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A Continuum of Minimal Pairs of Compact Convex Sets which are not Connected by Translations

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Dedicated to R. T. Rockafellar on his 60th Birthday

Pairs of compact convex sets naturally arise in quasidifferential calculus as the sub- and superdifferentials of the directional derivative of a quasidifferentiable function (see [1]). Since the sub- and superdifferential in a given point are not uniquely determined, minimal representations are of special importance. For the 2-dimensional case, equivalent minimal pairs of compact convex sets are uniquely determined up to translations (see [2],[13]). For the 3-dimensional case, J. Grzybowski [2] gave an example of finitely many equivalent minimal pairs of compact convex sets which are not connected by translations. In this paper we construct for the 3-dimensional case a continuum of equivalent minimal pairs of compact convex sets which are not connected by translation for different indices. Moreover, we present a more general method of reducing pairs of compact convex sets by hyperplanes as in [7].

Keywords : Convex Analysis, quasidifferential calculus, pairs of compact convex sets.

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1. Introduction

In this note we consider inclusion minimal representants for elements of the Rådström -Hörmander lattice of equivalence classes of pairs of nonempty compact convex sets (see [4], [6], [7], [9], [10]). As in [6] we denote for a real topological vector space X the set of all nonempty compact convex subsets by $\mathcal{K}(X)$ and the set of all pairs of nonempty compact convex subsets by $\mathcal{K}^2(X)$, i.e. $\mathcal{K}^2(X) = \mathcal{K}(X) \times \mathcal{K}(X)$. The equivalence relation between pairs of compact convex sets is given by the relation $(A, B) \sim (C, D)$ if and only if $A + D = B + C$ using the Minkowski-sum, and a partial order is given by the relation: $(A, B) \leq (C, D)$ if and only if $(A, B) \sim (C, D)$ and $A \subseteq C$, $B \subseteq D$.
Pairs of compact convex sets naturally arise in quasidifferential calculus as the sub- and superdifferentials of the directional derivative of a quasidifferentiable function (see [1], [11]).

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To be more precise: Let $U \subseteq X$ be an open subset of a real locally convex topological vector space. A continuous function $f : U \to \mathbb{R}$ is said to be *quasidifferentiable at* $x_0 \in U$ if and only if the following conditions are satisfied:

i) for every $h \in X$ the directional derivative

$$\left.\frac{df}{dh}\right|_{x_0} = \lim_{\substack{\alpha \to 0 \\ \alpha > 0}} \frac{f(x_0 + \alpha h) - f(x_0)}{\alpha}$$

exists;

ii) there exist two nonempty weak-*-compact convex subsets $\overline{\partial} f|_{x_0}, \underline{\partial} f|_{x_0} \subseteq X^*$ such that

$$\left.\frac{df}{dh}\right|_{x_0} =: df|_{x_0}(h) = \sup_{v \in \underline{\partial} f|_{x_0}} < v, h > + \inf_{v \in \overline{\partial} f|_{x_0}} < v, h > ,$$

where $< \ , \ >$ denotes the dual pairing.

Note that a quasidifferential $Df|_{x_0} = (\underline{\partial} f|_{x_0}, \overline{\partial} f|_{x_0})$ is not uniquely determined, which is a severe drawback as pointed out in [3].

Therefore a reduction or even a minimal representation of a quasidifferential is of special importance. In this connection, we mention the following two problems:

i) In [1], chapter 17, it is shown that for a quasidifferentiable function of max-min-type (i.e. a function generated by finitely many max- and min-operations over a finite set of C^1-functions) all steepest ascent- and descent directions can be determined from a quasidifferential by solving finitely many convex quadratic programs.

In fact, given a quasidifferential $Df|_{x_0} = (\underline{\partial} f|_{x_0}, \overline{\partial} f|_{x_0})$, then every steepest descent direction in $x_0 \in \mathbb{R}^n$ is given by

$$g^* := -\frac{w_0 + v_0}{\|w_0 + v_0\|}$$

with

$$\|w_0 + v_0\| = \sup_{\overline{\partial} f|_{x_0}} \inf_{\underline{\partial} f|_{x_0}} \|w + v\|.$$

An analogous formula holds for the steepest ascent-directions. For a function of max-min-type, the sets $\underline{\partial} f|_{x_0}$, $\overline{\partial} f|_{x_0}$ can be choosen as polytopes.

Since the steepest ascent- and descent directions are invariants of a function, the latter formulas led to the investigation of minimal representations of quasidifferentials.

ii) In [11] the relation between the Clarke subdifferential $\partial_{cl} f|_{x_0}$ and a quasidifferential $Df|_{x_0} = (\underline{\partial} f|_{x_0}, \overline{\partial} f|_{x_0})$ is investigated. In this connection two operations, considered as "set-differences" are introduced, namely:

$$\overline{\partial} f|_{x_0} \dot{-} \underline{\partial} f|_{x_0} \text{ and } \overline{\partial} f|_{x_0} \ddot{-} \underline{\partial} f|_{x_0}.$$

For a large class of locally Lipschitz quasidifferentiable functions, which contain all functions of max-min-type, the following inclusions are proved:

$$\overline{\partial} f|_{x_0} \dot{-} (-\underline{\partial} f|_{x_0}) \subseteq \partial_{cl} f|_{x_0} \subseteq \overline{\partial} f|_{x_0} \ddot{-} (-\underline{\partial} f|_{x_0})$$

While the first expression is independent of the specific choice of a quasidifferential, (since $\partial_{cl}(df|_{x_0}(h)\Big|_{h=0} = \overline{\partial}f|_{x_0} \dot{-} (-\underline{\partial}f|_{x_0})$) the second depends on it. To get a good upper bound for Clarke's subdifferential, it is therefore of considerable interest to work with a minimal representant of the quasidifferential.

In this paper, we will show that in Euclidean space $\mathbb{R}^n$ of dimension greater or equal to three there exists a continuous family of equivalent minimal pairs of compact convex sets which are not connected by translations. So the use of minimal pairs of compact convex sets may not necessarily lead to an essential reduction of the two problems mentioned above, i.e of determining the steepest descent- or ascent-directions or the upper estimates for the Clarke-subdifferential. However, the classification of minimal pairs of nonempty compact convex sets is an interesting problem in Convex Analysis (see [10])

Let us first set some notations: Let X be a real topological vector space, $f \in X^*$ be a continuous linear functional, and $K \subseteq X$ a nonempty compact convex set.

Then we denote by

$$H_f(K) := \{z \in K \mid f(z) = \max_{y \in K} f(y)\}$$

the face of K with respect to f. For the sum of the faces of two nonempty compact convex sets $A, B \subseteq X$ with respect to $f \in X^*$ the following identity holds:

$$H_f(A + B) = H_f(A) + H_f(B).$$

For $A \in \mathcal{K}(X)$ we denote by $\mathcal{E}(A)$ the set of extremal points. For two compact convex sets $A, B \subseteq X$ we will use the notation

$$A \vee B := \mathrm{conv}(A \cup B),$$

where the operation "conv" (see [10]) denotes the convex hull.

For a locally convex topological vector space X, A. Pinsker [8] proved the following identity about nonempty compact convex sets $A, B, C \subseteq X$:

$$(A + C) \vee (B + C) = C + (A \vee B).$$

We will use the abbreviation $A + B \vee C$ for $A + (B \vee C)$ and $C + d$ for $C + \{d\}$ for compact convex sets A, B, C and a point d.

Finally let us state explicitely the order cancellation law (see [4], [9],[14]).

Let X be real topological vector space and $A, B, C \subseteq X$ nonempty compact convex subsets. Then the inclusion

$$A + B \subseteq A + C$$

implies

$$B \subseteq C.$$

Thus from the algebraic point of view the set $\mathcal{K}(X)$ of all nonempty compact convex subsets of a real locally convex topological vector space X is a *commutative semi-ring with cancellation property* endowed with the "*addition* $\#$ " given by:

$$A \# B := A \vee B$$

and the "*multiplication* $*$ " given by:

$$A * B := A \dotplus B.$$

Within this context, the elements of $\mathcal{K}^2(X)$ with respect to the relation $\sim$ can be considered as *fractions.*

2. The Reduction of Pairs of Nonempty Compact Convex Sets by Cutting Hyperplanes

Let X be a real locally convex topological vector space, $A \subseteq X$ a nonempty compact convex set, and $f \in X^*$ a continuous linear functional. For a point $z \in X$ let us put

$$A^+_{f,z} := \{x \in A \mid f(x) \geq f(z)\}$$

$$A^-_{f,z} := \{x \in A \mid f(x) \leq f(z)\}$$

and

$$A_{f,z} := \{x \in A \mid f(x) = f(z)\}.$$

In [7] we proved the following result:

Lemma 2.1. *Let X be a real locally convex topological vector space, $A \in \mathcal{K}(X)$ a nonempty compact convex set, $f \in X^*$ a continuous linear functional, and let us assume that $A_{f,z} \neq \emptyset$ for an element $z \in X$.*

Then the pairs

$$(A, A^-_{f,z}) \quad \text{and} \quad (A^+_{f,z}, A_{f,z}) \in \mathcal{K}^2(X)$$

are equivalent.

Now we are able to show how to reduce pairs of compact convex sets by hyperplanes.

Theorem 2.2. *Let X be a locally convex topological vector space and $(B, D) \in \mathcal{K}^2(X)$.*

a) *Assume that there exist points $z_1, d_1 \in X$ and a continuous linear functional $f_1 \in X^* \setminus \{0\}$ such that $B_{f_1,z_1} \neq \emptyset$ and $B^-_{f_1,z_1+d_1} = D^-_{f_1,z_1} + \{d_1\}$.*

Then the pairs

$$(B, D) \quad \text{and} \quad (B^+_{f_1,z_1}\ ,\ D^+_{f_1,z_1+d_1})$$

are equivalent.

b) *Moreover, if there exist points $z_2, d_2 \in X$ and a continuous linear functional $f_2 \in X^* \setminus \{0\}$ such that $B_{f_2,z_2} \neq \emptyset$, $B^-_{f_1,z_1} \cap B^+_{f_2,z_2} = \emptyset$, $D^-_{f_1,z_1+d_1} \cap D^+_{f_2,z_2+d_2} = \emptyset$, and $D^+_{f_2,z_2+d_2} = B^+_{f_2,z_2} + \{d_2\}$.*

Then the pairs

$$(B, D) \quad \text{and} \quad (B^+_{f_1,z_1} \cap B^-_{f_2,z_2}\ ,\ D^+_{f_1,z_1+d_1} \cap D^-_{f_2,z_2+d_2})$$

are equivalent.

Remark 2.3. This is a generalization of a reduction method, which was introduced in [7]. In other words: all parts of two compact convex sets which can be translated onto each other can be cut off without leaving the equivalence class.

The situation is illustrated in figure 1.

Proof. To prove part a) we cut the set $D \in \mathcal{K}(X)$ by $\{x \in X \mid f_1(x) = f_1(z_1 + d_1)\}$. This gives that $D_{f_1,z_1+d_1} = B_{f_1,z_1} + \{d_1\} \neq \emptyset$. Now by Lemma 2.1 we have:

$$B + B_{f_1,z_1} = B^-_{f_1,z_1} + B^+_{f_1,z_1}$$

and

$$D + D_{f_1,z_1+d_1} = D^-_{f_1,z_1+d_1} + D^+_{f_1,z_1+d_1}.$$

Since $D^-_{f_1,z_1+d_1} = B^-_{f_1,z_1} + \{d_1\}$ we write the second equation as:

$$D + D_{f_1,z_1+d_1} = D^+_{f_1,z_1+d_1} + B^-_{f_1,z_1} + \{d_1\}.$$

Adding both equations i.e.

$$B^-_{f_1,z_1} + B^+_{f_1,z_1} + D + D_{f_1,z_1+d_1} = B + B_{f_1,z_1} + D^+_{f_1,z_1+d_1} + B^-_{f_1,z_1} + \{d_1\}$$

and simplifying the sum by $B^-_{f_1,z_1}$ and by $D_{f_1,z_1+d_1} = B_{f_1,z_1} + \{d_1\}$ we get

$$(B, D) \sim (B^+_{f_1,z_1}\ ,\ D^+_{f_1,z_1+d_1}).$$

Figure 1

To prove part b) let us first observe that, by symmetry with respect to the origin in X, the statement obtained in part a) remains true if we exchange the exponents $+$ and $-$.

Let us now apply the same technique to the sets $B' := B^+_{f_1,z_1}$ and $D' := D^+_{f_1,z_1+d1}$.

By assumption we have:

$$D'^+_{f_2,z_2+d_2} = D^+_{f_2,z_2+d_2} = B^+_{f_2,z_2} + \{d_2\} = B'^+_{f_2,z_2} + \{d_2\},$$

and hence $B'_{f_2,z_2} \neq \emptyset$.

Now the symmetric part of a) gives for the pair $(B', D') \in \mathcal{K}^2(X)$ that

$$D' + B'^-_{f_2,z_2} = B' + D'^+_{f_2,z_2+d_2}.$$

By definition of B' and D' we have:

$$B'^-_{f_2,z_2} = B^+_{f_1,z_1} \cap B^-_{f_2,z_2}$$

and

$$D'^-_{f_2,z_2+d_2} = D^+_{f_1,z_1+d_1} \cap D^-_{f_2,z_2+d_2}.$$

Hence

$$(B, D) \sim (B', D') \sim (B^+_{f_1,z_1} \cap B^-_{f_2,z_2}\, ,\; D^+_{f_1,z_1+d_1} \cap D^-_{f_2,z_2+d_2}),$$

which proves the assertion. □

3. Properties of Equivalent Minimal Pairs of Compact Convex Sets which are constructed by a General Frustum

In [12] G. T. Sallee studied a special type of convex sets, called general frusta. By definition, a general frustum is the convex hull of two convex sets that lie in different parallel hyperplanes. We will restrict ourselves to the case of compact general frusta which can be defined as follows:

Let X be a real locally convex topological vector space, $f \in X^*$ a continuous linear functional, $z \in X$, with $f(z) \neq 0$ and $E, F \in \mathcal{K}(X)$ a nonempty compact convex set with $E, F \subset f^{-1}(0)$. Then

$$A := E \vee (F + \{z\})$$

is called a *general frustum over E and F*.

Fixing the continuous linear functional, $f \in X^*$ and the point $z \in X$, with $f(z) \neq 0$, we use the notation

$$A =: \mathbb{F}(E, F) := E \vee (F + \{z\})$$

for a general frustum over E and F.

Now we will consider pairs of general frusta:

Theorem 3.1. *Let X be a real locally convex topological vector space, $f \in X^*$ a continuous linear functional, $z \in X$, with $f(z) \neq 0$ and for $i \in \{0, 1\}$ let $E_i, F_i, U_i, V_i \in \mathcal{K}(X)$*

be nonempty compact convex sets, with $E_i, F_i, U_i, V_i \subset f^{-1}(0)$*. Let* $A_i := \mathbb{F}(E_i, F_i) := E_i \vee (F_i + \{z\})$ *and* $B_i := \mathbb{F}(U_i, V_i) := U_i \vee (V_i + \{z\})$ *be general frusta.*
Then

$$(A_0, B_0) \sim (A_1, B_1)$$

if and only if

i) $(E_0, U_0) \sim (E_1, U_1)$
ii) $(E_0 + V_1) \vee (F_0 + U_1) = (E_1 + V_0) \vee (F_1 + U_0)$
iii) $(F_0, V_0) \sim (F_1, V_1)$

Proof. This equivalence is a direct consequence of the following representation of the sum of two general frusta. Let us omit for a moment the index $i \in \{0, 1\}$ and consider the general frusta

$$A = \mathbb{F}(E, F) \text{ and } B := \mathbb{F}(U, V).$$

Then the formula:

$$A + B = (E + U) \vee ([(E + V) \vee (U + F)] + \{z\}) \vee ((F + V) + \{2 \cdot z\}).$$

holds.

This formula can be shown as follows:
Obviously we have

$$A + B \supseteq (E + U) \vee ([(E + V) \vee (U + F)] + \{z\}) \vee ((F + V) + \{2 \cdot z\}),$$

since the righthand side of the inclusion contains only sums of subsets of A and B.

Since for the extremal points of a general frustum we have:
$\mathcal{E}(A) = \mathcal{E}(E) \cup \{\mathcal{E}(F) + \{z\}\}$ and $\mathcal{E}(B) = \mathcal{E}(U) \cup \{\mathcal{E}(V) + \{z\}\}$,
it follows that

$$\mathcal{E}(A + B) \subseteq \mathcal{E}(E + U) \cup ([\mathcal{E}(E + V) \cup \mathcal{E}(U + F)] + \{z\}) \cup (\mathcal{E}(F + V) + \{2 \cdot z\}).$$

Now $A + B$ is equal to the convex hull of $\mathcal{E}(A + B)$, which implies

$$A + B \subseteq (E + U) \vee ([(E + V) \vee (U + F)] + \{z\}) \vee ((F + V) + \{2 \cdot z\}),$$

and proves the formula.

The equivalence stated in the assertion follows now immediately from this formula. □

Let us remark that condition **ii)** of the above theorem can be formulated in the commutative semi-ring $\mathcal{K}(X)$ endowed with the "*addition #*" given by:

$$A \# B := A \vee B$$

and the "*multiplication* $*$" given by:

$$A * B := A + B.$$

by using the definition of the *permanent*, namely as:

$$perm \begin{bmatrix} E_0 & U_1 \\ F_0 & V_1 \end{bmatrix} = perm \begin{bmatrix} E_1 & U_0 \\ F_1 & V_0 \end{bmatrix},$$

i.e.

$$E_0 * V_1 \ \# \ F_0 * U_1 \ = \ E_1 * V_0 \ \# \ F_1 * U_0$$

4. An uncountable Family of Equivalent Minimal Pairs of Compact Convex Sets, which are not connected by Translations

In [5] we posed the question whether equivalent minimal pairs of nonempty compact convex sets are uniquely determined up to translations. An affirmative answer for the two-dimensional case was independently given by J. Grzybowski [2] and S. Scholtes [13]. Moreover J. Grzybowski [2] showed by a counterexample in $\mathbb{R}^3$ that this is not true for higher dimensional spaces. Using a different technique, we could show in [7] by modifying this counterexample that in every locally convex topological vector space X with dim $X \geq 3$ there exist equivalent minimal pairs of compact convex sets which are not related by translations and each of these sets is of full dimension.

In this paper we will construct a continuous family $(A_\alpha, B_\alpha) \in \mathcal{K}^2(\mathbb{R}^3)$, $\alpha \in [0, \infty)$ of equivalent minimal pairs of compact convex sets such that any two pairs with different indices are not related by translations.

The minimality follows from a criterium proved in [6] which we recall bellow:
Let X be a real locally convex topological vector space. For a nonempty compact convex set $A \subset X$ we consider a set $\mathcal{S} \subseteq X^* \setminus \{0\}$ such that

$$\overline{\operatorname{conv}(\bigcup_{f \in \mathcal{S}} H_f(A))} \ = \ A.$$

The sets $\mathcal{S} \subset X^* \setminus \{0\}$ of this type can be ordered by inclusion. A minimal element will be called a *shape of* A and will be denoted by $\mathcal{S}(A)$. For a shape $\mathcal{S}(A)$ we consider the subsets

$$\mathcal{S}_p(A) := \{f \in \mathcal{S}(A) \mid \operatorname{card}(H_f(A)) = 1\}$$

which may be empty and put

$$\mathcal{S}_l(A) := \mathcal{S}(A) \setminus \mathcal{S}_p(A);$$

i.e.

$$\mathcal{S}(A) \ = \ \mathcal{S}_l(A) \cup \mathcal{S}_p(A).$$

Using this notation the following result is proved in [6].

Theorem 4.1. *Let X a real locally convex topological vector space, and let $A, B \subset X$ be nonempty compact convex sets. Let us assume that there is a shape $\mathcal{S}(A)$ of A satisfying the following conditions:*

a) *for every $f \in \mathcal{S}(A)$, $card(H_f(B)) = 1$*

b) *for every $f \in \mathcal{S}_l(A)$ and every $b \in B$, the condition $\mathcal{S}_l(A) + (b - H_f(B)) \subseteq A$ implies $b \ = \ H_f(B)$.*

c) *for every $f \in \mathcal{S}_p(A)$, $H_f(A) - H_f(B) \in \mathcal{E}(A - B)$*

or conversely by interchanging A and B.

Then the pair $(A, B) \in \mathcal{K}^2(X)$ is minimal.

Remark 4.2. This is a sufficient crierium for minimality. In [6] we required for (b) a stronger condition. But the proof given in [6] is exactly the proof for the present formulation of this theorem.

In the following theorem we construct explicitely an uncountable family of equivalent minimal pairs $(A_\alpha, B_\alpha) \in \mathcal{K}^2(\mathbb{R}^3)$, $\alpha \in \mathbb{R}_+ := \{\tau \in \mathbb{R} | \tau \geq 0\}$ which are not connected by translations, i.e. for $\alpha, \beta \in \mathbb{R}_+$, $\alpha \neq \beta$ there exists no element $x \in \mathbb{R}^3$ such that

$$A_\alpha + x = A_\beta \quad \text{and} \quad B_\alpha + x = B_\beta,$$

namely:

Theorem 4.3. *Let $f \in (\mathbb{R}^3)^*$ be given by $f(x) := f((x_1, x_2, x_3)) := x_3$ and put $z := e_3 := (0,0,1) \in \mathbb{R}^3$. For $\alpha \geq 0$ define the following sets:*

i) $E_\alpha :=$ conv$\{(0,0,0),\ (1,1,0),\ (1+\alpha,0,0)\}$
ii) $F_\alpha :=$ conv$\{(0,1,0),\ (\alpha,0,0),\ (1+\alpha,1,0)\}$
iii) $U_\alpha :=$ conv$\{(0,0,0),\ (0,1,0),\ (1,1,0),\ (1+\alpha,0,0)\}$
iv) $V_\alpha :=$ conv$\{(0,1,0),\ (\alpha,0,0),\ (1+\alpha,0,0),\ (1+\alpha,1,0)\}$

Then the families of general frusta

$$A_\alpha := \mathbb{F}(E_\alpha, F_\alpha) := E_\alpha \vee (F_\alpha + \{z\})$$

$$B_\alpha := \mathbb{F}(U_\alpha, V_\alpha) := U_\alpha \vee (V_\alpha + \{z\})$$

form a family of equivalent minimal pairs

$$(A_\alpha, B_\alpha) \in \mathcal{K}^2(\mathbb{R}^3)$$

which are not connected by translations.

Remark 4.4. Before the proof of this theorem, let us draw a picture for illustration. Here, we identify the kernel of the linear functional $f \in (\mathbb{R}^3)^*$ with the subspace $\mathbb{R}^2$, i.e. *kern* $f = \mathbb{R}^2$.

Proof. The proof goes in several steps:
i) We first show that for every $\alpha \in \mathbb{R}_+ := \{\tau \in \mathbb{R} | \tau \geq 0\}$ the pairs

$$(A_0, B_0)\ ,\ (A_\alpha, B_\alpha) \in \mathcal{K}^2(\mathbb{R}^3)$$

are equivalent. Because

$$A_\alpha := \mathbb{F}(E_\alpha, F_\alpha) := E_\alpha \vee (F_\alpha + \{z\})$$

and

$$B_\alpha := \mathbb{F}(U_\alpha, V_\alpha) := U_\alpha \vee (V_\alpha + \{z\}),$$

therefore by Theorem 3.1 we have to show that

1) $(E_0, U_0) \sim (E_\alpha, U_\alpha)$,
2) $(E_0 + V_\alpha) \vee (F_0 + U_\alpha) = (E_\alpha + V_0) \vee (F_\alpha + U_0)$,
3) $(F_0, V_0) \sim (F_\alpha, V_\alpha)$.

The conditions 1) and 3) can be shown by using the reduction method, which is stated in Theorem 2.2, since for $\alpha \in \mathbb{R}_+ := \{\tau \in \mathbb{R} | \tau \geq 0\}$ the pairs

$$(E_\alpha, U_\alpha) \sim (F_0,\ \mathrm{conv}\{(0,0,0)\ ,\ (1,1,0)\})$$

and

$$(F_\alpha, V_\alpha) \sim (\mathrm{conv}\{(0,0,0)\ ,\ (1,1,0)\},\ E_0)$$

are equivalent.

Condition 2) can be checked by a direct calculation, since

$$\begin{aligned}
E_\alpha + V_0 &= \mathrm{conv}\{(0,0,0),(0,1,0),(1,2,0),(2,2,0),(2+\alpha,0,0),(2+\alpha,1,0)\}\\
F_\alpha + U_0 &= \mathrm{conv}\{(0,1,0),(0,2,0),(2,0,0),(2+\alpha,1,0),(2+\alpha,2,0)\}\\
E_0 + V_\alpha &= \mathrm{conv}\{(0,1,0),(1,2,0),(2+\alpha,2,0),(\alpha,0,0),(2+\alpha,0,0)\}\\
F_0 + U_\alpha &= \mathrm{conv}\{(0,0,0),(0,2,0),(2,2,0),(1+\alpha,0,0),(2+\alpha,1,0)\}
\end{aligned}$$

Hence

$$\begin{aligned}
&(E_0 + V_\alpha) \vee (F_0 + U_\alpha) = \\
&\mathrm{conv}\{(0,0,0),(0,2,0),(2+\alpha,0,0),(2+\alpha,2,0)\}\\
&= (E_\alpha + V_0) \vee (F_\alpha + U_0)
\end{aligned}$$

ii) Next we show, that for every $\alpha \in \mathbb{R}_+ := \{\tau \in \mathbb{R} | \tau \geq 0\}$ the pair

$$(A_\alpha, B_\alpha) \in \mathcal{K}^2(\mathbb{R}^3)$$

is minimal. We shall use Theorem 4.1.

Denote

$$a_0 := (0,0,0)\ ,\ b_0 := (1,1,0)\ ,\ c_0 := (0,1,1)\ ,\ d_0 := (1,0,1)$$

and

$$a_1 := (1,1,1)\ ,\ b_1 := (0,0,1)\ ,\ c_1 := (1,0,0)\ ,\ d_1 := (0,1,0)$$

and for $\alpha \in \mathbb{R}_+ := \{\tau \in \mathbb{R} | \tau \geq 0\}$ put

$$s_\alpha := \alpha \cdot c_1 = (\alpha, 0, 0)$$

Now we define the following sets:

$$G_1 := a_0 \vee b_0 \vee c_0 \quad \text{and} \quad G_2 := a_1 \vee b_1 \vee c_1$$

and notice that

$$A_\alpha := G_1 \vee (G_2 + s_\alpha)$$

and

$$B_\alpha := (d_1 \vee G_1) \vee ((G_2 \vee d_0) + s_\alpha).$$

Observe that as a shape for

$$A_\alpha = G_1 \vee (G_2 + s_\alpha),$$

we can choose the set

$$\mathcal{S}(A_\alpha) := \mathcal{S}_l(A_\alpha) = \{f_1, f_2\}$$

with

$$H_{f_1}(A_\alpha) = G_1 \quad \text{and} \quad H_{f_2}(A_\alpha) = G_2 + s_\alpha.$$

Since

$$\mathcal{S}_p(A_\alpha) = \emptyset,$$

we need only to check the conditions i) and ii) of Theorem 4.1.

Condition i) of Theorem 4.1 follows easily from the equalities

$$H_{f_1}(B_\alpha) = \{d_1\}$$

and

$$H_{f_2}(B_\alpha) = \{d_0\}.$$

Furthermore, condition ii) can be deduced from the facts, that for $x, y \in \mathbb{R}^3$:

$$x + G_1 \subseteq A_\alpha \quad \text{implies} \quad x = 0$$

and

$$y + (G_2 + s_\alpha) \subseteq A_\alpha \quad \text{implies} \quad y = 0.$$

It is obvious that for $\alpha, \beta \in \mathbb{R}_+$ with $\alpha \neq \beta$ the pair $(A_\alpha, B_\alpha) \in \mathcal{K}^2(\mathbb{R}^3)$ is not a translation of the pair $(A_\beta, B_\beta) \in \mathcal{K}^2(\mathbb{R}^3)$ which completes the proof. $\square$

This example generalizes essentially the example of J. Grzybowski [2] and suggests to restrict the definition of minimality.

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