sequence pointwisely $(w_n(t))_n$ converges to $w(t)$ in norm topology.

By (3.7) all trajectories w_n are monotone with respect to the preorder P, since P has closed graph also w is monotone with respect to P.

To complete the proof it remains thus to show that $w'(t) \in F(t, w(t))$ for almost every $t \in [0,T]$. According to (3.7) we then apply Lemma 2.1 to $(w_n')_n$, $(w_n)_n$ and to F so that we obtain $w'(t) \in F(t, w(t))$ a.e. as desired. □

Let us mention that when F is only scalarly measurable on $[0,T]$ for each fixed $x \in X$ and scalarly upper semicontinuous on X for each fixed $t \in [0,T]$, the conclusion of Theorem 3.3 does not hold (see e.g. Bothe [3], Example 2). We deal now with a stronger tangential condition and a weaker measurable assumption. Namely we have the following version of Theorem 3.3.

Theorem 3.4. *Let X be a compact subset of a separable Banach space E, $P : X \to X$ a given continuous preorder and $F : [0,T] \times X \to cwk(E)$ a correspondence satisfying*

(i) *$F(\cdot, x)$ admits a $\tau_\lambda([0,T])$-measurable selection, for all $x \in X$;*

(ii) *$F(t, \cdot)$ is scalarly upper semicontinuous on X, for any $t \in [0,T]$;*

(iii) *there is a balanced convex weakly compact subset K of E such that $F(t,x) \subset K \cap C_{P(x)}(x)$ for each $(t,x) \in [0,T] \times X$.*

Then, for every $x_0 \in X$, there exists a Lipschitz function $w : [0,T] \to X$ satisfying

$$\begin{array}{l} w(0) = x_0, \quad w'(t) \in F(t, w(t)) \textit{ for almost all } t \in [0,T] \\ w \textit{ is monotone with respect to the preorder } P. \end{array}$$

Proof. In consequence of (ii), $F(t, \cdot)$ is for each t upper semicontinuous from X to E_σ, where E_σ denotes the vector space E endowed with the weak $\sigma(E, E')$-topology (see [6], Thm.II.20); hence it is obvious that graph $F_t = \{(x, y) \in X \times K : y \in F(t, x)\}$ is compact in the compact metrizable space $X \times K_\sigma$, where K_σ denotes K with the metric associated to the weak topology, for which it is a compact convex set.

By virtue of Theorem 2.3, there exists a multifunction $F_0 : [0, T] \times X \to cwk(K) \cup \{\emptyset\}$ which has the properties (1)-(3) of Theorem 2.3, that is

(1) there is a λ-null set M, independent of (t, x), such that

$$F_0(t, x) \subset F(t, x), \text{ for all } t \notin M \text{ and } x \in X;$$

(2) if $u : [0, T] \to X$ and $v : [0, T] \to K$ are $\tau_\lambda([0, T])$-measurable functions with $v(t) \in F(t, u(t))$ a.e., then $v(t) \in F_0(t, u(t))$ a.e.;

(3) for every $\epsilon > 0$, there is a compact subset $I_\epsilon \subset [0, T]$ such that $\lambda([0, T] \setminus I_\epsilon) < \epsilon$, the graph of the restriction $F_{0 \setminus I_\epsilon \times X}$ is closed in $I_\epsilon \times X \times K_\sigma$ and $\emptyset \neq F_0(t, x) \subset F(t, x)$, for all $(t, x) \in I_\epsilon \times X$.

By property (3), one gets a sequence of compact sets $I_n \subset [0, T]$ with $\lambda([0, T] \setminus I_n) = \epsilon_n \to 0$ such that the restriction of F_0 to $I_n \times X$ has compact graph in $I_n \times X \times K_\sigma$; we may also assume that $(I_n)_n$ is increasing.

Since, for each n, F_0 is upper semicontinuous from $I_n \times X$ to E_σ, by Theorem 2.4 it admits an upper semicontinuous extension $\tilde{F}_n$ from $[0, T] \times X$ to E_σ with nonempty convex compact values in E_σ satisfying

$$\tilde{F}_n(t, x) \subset K \cap C_{P(x)}(x) \quad \text{for all } (t, x) \in [0, T] \times X.$$

Let $\alpha = \sup\{\|k\| : k \in K\}$; notice that each $\tilde{F}_n$ is also scalarly upper semicontinuous (see [6], Thm.II.20), hence, for every n, we can apply Theorem 3.2 in order to obtain an α-Lipschitz function $w_n : [0, T] \to X$ such that

$$\begin{aligned} &w_n(0) = x_0, \\ &w_n'(t) \in \tilde{F}_n(t, w_n(t)) \text{ a.e.} \\ &w_n \text{ is monotone with respect to the preorder } P. \end{aligned} \tag{3.8}$$

By the construction of $\tilde{F}_n$, (3.8) implies

$$w_n'(t) \in F_0(t, w_n(t)), \text{ for all } t \in I_n \setminus M_n \tag{3.9}$$

with $\lambda(M_n) = 0$.

Since K is convex weakly compact and $w_n'(t) \in K$ a.e., repeating the same arguments given in the proof of Theorem 3.3, we can extract a subsequence, again denoted $(w_n')_n$, such that w_n' converges for the $\sigma(L^1, L^\infty)$-topology to $w' \in S_K$ and for all t

$\lim\limits_{n \to +\infty} w_n(t) = w(t) = x_0 + \int_0^t w'(s)\, ds$ with respect to the norm topology.

Let $M_0 = ([0, T] \setminus \cup_n I_n) \cup \cup_n M_n$, which is a set with zero measure. If $t \notin M_0$, then there is $p = p(t)$ such that $t \in I_n \setminus M_n$ for all $n \geq p$, so that by (3.9) $w_n'(t) \in F_0(t, w_n(t))$, hence follows $\langle e', w_n'(t) \rangle \leq \delta^*\Big(e', F_0(t, w_n(t))\Big)$ for all $e' \in E'$.

For $t \notin M_0$ and $e' \in E'$, since F_0 is scalarly upper semicontinuous in $I_p \times X$ and $w_n(t) \to w(t)$ in norm topology, we have

$$\limsup_{n\to+\infty} \delta^*\Big(e', F_0(t, w_n(t))\Big) \le \delta^*\Big(e', F_0(t, w(t))\Big)$$

so that, for such t

$$\limsup_{n\to+\infty} \langle e', w_n'(t)\rangle \le \delta^*\Big(e', F_0(t, w(t))\Big)$$

where the right-hand side is a measurable function.
For all measurable sets $A \subset [0,T]$ and every $e' \in E'$, by Fatou's Lemma, it follows

$$\int_A \langle e', w'(t)\rangle\, dt = \lim_{n\to+\infty} \int_A \langle e', w_n'(t)\rangle\, dt \le \int_A \delta^*\Big(e', F_0(t, w(t))\Big)\, dt.$$

Since E is separable, this is known to imply that

$$w'(t) \in F_0(t, w(t)) \subset F(t, w(t)) \text{ a.e.}$$

Finally, we recall that P has closed graph, by (3.8) w is then monotone with respect to P and this completes the proof. $\square$

4. Comments

It is worth to compare the results obtained here with those given by Aubin-Cellina [1], Deimling [8] and Haddad [9].

For dim E $< \infty$, Haddad [9] proved a necessary and sufficient condition for the existence of monotone solutions of (1.1).

Via a constructive algorithm based on Lemma 2.2, Theorem 3.3 extends in two directions the sufficient condition given by Haddad, since dim E $= \infty$ and F is globally scalarly measurable and scalarly upper semicontinuous in $x \in X$.

Theorem 3.3 extends even a result given by Aubin-Cellina ([1], Thm.4.2.3) dealing with the case when E is a Hilbert space and F is globally upper semicontinuous, with norm-compact convex values.

Deimling obtained an existence result ([8], Thm.4) in Banach spaces when X is only closed, P has closed graph, F is independent of t and satisfies the normal growth conditions; he also assumed the following compactness type condition $\alpha(F(B)) \le k\alpha(B)$ for some $k \ge 0$ and all bounded $B \subset X$, where α denotes the Kuratowski's measure of noncompactness; such condition implies the compactness, with respect to the norm topology, of $F(x)$ for all $x \in X$ and consequently Deimling's Thm.4 [8] and Theorem 3.1 are not comparable. Finally Deimling proved his result with a different method based on the use of a measure of noncompactness and Zorn's Lemma.

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