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A New Tool in the Theory of Integral Representation

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1. Introduction

Let E be an l.c.s.. (*Hausdorff locally convex space*), E' its *dual* and $X \subset E$ a *proper convex cone*, not necessarily closed. Recall that an *open ray* δ of X is said to be *extreme* if ($x \in \delta$ and $x = y + z$ with $y, z \in X \setminus 0$) implies ($y, z \in \delta$); we denote by $\mathcal{E}_g(X)$ the *union of all the extreme open rays of* X. One of the problems of the *theory of integral representation* is to give conditions on X in order that for each $x \in X$, there is at least one *positive Radon measure* m on $\mathcal{E}_g(X)$, such that:

1. Each $\ell \in E'$ is m-integrable.
2. One has $m(\ell) = \ell(x)$, for each $\ell \in E'$.

Then we say that x is the *resultant* of m and we write $x = r(m)$.

Definition 1.1. When the cone X has the preceding properties, we say that X has the property of *integral representation*, denoted by I.R..

Recently E. Thomas ([8]) proved a theorem of integral representation for a class of convex cones, called *conuclear*. The aim of this work is to give a quite different presentation of his results, with the help of other tools, one of which is new (*the pseudo-caps*, introduced in 3.1), allowing to avoid some of his hypotheses.

The plan is as follows:
In part 2 we recall some results showing the progression of the theory of integral representation and how the present work fits with this theory.
In part 3 we prove the main results of this work and we introduce the pseudo-caps.
In part 4 we establish the link with the conuclear cones of E. Thomas.
In part 5 we discuss some examples.
The main results are Theorems 3.11 and 4.8.

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2. The theory of Integral Representation

When the cone X has a weakly compact metrizable base then X has I.R.: This is the classical theorem of integral representation for convex compact metrizable sets, proved by G. Choquet. ([7], page 19)
The notion of *cap* of a convex cone, introduced by G. Choquet, permits to cope with more general cones:

Definition 2.1. ([7], page 87)
A subset K of X is called a *cap* of X, when it is non-empty, convex, weakly compact and such that $X \setminus K$ is also convex.
If X is the *union of its caps* then X is said to be *well-capped.*

For what follows, it is useful to recall the following characterization of caps of G. Choquet:

Lemma 2.2. ([7], page 89)
Let K be a convex subset of X, containing 0; the two following properties are equivalent:

1. *K is a cap of X.*
2. *The gauge j of K is a lower semi-continuous function, defined on X, which is additive and positive-homogeneous, with values in $[0, +\infty]$ such that $K = j^{-1}([0, 1])$ is weakly compact.*

The following result of G. Choquet is fundamental:

Theorem 2.3. ([7], page 90)
For each $x \in X$, contained in a metrizable cap of X, there is a Radon measure $m \geq 0$ on $\mathcal{E}_g(X)$ such that $m(\ell) = \ell(x)$, for each $\ell \in E'$, or in other words $x = r(m)$.

For example, the cone $l^1_+(N)$ does not admit a compact base for the duality with $c_o(N)$, but it is generated by a cap, namely $K = (x : x = (x_n) \in l^1_+(N)$ and $\sum_0^\infty x_n \leq 1)$, which is metrizable. By Theorem 2.3 this cone has property I.R..

Now here is a theorem concerning a class of cones, studied in [6] and in [3], which are not necessarily well-capped.

Theorem 2.4. *Let X be a convex cone, generated by a compact metrizable set $K \subset X$, containing 0, for which there exists an affine function ϕ, defined on X, such that for some positive constant M:*
$j_K(x) \leq \phi(x) \leq M.j_K(x)$, for each $x \in X$, where j_K is the gauge of K.
Then the cone X has property I.R..

In fact, this work can be seen as an extension of the preceding theorem.
The results that we have recalled yield positive Radon measures; but they can be formulated in an apparently more abstract way which will lead us to a very useful tool.

Definition 2.5. (Weakly complete cones and conical measures) ([4], par.30)
Let $\mathcal{S}$ be the class of weakly complete convex proper cones contained in an l.c.s. E. And let $X \in \mathcal{S}$. We denote by $h(X)$ the Riesz vector space of functions on X generated by $E' \mid_X$ and by $s(X)$ the subcone of $h(X)$ consisting of all finite suprema of elements of $E' \mid_X$. The space $h(X)$ is ordered by the set of its positive elements.
A positive linear form on $h(X)$ is called *a positive conical measure on X*. We denote by $M^+(X)$ their set.

Each $\mu \in M^+(X)$ has a *resultant* $r(\mu) \in X$ such that:
$\mu(\ell) = \ell(r(\mu))$ for each $\ell \in E'$.
The cone $M^+(X)$ is ordered by:
$(\lambda \prec \mu)$ iff ($\lambda(f) \leq \mu(f)$ for each $f \in s(X)$).

The following theorem is fundamental:

Theorem 2.6. ([4], 30.13)
For each $x \in X$ there is a $\mu \in M^+(X)$, which is maximal for the order $\prec$, and we have $r(\mu) = x$.

The following characterization of maximal conical measures will be very useful:

Theorem 2.7. ([4], 30.pb1)
If $\mu \in M^+(X)$ the three following properties are equivalent:

1. *μ is maximal for the order $\prec$.*
2. *For each $f \in h(X)$ then: $\mu(f) = inf(\mu(g) : g \in -s(X)$ and $g \geq f$ on X).*
3. *Same condition as in 2. but only for each $f \in s(X)$.*

Finally we recall a theorem of H. Fakhoury ([5]) which will be useful for us:

Theorem 2.8. *Let $Z \subset E$ be a proper convex cone, covered by an increasing sequence of weakly compact convex sets (Z_n), which are hereditary for the order of Z.*
Then $Z \in \mathcal{S}$ for the duality with the space of all affine and homogeneous functions on Z, whose restriction to each Z_n is continuous.

3. Pseudo-caps and main results

We introduce now the notion of *pseudo-cap* of a cone, which will show that this work is in some sense parallel to the theory of caps.

Definition 3.1. (The pseudo-caps)
Let X be a proper convex cone, contained in a vector space V, not necessarily equipped with a topology. We call *pseudo-cap* of X any non-empty convex set $K \subset X$, such that $X \setminus K$ is also convex and that each ray of X intersects K along a compact interval.

We shall give in 5.2 an example of a pseudo-cap K of a cone contained in an l.c.s., such that $\overline{K}$ is compact, but is not a cap.
Here is an analytical characterization of pseudo-caps, analoguous to that of caps:

Proposition 3.2. *Let K be a pseudo-cap of a convex cone X; the gauge of K is additive and homogeneous on X, with values in $[0, \infty]$, and it is > 0 on $X \setminus 0$.*
Reciprocally, for each function f, defined on X, with values in $[0, \infty]$, which is additive and homogeneous, and which is > 0 on $X \setminus 0$, the set $K = f^{-1}([0, 1])$ is a pseudo-cap of X and f is the gauge of K.

Proof. In order to show that the gauge j of K is additive on X, it is sufficient to consider the two dimensional sub-cones of X: Let Y be such a sub-cone, generated by $x, y \in X$; it is sufficient to prove that: $K \cap Y = conv((K \cap R^+.x) \cup (K \cap R^+.y))$, which is easy to prove since each ray of X intersects K along a segment $[0, a]$ and hence $K \cap (R^+.x)$ and $K \cap (R^+.y)$ are compact intervals.

One has $j(x) > 0$ for each $x \neq 0$, since each ray of X intersects K along a segment $[0, a]$. The remainder of the proposition is easy. □

Remark 3.3. Here is an example proving that, in the preceding proposition, the condition that the intersection of K with each ray of X is a compact interval cannot be removed:
In R^2, let X be the cone $((x, y) : x, y \geq 0)$.
If $K = ((x, y){:}(x, y) \in X$ and $y \geq x - 1)$ obviously K is convex, contains 0 and $X \setminus K$ is also convex. But the gauge j of K is not additive:
Indeed, one has $j((1, 0)) = 1$ and $j((0, 1)) = 0$, but $j((1, 1)) = 0$.

Recall that a *face* of a proper convex cone X is a convex sub-cone of X which is hereditary for the order of X.

Proposition 3.4. *For each pseudo-cap K of X the cone X_K generated by K is a face admitting $(x : j_K(x) = 1)$ as a base.*

Proof. One has $X_K = (x : x \in X$ and $j_K(x) < \infty)$ and j_K is additive on X. □

The two following theorems are fundamental; in particular the following theorem is analogous to Theorem 2.3.

Theorem 3.5. *Let X be a convex proper cone, not necessary closed, contained in an l.c.s. E, equipped with its weak topology. Let K be a pseudo-cap of X, contained in a compact convex set S of X, which is metrizable and hereditary for the order associated to X.*
Then, for each $x \in X_K$, there is a positive Radon measure m on $\mathcal{E}_g(X)$, integrating all the elements of E', such that $r(m) = x$.

Proof. There are three steps. We suppose that $x \neq 0$.
1. Let Y be the cone generated by S. Using Theorem 2.8, if we denote by F the space $Y - Y$, there is $F' \subset F^*$, containing $E' |_F$, such that $Y \in \mathcal{S}$ for the duality with F'.
Moreover, S is compact and metrizable for this topology by Theorem 2.8. Since S is hereditary in X, the cone Y is a face of X and hence $\mathcal{E}_g(Y) \subset \mathcal{E}_g(X)$. Hence we can use the cone Y to prove the theorem.
2. Since $Y \in \mathcal{S}$, there exists, by Theorem 2.6, a conical measure $\mu \in M^+(Y)$, for the duality with F', which is maximal and such that $x = r(m)$.
Since $Y \in \mathcal{S}$, there exists an ultrafilter $\mathcal{U}$ on the set of all conical measures ν on Y such that:
$\nu = \sum_1^n \varepsilon_{x_i}$, with $\sum_1^n x_i = x$, and $x_i \neq 0$, for each i, and this ultrafilter converges to μ for the duality with $h(F)$. This follows from ([4], 30.9).
Let j be the gauge of K; since one has $j > 0$ on $X \setminus 0$ and since $j(x) < \infty$, one can write:
$\sum_1^n \varepsilon_{x_i} = \sum_1^n j(x_i).\varepsilon_{y_i}$, with $y_i = x_i/j(x_i)$.
The Radon measure $\sum_1^n j(x_i).\varepsilon_{y_i}$, on S, converges, with respect to $\mathcal{U}$, to a Radon measure m on S, since $\sum_1^n j(x_i) = j(x)$, so that one has $\mu = m$ on $h(F)$.
3. Let us show that m is carried by $\mathcal{E}_g(X)$:
For each $f \in h(Y)$, let $\hat{f}$ be the function on Y defined by:
$\hat{f}(y) = \inf(g(y) : g \in -s(Y)$ and $g \geq f$ on Y), for each $y \in Y$.

Since μ is maximal on Y, one has $m(\hat{f}) = m(f)$ for each $f \in h(F)$, by Theorem 2.7.
Since S is metrizable for the duality with F', it is easy to find a sequence (f_n) of $h(F)$, such that $(X \setminus 0) \cap (\hat{f_n} = f_n) = \mathcal{E}_g(Y)$.
Since Y is a face of X, one has $\mathcal{E}_g(Y) \subset \mathcal{E}_g(X)$, hence the result. □

Theorem 3.6. (With the hypotheses and notations of the preceding theorem)
Suppose that for each $x \in X_K$ all the positive Radon measures on $\mathcal{E}_g(X)$ having x as resultant, induce the same conical measure on X.
Then X_K is a Riesz cone for its own order.

Proof. By the hypotheses, for each $x \in X_K$, there is only one maximal conical measure $\mu \in M^+(Y)$ having x as resultant.
Since X_K is a face of Y, for each $\lambda \in M^+(Y)$, such that $\lambda \leq \mu$, one has $r(\lambda) \in X_K$.
Hence X_K is isomorphic to a face of $M^+(Y)$ and so X_K is a Riesz cone. □

Remark 3.7. The converse of the preceding theorem is false, even for a well-capped closed cone. We shall give an example in 5.1.

The following elementary lemma will permit us to use Theorem 3.5:

Lemma 3.8.
Let X be a convex proper cone, not necessarily closed, contained in an l.c.s. E, equipped with its weak topology.
Let K_1 and K_2 be two pseudo-caps of X such that $\overline{K_1}$ and $\overline{K_2}$ are two compact subsets of X, satisfying $\overline{K_1} \subset K_2$.
Then, the hereditary envelope of $\overline{K_1}$ in X, denoted by $s(\overline{K_1})$, is a convex compact subset of X, contained in $\overline{K_2}$.

Proof. Suppose $x \in s(\overline{K_1})$; there are $y \in \overline{K_1}$ and $z \in X$ such that $x + z = y$; one has also $y \in K_2$ since $\overline{K_1} \subset K_2$, hence $x, z \in K_2$, since K_2 is a pseudo-cap.
Hence the lemma, since $\overline{K_1}$ and $\overline{K_2}$ are compact convex subsets of X:
Using an ultrafilter $\mathcal{U}$ on $s(\overline{K_1})$, it is immediate that y converges to an element of $\overline{K_1}$ and x, z to elements of $\overline{K_2}$. □

Here is a first consequence of Theorem 3.5 and of the preceding lemma:

Theorem 3.9. *Let X be a proper convex cone, not necessarily closed, contained in an l.c.s. E, equipped with its weak topology.*
Let K_1 and K_2 be two pseudo-caps of X, such that $\overline{K_1}$ and $\overline{K_2}$ are compact metrizable subsets of X, satisfying $\overline{K_1} \subset K_2$.
Then, for each $x \in X_{K_1}$, there is a Radon measure $m \geq 0$ on $\mathcal{E}_g(X)$, integrating all the elements of E', and such that $r(m) = x$.

Proof. We apply Theorem 3.5, with $K = K_1$ and $S = s(\overline{K_1})$, which is a metrizable compact convex subset of X, by Lemma 3.8. □

The preceding theorem suggests the following definition, which can be compared with Definition 2.1 of *well-capped* cones.

Definition 3.10. (Pseudo-well-capped cones)
Let X be a proper convex cone, not necessarily closed, contained in an l.c.s. E, equipped with its weak topology.
We say that X is *pseudo-well-capped* by $\mathcal{K}$, if there is a family $\mathcal{K}$ of pseudo-caps of X such that:

1. $\cup K = X$, where $K \in \mathcal{K}$.
2. For each $K \in \mathcal{K}$, then $\overline{K}$ is a compact subset of X.
3. For each $K_1 \in \mathcal{K}$ there is $K_2 \in \mathcal{K}$, such that $\overline{K_1} \subset K_2$.

Of course this definition remains meaningfull if E is an l.c.s. not necessarily equipped with its weak topology. In what follows we shall never use such an extension of the definition.

Now we are able to prove the following theorem of integral representation which is the main result of this work.

Theorem 3.11. (With the notations of Definition 3.10)
Suppose E is an l.c.s. equipped with its weak topology.
Let $X \subset E$ be a cone which is pseudo-well-capped by $\mathcal{K}$, and such that each $K \in \mathcal{K}$ is metrizable. Then the cone X has I.R.

Proof. Apply Theorem 3.9. □

Remark 3.12. We shall give in 5.2 an example, due to A. Goullet de Rugy, of a pseudo-cap K of a cone X, such that $\overline{K}$ is a compact subset of X, but not a cap of X.

The following proposition concerns *the stability* of the class of pseudo-well-capped cones under some operations:

Proposition 3.13.

1. *Any (relatively) closed convex subcone of a pseudo-well-capped cone is itself pseudo-well-capped.*
2. *Any denumerable product of pseudo-well-capped cones is also pseudo-well-capped.*

Proof.
1. Obvious.
2. Let $X_n \subset E_n$ be a sequence of pseudo-well-capped cones and let $\mathcal{K}_n$ be a family of pseudo-caps satisfying the three conditions of Definition 3.10. Let $E = \prod E_n$ and let $X = \prod X_n$. Let $\mathcal{K}$ be the family of all pseudo-caps K of X such that, for each $x = (x_n)$, one has:
$j_K(x) = \sum_1^\infty a_n.j_n(x_n)$, where j_n is the gauge of $K_n \in \mathcal{K}_n$ and each a_n is > 0.
It is immediate that K is a pseudo-cap of X; let us prove that the family $\mathcal{K}$ satisfies the three conditions of Definition 3.10:

1. $\cup K = X$, where $K \in \mathcal{K}$: It is immediate.
2. If $K \in \mathcal{K}$, then $K \subset \prod K_n/a_n$, hence $\overline{K}$ is a compact subset of X, since $\overline{K_n}$ is a compact subset of X_n.
3. If $j_K(x) = \sum_1^\infty a_n.j_n(x_n)$, using the preceding notations, for each integer n, choose $K'_n \in \mathcal{K}_n$, such that $\overline{K_n} \subset K'_n$; we denote by j'_n the gauge of K'_n.

For each $x = (x_n)$ we set:
$j'(x) = \sum_1^\infty 2^{-n}.a_n.j'_n(x_n)$; it is immediate that j' is the gauge of some $K' \in \mathcal{K}$; then we have:

$\overline{K} \subset \prod \overline{K_n}/a_n \subset \prod K'_n/a_n \subset K'$.
The proof is now complete. □

Recall the following definition which is useful for Theorem 3.15.

Definition 3.14. (Localizable conical measures)
Let $X \in \mathcal{S}$; a conical measure $\mu \in M^+(X)$ is said to be *representable* or *localizable* on $X \setminus 0$ if there is a positive Radon measure on $X \setminus 0$ which is equal to μ on $h(X)$.

Theorem 3.15. (With the notations of Definition 3.10)
Suppose E is an l.c.s. equipped with its weak topology.
Let $X \subset E$ be a pseudo-well-capped cone. Let (K_n) be a sequence of elements of $\mathcal{K}$ such that $\overline{K_n} \subset K_{n+1}$ for each n. Then the cone Z, generated by the union of the sets K_n, has the three following Properties:

1. *Z is a face of X.*
2. *$Z \in \mathcal{S}$, for the duality with the space A_0 consisting of all affine homogeneous functions on Z, whose restrictions to each Z_n are continuous.*
3. *Each positive conical measure on Z, for the duality with A_0, is localizable on $Z \setminus 0$.*

Proof.
1. Obvious by Proposition 3.4.
2. Obvious by Theorem 2.8 and Lemma 3.8.
3. The proof of part 2 of Theorem 3.5 works also for a conical measure which is not maximal:
Indeed, if $r(\mu) \in K_n$, then μ is localizable on $\overline{K_n}$. □

4. Pseudo-well-capped cones and conuclear cones

In this part, we will compare the class of pseudo-well-capped cones and that of conuclear cones of E. Thomas ([8]). Indeed, in Theorem 4.8 we shall prove that these two classes of cones are identical when the space is equipped with its weak topology.
First we need the following definition:

Definition 4.1. Let X be a proper convex cone, contained in a vector space V, not necessarily equipped with a topology. Let A and B be two convex subsets of X, containing 0; we say that A is *capped* by B, and we note $A \prec B$, if, for each finite sequence $(x_1, \ldots, x_n)$ of X, such that $\sum_1^n x_i \in A$, one has: $\sum_1^n j_B(x_i) \leq 1$.
It is immediate that any pseudo-cap of X is capped by itself.

The following elementary lemma enables us to consider only hereditary subsets.

Lemma 4.2. (With the notations of the preceding definition)
If $A \prec B$ then $\hat{A} \prec \hat{B}$, if we denote by $\hat{A}$ and by $\hat{B}$ the hereditary envelopes of A and B, for the order of X.

Proof. It is sufficient to prove that $\hat{A} \prec B$, since $j_{\hat{B}} \leq j_B$.
If $\sum_1^n x_i \in \hat{A}$, there is $x_0 \in X$ such that $(x_0 + \sum_1^n x_i) \in A$.
The lemma follows immediately. □

We will give an interpretation of the relation $A \prec B$ in terms of pseudo-caps. For this purpose we shall use a technique of L. Asimow ([1], prop.2.3. and [2], prop.5.).

Lemma 4.3. *Let X be a proper convex cone, not necessarily closed, contained in an l.c.s. E, equipped with its weak topology. Let A and B be two compact convex subsets of X, containing 0. We suppose that A is hereditary for the order of X.*
Then the two following properties are equivalent :

(1) *One has $A \prec B$.*
(2) *For each $\varepsilon > 0$, there is an algebraic hyperplane of the vector subspace E_A generated by A, which separates A and $E_A \cap j_B^{-1}(1+\varepsilon)$.*

Proof.
(1) $\Rightarrow$ (2) (The heredity of A will not be used here).
Let q be the norm on E_A such that the associated unit ball is $(A - A)$. Let us show that the two following sets are disjoint:
$B_\varepsilon = conv(E_A \cap j_B^{-1}(1+\varepsilon))$ and $A_\varepsilon = A + \varepsilon/2.(A - A)$.
It is sufficient to prove that the following equality is impossible:
$\sum_1^n \alpha_i.x_i = a_0 + \varepsilon/2.(a_1 - a_2)$, where each x_i is an element of X, $j_B(x_i) = 1+\varepsilon$, and each α_i is ≥ 0, with sum 1 and $a_0, a_1, a_2 \in A$:
Indeed, suppose that: $\varepsilon/2.a_2 + \sum_1^n \alpha_i.x_i = a_0 + \varepsilon/2.a_1$. Since $A \prec B$ we have $(1+\varepsilon) \leq 1 + \varepsilon/2$, which is a contradiction.
Now it is sufficient to apply the separation theorem of Hahn-Banach to B_ε and A_ε.
(2) $\Rightarrow$ (1)
Let l_ε be a linear form on E_A such that $(l_\varepsilon = 1)$ is the hyperplane mentioned in (2).
If $\sum_1^n x_i \in A$, where each x_i is an element of X, then $x_i \in E_A$ for each i, since A is hereditary, hence $\sum_1^n l_\varepsilon(x_i) \leq 1$.
Consequently one has $(\sum_1^n j_B(x_i)/(1+\varepsilon) \leq \sum_1^n l_\varepsilon(x_i) \leq 1$.
Hence the result when $\varepsilon \to 0$. □

The following lemma is a translation, in terms of pseudo-caps, of the preceding lemma:

Lemma 4.4. (With the hypotheses of the preceding lemma)
The two following properties are equivalent:

(1) *One has $A \prec B$.*
(2) *For each $\varepsilon \geq 0$ there is a pseudo-cap K_ε of X, such that $A \subset K_\varepsilon \subset (1+\varepsilon).B$.*

Proof.
(1) $\Rightarrow$ (2)
Since A is hereditary in X, the set $E_A \cap X$ is a face of X; consequently the set $(x : x \in E_A \cap X \textit{ and } l_\varepsilon(x) \leq 1)$ is a pseudo-cap of X, denoted by K_ε, with property (2).
(2) $\Rightarrow$ (1)
This follows from (2) $\Rightarrow$ (1) of the preceding lemma, using the gauge of the pseudo-cap K_ε. □

We are now able to give the definition of the conuclear cones of E. Thomas ([8]):

Definition 4.5. (Conuclear cones)
Let X be a proper convex cone, not necessarily closed, contained in an l.c.s. E, not necessarily equipped with its weak topology. The cone X is called *conuclear* if there is a

family Θ of compact convex subsets of X, containing 0, whose union is X, and such that for each $A \in \Theta$, there is $B \in \Theta$ such that $A \prec B$.
Of course X remains conuclear if E is equipped with its weak topology. Indeed E. Thomas uses this definition even when E is not equipped with its weak topology. We shall see why in Remarks 4.10.

The following elementary lemma is useful, for the use of hereditary subsets:

Lemma 4.6. *For each $A \in \Theta$, its hereditary envelope, denoted by $\hat{A}$, is a compact convex subset of X.*

Proof. Let $x \in \hat{A}$; there are $y \in A$ and $z \in X$ such that $y = x + z$. Choose $B \in \Theta$ such that $A \prec B$; one has $x, z \in B$, hence the result, since B is compact. □

Lemma 4.7. *If X is conuclear for the family Θ, it is also conuclear for the following family: $\hat{\Theta} = \{\hat{A} : A \in \Theta\}$.*

Proof. Obvious by Lemmas 4.2 and 4.6. □

Here now is the identification we were looking for:

Theorem 4.8. *Let X be a proper convex cone, not necessarily closed, contained in an l.c.s. E equipped with its weak topology. The two following properties are equivalent:*

1. *X is pseudo-well-capped.*
2. *X is conuclear.*

Proof.
(1) $\Rightarrow$ (2)
Let $\mathcal{K}$ be a family of pseudo-caps of X, satisfying the conditions of Definition 3.10:
Let $\Theta = (\overline{K} : K \in \mathcal{K})$; it is immediate that $\overline{K_1} \prec \overline{K_2}$ if $\overline{K_1} \subset K_2$. Hence X is conuclear for the family Θ.
(2) $\Rightarrow$ (1)
Let Θ be a family of convex compact subsets of X, satisfying the conditions of definition 4.5:
By Lemma 4.7, we may assume that the menbers of Θ are hereditary and that Θ is stable by positive scalar multiplications. Let $\mathcal{K}$ be the family of all pseudo-caps K of X such that there is $A, B \in \Theta$ satisfying $A \subset K \subset B$.
Then, it is immediate that X is pseudo-well-capped, by Lemma 4.4. □

As a consequence of the preceding theorem, Theorem 3.11 can be given the following formulation:

Theorem 4.9. (With the notations of Definition 4.5)
Suppose E is an l.c.s. not necessarily equipped with its weak topology.
Let $X \subset E$ be a conuclear cone for a family Θ such that each $A \in \Theta$ is metrizable. Then X has I.R.

Proof. Obvious, since X is also conuclear when E is equipped with its weak topology.
□

Remark 4.10. E. Thomas ([8]) proved the preceding theorem using two other hypotheses on X:
1. X is closed.
2. Condition ⟪*Convex Envelope*⟫, or (C.E.): Each compact subset of X is contained in a convex compact subset of X.

Note that X can satisfy condition (2) for the topology of E but not for its weak topology and compare with Definition 4.5.

The two hypotheses of the preceding remark were used by E. Thomas ([8]) to prove the following lemma, not proved here, which can be compared with part 2 of the proof of Theorem 3.5:

Lemma 4.11. *Let X be a closed convex cone, having (C.E), contained in an l.c.s. E. For each positive Radon measure m on $X \setminus 0$, integrating all the elements of E', such that $r(m) \in X$, the conical measure μ_m induced by m, is the limit, for the duality with $h(X)$, of a net of conical measures of the following form:*
$\sum_1^n \varepsilon_{x_i}$, *with $x_i \in X$ for each i, and $\sum_1^n x_i = r(m)$.*

E. Thomas ([8]) proved also the following theorem not proved here and which can be compared with Theorem 3.6:

Theorem 4.12. (With the hypotheses of Theorem 4.9)
Suppose that X is closed and has (C.E). If X is a Riesz cone then, for each $x \in X$, all the positive Radon measures on $\mathcal{E}_g(X)$ having x as resultant, induce the same conical measure on X.

Remark 4.13. We shall give in 5.1 an example of a cone X, showing that, in the preceding theorem, the condition (C.E.) cannot be dropped.

5. Examples

Here are a few examples, interesting for various reasons:

Example 5.1. (Showing that condition (C.E.) cannot be dropped in Theorem 4.12)
Let $T = [0, 1]$. Let E be the vector space generated by all the Dirac measures ε_x, where $x \in T$, and by the Lebesgue measure λ. We equip E with the weak* topology: Hence E' is not the dual of E for the norm topology.
Let X be the convex subcone of E generated by all the Dirac measures ε_x and by λ:
It is immediate that X is a Riesz cone, but λ is the resultant of two different conical measures, localizable on $\mathcal{E}_g(X)$:
λ is the resultant of ε_λ and of $\int_T \varepsilon_x.\lambda(dx)$.
The cone X does not enjoy condition (C.E.), although it is the union of its finite dimensional caps and it is closed in E.

Example 5.2. (A. Goullet de Rugy [6])
Let $\mathcal{M}$ be the Banach space of all Radon measures on $[0, 1]$, equipped with the duality with the space $\mathcal{C} = \mathcal{C}\ ([0, 1])$; we denote by $\mathcal{M}_1$ the unit ball of $\mathcal{M}$.
Choose $a \in [0, 1]$ and let X be the cone generated by the following compact convex set $A = \{\lambda - \varepsilon_a : \lambda \in \mathcal{M}_1\}$. We set $X_1 = X \cap \mathcal{M}_1$.

We will recall the main properties of X, giving only sketchy proofs. Here is an easy key lemma without proof:

Lemma 5.3. *One has:*

$$X = \bigcup_{k \geq 0} k((\lambda - \varepsilon_a) : \|\lambda\| \leq 1; \lambda(\{a\}) = 0)$$

Theorem 5.4.
1. *Let ϕ be the linear form on $\mathcal{M}$ defined by $\phi(\nu) = -2.\nu(\{a\})$; then:* $\|x\| \leq \phi(x) \leq 2.\|x\|$, *for each $x \in X$.*
2. *One has $X \in \mathcal{S}$, for the duality with $\mathcal{C}$.*
3. *The element -ε_a does not belong to any cap of X.*

Proof.
1. It is an immediate consequence of the preceding lemma.
2. It is a consequence of Theorem 2.4, since X_1 contains a ball of $\mathcal{M}$.
3. Let j be the gauge of a cap containing -ε_a; since X contains a ball of center -ε_a it is immediate that j is finite and continuous, hence X would have a compact base for the duality with $\mathcal{C}$. One can prove that this is impossible, since a is not an isolated point of T. □

As an immediate consequence of the preceding theorem we have:

Theorem 5.5. *The cone X is pseudo-well-capped and has I.R.*

Proof.
X is pseudo-well-capped by the family of all the sets: $k.\{x : x \in X \textit{ and } \phi(x) \leq 1\}$ where k is a positive scalar.
X has I.R. by Theorem 3.11, since X_1 is compact and metrizable for the duality with $\mathcal{C}$.
□

Remark 5.6. The construction of example 5.2 can be carried out starting from any compact set T and any non isolated point a of T:
Theorem 5.4 remains true for the cone constructed in this way; Theorem 5.5 also, if T is metrizable.

5.7. Other examples

Proposition 3.13 shows the existence of a lot of pseudo-well-capped cones:
For example, if $X^1, X^2, \ldots$ denotes a sequence of cones built as in 5.2 and 5.6, starting from $T_1, T_2, \ldots$ and from $a_1, a_2, \ldots$, where for each n, $a_n \in T_n$ and is not isolated, one has:
Any closed convex subcone of the product $\prod X^n$ is pseudo-well-capped and it has I.R., if the spaces T_n are metrizable.

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